We study the behavior of strategic customers in an open-routing service network with multiple stations. When a customer enters the network, she is free to choose the sequence of stations that she visits, with the objective of minimizing her expected total system time. For the two-station game with deterministic service times, we prove that the game is supermodular. By applying the supermodularity result, we observe that strategic customers “herd,” i.e., in equilibrium all customers choose the same route. We then identify a broad class of learning rules—which includes both fictitious play and Cournot best-response—that converges to herding in finite time. We also find that the herding behavior is prevalent in many other open-routing service networks, including those with stochastic service times and those with more than two stations.

Key words: service networks, herding, game theory, queueing

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1. Introduction

In many large entertainment or commercial service environments, various services are provided at different stations. Most of these environments exhibit the open routing characteristic, i.e., customers are self-interested and can visit the service stations in any sequence. For example, upon entering the Magic Kingdom at Walt Disney World®, customers may visit their preferred attractions in any order that they wish (Bleiberg 2007). Similarly, patrons arriving to a shopping mall often plan to visit multiple stores, but they can choose any sequence. In both cases, a customer wishing to minimize her waiting time must account not only for her own routing decision, but also for the decisions made by others. The dynamics of such networks are more complex than, say, those of