A \(O(\log n)\)-OPTIMAL POLICY FOR THE ONLINE SELECTION OF A MONOTONE SUBSEQUENCE FROM A RANDOM SAMPLE

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Abstract. Given a sequence of \(n\) independent random variables with common continuous distribution, we propose a simple adaptive online policy that selects a monotone increasing subsequence. We show that the expected number of monotone increasing selections made by such policy is within \(O(\log n)\) of optimal. Our construction provides a direct and natural way for proving the \(O(\log n)\)-optimality gap. An earlier proof of the same result made crucial use of a key inequality of Bruss and Delbaen (2001) and of de-Poissonization.

Key Words. monotone subsequence, online selection, adaptive policy, Markov decision problem, dynamic programming.

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1. Introduction

In the problem of online selection of a monotone increasing subsequence, a decision maker observes sequentially a sequence of independent non-negative random variables \(\{X_1, X_2, \ldots, X_n\}\) with common continuous distribution \(F\) and seeks to construct a monotone subsequence

\[(1) \quad X_{\tau_1} \leq X_{\tau_2} \leq \cdots \leq X_{\tau_j}\]

where the indices \(1 \leq \tau_1 < \tau_2 < \cdots < \tau_j \leq n\) are stopping times with respect to the \(\sigma\)-fields \(F_i = \sigma\{X_1, X_2, \ldots, X_i\}\), \(1 \leq i \leq n\), and the trivial \(\sigma\)-field \(F_0\). Because the indices are required to be stopping times, all selection/rejection decisions are terminal. That is, if the decision maker chooses not to select the value \(X_i\) at time \(i\), then that value is lost forever. Similarly, if \(X_i\) is selected at time \(i\), then that selection cannot be changed in the future.

We call a sequence of stopping times \(1 \leq \tau_1 < \tau_2 < \cdots < \tau_j \leq n\) such that (1) holds a feasible policy, and we denote the set of all such policies by \(\Pi(n)\). For any \(\pi \in \Pi(n)\), we then let \(L_n(\pi)\) be the random variable that counts the number of selections made by policy \(\pi\) for the sample \(\{X_1, X_2, \ldots, X_n\}\). That is,

\[L_n(\pi) = \max\{j : X_{\tau_1} \leq X_{\tau_2} \leq \cdots \leq X_{\tau_j} \text{ where } 1 \leq \tau_1 < \tau_2 < \cdots < \tau_j \leq n\}.\]
Samuels and Steele (1981) first studied this selection problem and found that for each \( n \geq 1 \) there is a unique policy \( \pi^*_n \in \Pi(n) \) such that
\[
E[L_n(\pi^*_n)] = \sup_{\pi \in \Pi(n)} E[L_n(\pi)],
\]
and for such optimal policies one has the asymptotic estimate
\[
E[L_n(\pi^*_n)] \sim (2n)^{1/2} \quad \text{as } n \to \infty.
\]

Over the last few decades the understanding of policy \( \pi^*_n \) has substantially evolved. For instance, by formulating (2) as a finite-horizon Markov decision problem one sees that the optimal policy \( \pi^*_n \) is characterized by time and state dependent acceptance intervals. Furthermore, the asymptotic estimate (3) was refined with the much tighter bounds
\[
(2n)^{1/2} - O(\log n) \leq E[L_n(\pi^*_n)] \leq (2n)^{1/2} \quad \text{as } n \to \infty.
\]
The upper bound first appeared in Bruss and Robertson (1991) who studied a version of Wald’s lemma motivated by the dynamic and stochastic knapsack problem of Coffman et al. (1987). Gnedin (1999) obtained the same upper bound while considering the sequential selection of a monotone increasing subsequence from a random sample with random size.

The \( O(\log n) \) lower bound is much more recent, and it is the culmination of several successive improvements. In the context of the random-sized sample problem, Gnedin (1999) proved that the difference \( (2n)^{1/2} - E[L_n(\pi^*_n)] = O(n^{1/4} (\log n)^{1/2}) \) as \( n \to \infty \), and the delicate analyses of Rhee and Talagrand (1991) and Arlotto and Steele (2011) reduced the gap estimate to \( O(n^{1/4}) \). Finally, only recently and by leveraging substantially on previous work, Arlotto et al. (2015) settled on the \( O(\log n) \) estimate in (4). Arlotto et al. (2015) also proved the asymptotic normal convergence
\[
\frac{2^{1/2} \{L_n(\pi^*_n) - (2n)^{1/2}\}}{(2n)^{1/4}} \Rightarrow N(0,1) \quad \text{as } n \to \infty,
\]
but their analysis has a peculiar feature. While the proof of the central limit theorem relies on a detailed analysis of the dynamic programming formulation associated with the optimization problem (2), the proof of the lower bound (4) comes from de-Poissonizing a key inequality of Bruss and Delbaen (2001).

In two noteworthy papers, Bruss and Delbaen (2001, 2004) studied the mean-optimal sequential selection of a monotone increasing subsequence when the observations \( X_1, X_2, \ldots \) are revealed at the arrival epochs of a unit-rate Poisson process on \([0, n]\). The Bruss and Delbaen’s results are highly non-trivial and their mean lower bound is crucial to the de-Poissonization argument. As such, it is natural to ask if one can construct an adaptive online policy that attains the lower bound (4) or, equivalently, that is \( O(\log n) \) optimal. In the earlier analyses that lead to the proof of the \( O(n^{1/4}) \)-optimality gap, Rhee and Talagrand (1991) and Arlotto and Steele (2011) considered the suboptimal policy that selects values that are within \((2n)^{1/2}\) of the last selected observation. Such a suboptimal policy is non-adaptive — i.e., the threshold \((2n)^{1/2}\) used to make selections does not depend on the last selected value and on the number of observations that are yet to be seen — and its optimality gap is too crude. In fact, the expected number of monotone increasing selections made by a such non-adaptive policy cannot even be used to center the random variable \( L_n(\pi^*_n) \) around \((2n)^{1/2}\) in the weak law (5).
In this paper, we construct a simple adaptive online policy \( \hat{\pi}_n \) that is \( O(\log n) \) optimal. The policy is characterized by a sequence of functions \( \hat{h}_n, \hat{h}_{n-1}, \ldots, \hat{h}_1 \) such that if the value of the last selection up to and including time \( i \) is \( s \) and if \( k = n - i \) observations remain to be seen then the value \( X_{i+1} \) is selected if and only if \( X_{i+1} \) falls in the acceptance interval \( [s, \hat{h}_{n-i}(s)] \). In terms of the stopping times, the policy \( \hat{\pi}_n \) corresponds to setting \( \hat{\tau}_0 = 0, X_{\hat{\tau}_0} = 0 \), and then defining the stopping times \( \hat{\tau}_1 < \hat{\tau}_2 < \cdots < \hat{\tau}_j \) recursively as

\[
\hat{\tau}_j = \min\{ \hat{\tau}_{j-1} < i \leq n : X_i \in [X_{\hat{\tau}_{j-1}}, \hat{h}_{n-i+1}(X_{\hat{\tau}_{j-1}})] \} \quad \text{for } 1 \leq j \leq n,
\]

with the convention that if the set of indices on the right-hand side is empty, then \( \hat{\tau}_j = \infty \). The random variable \( L_n(\hat{\pi}_n) \) denotes the number of monotone increasing selections made by policy \( \hat{\pi}_n \), and the expected value of \( L_n(\hat{\pi}_n) \) satisfies the two bounds given in the next theorem.

**Theorem 1** (\( O(\log n) \)-Optimal Policy). For each \( n \geq 1 \) one has that

\[
(2n)^{1/2} - 2\{\log(n) + 1\} \leq \mathbb{E}[L_n(\hat{\pi}_n)] \leq (2n)^{1/2}.
\]

We discuss the structure of policy \( \hat{\pi}_n \) and of the functions \( \{\hat{h}_k : 1 \leq k < \infty\} \) in Section 2 and then we turn to the proof of Theorem 1. The upper bound in (6) immediately follows from (4) and the optimality of policy \( \pi^*_n \), but our analysis requires a generalization of the upper bound (4) which we study in Section 3. The proof of the lower bound (6) then follows in Sections 4 and 5. Finally, in Section 6 we make concluding remarks and underscore some open problems.

### 2. Policy \( \hat{\pi}_n \) and Its Value Function

For any feasible policy \( \pi \) and any continuous distribution \( F \), we see that the number of selections made by \( \pi \) for the sample \( \{X_1, X_2, \ldots, X_n\} \) is unchanged if we replace each \( X_i \) by its monotone transformation \( F^{-1}(X_i) \). Thus, we can assume without loss of generality that the \( X_i \)'s are uniformly distributed on \([0, 1]\). Next, we let

\[
\hat{h}_k(s) = \min\left\{ s + [2k^{-1}(1 - s)]^{1/2}, 1 \right\} \quad \text{for all } s \in [0, 1] \text{ and all } k \geq 1,
\]

and we use the sequence of functions \( \{\hat{h}_k : 1 \leq k < \infty\} \) to construct appropriate acceptance intervals. Specifically, if \( s \) denotes the value of the last selection when \( k \) observations are yet to be seen and \( x \) is the \( k \)-to-last presented value, then \( x \) is selected as element of the subsequence that is under construction if and only if \( x \in [s, \hat{h}_k(s)] \).

We now define the critical value \( s_k = \max\{1 - 2k^{-1}, 0\} \) and we note that for \( s \in [0, s_k] \) the decision maker is conservative and selects the \( k \)-to-last value \( x \) if and only if it is within \( \{2k^{-1}(1 - s)\}^{1/2} \) of the most recent selection \( s \). On the other hand, if \( s \in [s_k, 1] \) the decision maker is greedy and accepts any \( k \)-to-last value \( x \) that is larger than the most recent selection \( s \).

If \( k \) denotes the number of observations that are yet to be seen and \( s \) is the value of the most recent selection, then we let \( \hat{v}_k(s) \) denote the expected number of monotone increasing selections made by the acceptance interval policy characterized by the functions \( \hat{h}_k, \hat{h}_{k-1}, \ldots, \hat{h}_1 \). The functions \( \{\hat{v}_k : 1 \leq k \leq n\} \) are the
value functions associated with policy $\hat{\pi}_n$ and they can be obtained recursively. Specifically, if $\hat{v}_0(s) = 0$ for all $s \in [0, 1]$, then for $k \geq 1$ we have the recursion

$$\hat{v}_k(s) = \{1 - \hat{h}_k(s) + s\}\hat{v}_{k-1}(s) + \int_s^{\hat{h}_k(s)} \{1 + \hat{v}_{k-1}(x)\} dx. \tag{8}$$

To see why this recursion holds, we condition on the $k$-to-last uniform random value. With probability $1 - \hat{h}_k(s) + s$ the newly presented value $x$ does not fall in the acceptance interval $[s, \hat{h}_k(s)]$. In this case no selection is made and we are left with $k - 1$ remaining observations and with the value of the most recent selection $s$ unchanged. This amounts to an expected number of remaining selections equal to $\hat{v}_{k-1}(s)$ and it justifies the first summand of our recursion (8). On the other hand, with probability $\hat{h}_k(s) - s$ the newly presented value $x$ falls in the acceptance interval and we obtain a reward of one for selecting $x$ plus the expected number of remaining selections over the next $k - 1$ observations when the value of the most recent selection changes to $x$. Integrating this over all $x \in [s, \hat{h}_k(s)]$ gives us the second summand of the recursive equation (8). The value functions $\{\hat{v}_k : 1 \leq k < \infty\}$ are all continuous on $[0, 1]$ and their behavior is well summarized by the following theorem.

**Theorem 2 (\(\hat{v}_k\) Bounds).** For all $k \geq 1$ and all $s \in [0, 1]$ one has that

$$\{2k(1 - s)\}^{1/2} - 2\{\log(k) + 1\} \leq \hat{v}_k(s) \leq \{2k(1 - s)\}^{1/2}. \tag{9}$$

Because policy $\hat{\pi}_n$ is characterized by the thresholds $\hat{h}_n, \hat{h}_{n-1}, \ldots, \hat{h}_1$ and by the initial state $s = 0$, one then has the equivalence

$$E[L_n(\hat{\pi}_n)] = \hat{v}_n(0) \quad \text{for all } n \geq 1.$$

Hence, Theorem 1 is an immediate corollary of Theorem 2, but the upper bound (9) is a refinement of (4) to an arbitrary initial state. In fact the same upper bound holds for all feasible policies based on acceptance intervals, including the optimal one.

### 3. A Refined Prophet Upper Bound

In this section, we prove the upper bound (9) by showing that it holds for all policies that are based on acceptance intervals. The heuristic $\hat{\pi}_n$ and the unique optimal policy $\pi^*_n$ both have this property. The argument we provide draws substantially from the earlier analyses of Gnedin (1999) and Bruss and Delbaen (2001), but it takes advantage of the flexibility that comes from allowing for an arbitrary initial state. Specifically, here we assume that the first subsequence element can be selected only if it is larger than an arbitrary value $s \in [0, 1]$. In contrast, in the classical formulation one always takes the initial state $s = 0$.

For any $k \geq 1$, an arbitrary acceptance interval policy $\pi_k$ is given by a sequence $h_k, h_{k-1}, \ldots, h_1$ of functions such that

$$s \leq h_j(s) \leq 1 \quad \text{for all } 1 \leq j \leq k \text{ and all } s \in [0, 1].$$

If $s$ denotes the initial state or the value of the last observation selected prior to being presented with the $j$-to-last value $x$, then $x$ is selected if and only if $x \in [s, h_j(s)]$. Next, for any $s \in [0, 1]$ we set $M_0 = s$ and we let $M_j$ denote the maximum between $M_0$ and the largest of the elements of the subsequence that
have been selected up to and including time \( i \). The number of selections from \( \{X_1, X_2, \ldots, X_k\} \) when \( M_0 = s \) is then given by the random variable

\[
L_k(\pi_k, s) = \sum_{i=1}^{k} 1(X_i \in [M_{i-1}, h_{k-i+1}(M_{i-1})]).
\]

If we now take expectations on both sides and use twice the Cauchy-Schwarz inequality, we obtain

\[
E[L_k(\pi_k, s)] = \sum_{i=1}^{k} E[h_{k-i+1}(M_{i-1}) - M_{i-1}]
\]

\[
\leq k^{1/2} \left\{ \sum_{i=1}^{k} \left( E[h_{k-i+1}(M_{i-1}) - M_{i-1}]^2 \right) \right\}^{1/2}
\]

\[
\leq k^{1/2} \left\{ \sum_{i=1}^{k} \left[ (h_{k-i+1}(M_{i-1}) - M_{i-1})^2 \right] \right\}^{1/2}.
\]

The definition of \( M_i \) as the maximum between \( M_0 \) and the largest of the elements of the subsequence that have been selected up to and including time \( i \) tells us that

\[
M_i = \begin{cases} 
M_{i-1} & \text{if } X_i \notin [M_{i-1}, h_{k-i+1}(M_{i-1})] \\
X_i & \text{if } X_i \in [M_{i-1}, h_{k-i+1}(M_{i-1})]
\end{cases}
\]

and, because \( X_i \) is uniformly distributed on the unit interval, we have the identity

\[
E[M_i - M_{i-1} | F_{i-1}] = \int_{M_{i-1}}^{h_{k-i+1}(M_{i-1})} (x - M_{i-1}) \, dx = \frac{1}{2} (h_{k-i+1}(M_{i-1}) - M_{i-1})^2.
\]

Taking the total expectation then gives

\[
E[\{h_{k-i+1}(M_{i-1}) - M_{i-1}\}^2] = 2E[M_i] - E[M_{i-1}],
\]

so by our first bound and telescoping we find

\[
E[L_k(\pi_k, s)] \leq k^{1/2} \left\{ 2 \sum_{i=1}^{k} \left( E[M_i] - E[M_{i-1}] \right) \right\}^{1/2} = (2k)^{1/2} \{E[M_k] - E[M_0]\}^{1/2}.
\]

Here we have that \( E[M_k] \leq 1 \) and \( M_0 = s \) so we finally obtain that

\[
E[L_k(\pi_k, s)] \leq \{2k(1-s)\}^{1/2} \quad \text{for all } k \geq 1 \text{ and all } s \in [0, 1],
\]

just as needed to complete the proof of the upper bound (5).

4. RESIDUAL FUNCTIONS AND RESIDUAL DIFFERENCES

The upper bound (5) tells us that \( \hat{v}_k(s) \leq \{2k(1-s)\}^{1/2} \) for all \( s \in [0, 1] \) and all \( 1 \leq k < \infty \), so if we consider the residual function

\[
r_k(s) = \{2k(1-s)\}^{1/2} - \hat{v}_k(s) \quad \text{for } s \in [0, 1] \text{ and } 1 \leq k < \infty,
\]

we can obtain a recursion for \( \hat{v}_k \) that is equivalent to (3). Specifically, when we substitute the value function \( \hat{v}_{k-1}(y) \) with the difference \( [2(k-1)(1-y)]^{1/2} - r_{k-1}(y) \)

\[
\hat{v}_k(y) = \frac{1}{2} \left( h_{k-1}(y) + h_{k-2}(y) \right) + \int_y^{h_{k-1}(y)} \hat{v}_{k-1}(x) \, dx.
\]

\[
E[L_k(\pi_k, s)] \leq \{2k(1-s)\}^{1/2} \quad \text{for all } k \geq 1 \text{ and all } s \in [0, 1],
\]

just as needed to complete the proof of the upper bound (5).
we obtain that for all $y$ the conservative and greedy regimes. Nevertheless, the function $\delta$
This inequality implies that the residual difference $r$
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Proposition 3 (Residual Differences Bound)
(18) $r$
k bounded for each $s$
so when we plug-in the recursion (13) on the right-hand side of (12) and use the definition of $\delta$
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so when we plug-in the recursion (13) on the right-hand side of (12) and use the definition of $\delta$
(16) $r_k = \max_{0 \leq s \leq 1} r_k(s)$ for $1 \leq k < \infty$.
If we now substitute the residual functions on the right-hand side of (15) with their maximal value, we find the inequality
This inequality implies that the residual difference $r_k(s) - r_{k-1}$ is bounded above by $\delta_k(s)$, and we will shortly see that the behavior of the function $\delta_k$ changes in the conservative and greedy regimes. Nevertheless, the function $\delta_k$ is appropriately bounded for each $k \geq 1$.
Proposition 3 (Residual Differences Bound). For each $k \geq 1$ one has that
(18) $r_k(s) - r_{k-1} \leq \delta_k(s) \leq \frac{2}{k}$ for all $s \in [0, 1]$.
The first inequality of (18) is just a rearrangement of (17), so we focus on the second inequality. Its proof is given in the next two sections. Section 4.1 studies the conservative regime, while Section 4.2 deals with the greedy regime.
4.1. The Function $\delta_k$ in the Conservative Regime
We begin our analysis of the function $\delta_k$ in the conservative regime by stating the following monotonicity lemma.
Lemma 4 (Monotonicity of the $\delta$-Functions in the Conservative Regime). For each $k \geq 3$ and $s_k = 1 - 2k^{-1}$, the function $\delta_k : [0, 1] \to \mathbb{R}$ defined by (14) is non-decreasing on $[0, s_k]$. Moreover, we have that
(19) $\delta_k(s) \leq \delta_k(s_k) \leq \frac{4}{3k}$ for all $s \in [0, s_k]$. 
We prove the monotonicity of $\delta_k$ on $[0,s_k]$ by showing that the derivative $\delta_k'$ is non-negative on $(0,s_k)$. For $k \geq 3$ and $s \in [0,s_k]$, we see from (17) that the function $\tilde{h}_k(s) = s + [2k^{-1}(1-s)]^{1/2}$, and we obtain from (14) that $\delta_k$ is continuous and differentiable on $(0,s_k)$. Furthermore, we also have that $0 \leq [k(1-s)]^{-1} \leq 2^{-1}$, so when we differentiate both sides of (14) and rearrange, we obtain that

$$\delta_k'(s) = \left[\frac{(k-1)(1-s)}{2}\right]^{1/2} \gamma_k \left(\frac{1}{k(1-s)}\right),$$

where $\gamma_k$ is defined for all $y \in [0,2^{-1}]$ by

$$\gamma_k(y) = 2+yk \left[1-(1-k^{-1})^{1/2}\right] - 2(1-(2y)^{1/2})^{1/2} \left\{2 - [1-(2y)^{1/2}]^{1/2}\right\}.$$

Of course, here we have that the first factor of (20) is non-negative, and we prove the non-negativity of the second factor by showing that $\gamma_k$ is non-negative on the whole interval $[0,2^{-1}]$. As first step we recall the inequality

$$\frac{1}{2} = k \left[1 - \left(1 - \frac{1}{2k}\right)\right] \leq k \left[1 - (1-k^{-1})^{1/2}\right],$$

and we replace the second summand of $\gamma_k$ with this last lower bound. This then implies that for all $y \in [0,2^{-1}]$, one has that

$$2 + \frac{1}{2} y - 2[1-(2y)^{1/2}]^{1/2} - 2(y)^{1/2} \left\{2 - [1-(2y)^{1/2}]^{1/2}\right\} \leq \gamma_k(y).$$

The next lemma confirms that the left-hand side of this last inequality is non-negative, completing the proof of the monotonicity property in Lemma 4.

**Lemma 5.** If $y \in [0,2^{-1}]$, then one has the inequality

$$\left(2y\right)^{1/2} \left\{2 - \left[1-(2y)^{1/2}\right]^{1/2}\right\} \leq 2 + \frac{1}{2} y - 2\left[1-(2y)^{1/2}\right]^{1/2}.$$

The proof of Lemma 5 uses a infinite series representation of the two quantities that appear on each side of inequality (21). The next lemma represents and intermediate step, and it isolates an important property of the coefficients that appear in such series.

**Lemma 6 (Coefficient Difference Inequality).** If

$$a_j = \frac{(2j)!}{(2j-1)(j!)^22^{j/2}},$$

then one has the inequality

$$2^{1/2}a_j - a_{j-1} \geq 0 \quad \text{for all } j \geq 3.$$

**Proof.** By definition of $a_j$ we have $a_j \geq 0$ for all $j \geq 1$. Moreover, we also have the recursion

$$a_j = \frac{(2j-3)}{2^{(j/2)}} a_{j-1}.$$

It then follows that the difference

$$2^{1/2}a_j - a_{j-1} = \left(\frac{2j-3}{j}\right) a_{j-1} - a_{j-1} = \left(\frac{j-3}{j}\right) a_{j-1} \geq 0 \quad \text{for all } j \geq 3,$$

just as needed. \(\square\)
With the inequality of Lemma 6 at our disposal, we now carry out the proof of Lemma 5.

**Proof of Lemma 5.** For \( j \geq 1 \), we let
\[
a_j = \frac{(2j)!}{(2j-1)!(2j)^{3j/2}},
\]
and we note that for \( y \in [0, 2^{-1}] \) we have the infinite series representation
\[
-\left[1 - (2y)^{1/2}\right]^{1/2} = 1 + \sum_{j=1}^{\infty} a_j y^{j/2}.
\]

The infinite series representation (22) and the fact that \( a_1 = 2^{-1/2} \) implies that the left-hand side of (21),
\[
\ell = (2y)^{1/2} \left\{2 - \left[1 - (2y)^{1/2}\right]^{1/2}\right\},
\]
can be written as
\[
\ell = (2y)^{1/2} + y + \sum_{j=2}^{\infty} 2^{1/2} a_j y^{(j+1)/2}.
\]

Similarly, the infinite series representation (22) and the estimates \( a_1 = 2^{-1/2} \) and \( a_2 = 4^{-1} \) tell us that the right-hand side of (21),
\[
r = 2 + 2^{-1} y - 2 \left[1 - (2y)^{1/2}\right]^{1/2},
\]
can be expressed as
\[
r = (2y)^{1/2} + y + \sum_{j=3}^{\infty} 2 a_j y^{j/2}.
\]

In turn, one obtains the difference
\[
r - \ell = \sum_{j=3}^{\infty} 2 a_j y^{j/2} - \sum_{j=2}^{\infty} 2^{1/2} a_j y^{(j+1)/2} = 2^{1/2} \sum_{j=3}^{\infty} \left\{2^{1/2} a_j - a_{j-1}\right\} y^{j/2},
\]
and one has that \( r - \ell \geq 0 \) because of the non-negativity of the differences \( \{2^{1/2} a_j - a_{j-1} : j \geq 3\} \) established in Lemma 6.

The upper bound (19) then follows after one notices that
\[
\delta_k(s_k) = 2 \left[1 - \left(1 - \frac{1}{k}\right) \right]^{1/2} - \frac{4}{3k} \left(\frac{k-1}{k}\right)^{1/2}.
\]

For any \( y \in [0, 1] \) we know that \( 1 - (1 - y)^{1/2} \leq y \) so we see that the first summand on the right-hand side is non-positive while the second summand is simply bounded by \( 4(3k)^{-1} \). This then completes the proof of the upper bound in Lemma 6.

### 4.2. The Function \( \delta_k \) in the Greedy Regime

If \( s \in [s_k, 1] \) for each \( k \geq 1 \) then the decision maker selects any feasible observation and the function \( \hat{h}_k(s) = 1 \). Thus, we see from carrying the integration in (14) that \( \delta_k \) can be written as
\[
(23) \quad \delta_k(s) = (1-s)^{1/2} \left\{(2k)^{1/2} - [2(k-1)]^{1/2}\right\} + (1-s) \left\{\frac{1}{3} [2(k-1)(1-s)]^{1/2} - 1\right\}.
\]

We now recall that \( s_k = \max\{0, 1 - 2k^{-1}\} \) and that for \( s \in [s_k, 1] \) one has that
\[0 \leq 1 - s \leq 2k^{-1}.

If we now replace the two factors \((1-s)^{1/2}\) in (23) with their upper bound \((2k^{-1})^{1/2}\) and rearrange, we obtain the inequality
\[
\delta_k(s) \leq 2 \left[ 1 - \left( 1 - \frac{1}{k} \right)^{1/2} \right] + (1-s) \left[ \frac{2}{3} \left( \frac{k-1}{k} \right)^{1/2} - 1 \right].
\]
Again, for any \(y \in [0,1]\) one has that \(1 - (1-y)^{1/2} \leq y\) so the first summand on the right-hand side is bounded by \(2k^{-1}\). Moreover, we also see that the second summand is non-positive and we finally obtain that
\[
\delta_k(s) \leq \frac{2}{k} \quad \text{for all } s \in [s_k, 1].
\]
This last bound and inequality (19) together complete the proof of (18).

5. Completion of the Proof of the Lower Bound

Inequality (18) in Proposition 3 plays a crucial role in the proof of the lower bound (9). In fact, it suffices to recall from the definition of the maximal residual (16) that
\[
r_k = \max_{0 \leq s \leq 1} \{ [2k(1-s)]^{1/2} - \hat{v}_k(s) \}
\]
and to write \(r_k\) as the telescoping sum
\[
r_k = \sum_{j=1}^{k} (r_j - r_{j-1}).
\]
The bound (18) tells us that each summand on the right-hand side is bounded by \(2j^{-1}\) so we obtain that
\[
r_k \leq 2 \{ \log(k) + 1 \} \quad \text{for all } k \geq 1,
\]
completing the proof of Theorem 2.

6. Connections and Observations

Policy \(\hat{\pi}_n\) is a simple adaptive online policy that selects a monotone increasing subsequence. It is then easy to see how it could be generalized to the sequential selection of a unimodal or \(d\)-modal subsequence considered by Arlotto and Steele (2011). For instance, in the unimodal case with \(n\) observations, one could set the turning time \(\bar{n} = \lfloor n/2 \rfloor\) and run policy \(\hat{\pi}_n\) to construct an increasing segment with the first \(\bar{n}\) observations, and a decreasing version of \(\hat{\pi}_{n-\bar{n}}\) to obtain a decreasing segment over the next \(n-\bar{n}\) observations. Theorem 4 then would immediately apply to this construction, proving a \(O(\log n)\)-optimality gap for the sequential selection of unimodal and \(d\)-modal subsequences.

It is also reasonable to expect that policy \(\hat{\pi}_n\) could generalize to the sequential selection of coordinatewise increasing subsequences from a uniform random sample on the \(m\)-dimensional hypercube. Thus far, the best optimality-gap estimate comes from the work of Baryshnikov and Gnedin (2000) who use a non-adaptive policy to derive a \(O(n^{1/(2m+2)})\) bound. The adaptive character of policy \(\hat{\pi}_n\) could be fruitful one more time and help establish a \(O(\log n)\)-optimality gap for this multidimensional problem.

Many interesting questions however remain open. Theorem 1 tells us that
\[
\mathbb{E}[L_n(\pi_n)] \leq \mathbb{E}[L_n(\hat{\pi}_n)] + 2 \{ \log(n) + 1 \} \quad \text{for all } n \geq 1,
\]
but the actual expected performance of policy $\hat{\pi}_n$ seems to be much tighter. Based on an extensive numerical analysis\(^1\) we conjecture that there is a constant $0 < c < \infty$ such that

$$\mathbb{E}[L_n(\pi^*_n)] \leq \mathbb{E}[L_n(\hat{\pi}_n)] + c \quad \text{for all } n \geq 1.$$ 

Of course, this would be a substantial improvement and it would also imply that the functions $\hat{h}_n, \hat{h}_{n-1}, \ldots, \hat{h}_1$ are remarkably close to their analogues that arise when implementing the optimal dynamic programming algorithm.

Furthermore, it would also be interesting to study in greater detail the difference

$$g(n) = (2n)^{1/2} - \mathbb{E}[L_n(\pi^*_n)].$$

In the closely related problem in which the sequence $X_1, X_2, \ldots, X_n$ is given by a uniform random permutation of the integers $\{1, 2, \ldots, n\}$, [Peng and Steele (2016)] showed that the corresponding difference is $O(\log n)$ as $n \to \infty$. In our context with independent and identically distributed observations, the simplest open question is whether $g(n)$ diverges as $n$ grows to infinity, but its answer is unlikely to be easy.

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**References**


\(^1\)We estimated numerically the value functions that solve the recursion \([\text{8}]\) and its optimal counterpart on a discretized state space with a grid size of $10^{-5}$ and with $k$ ranging from 1 to $10^4$. 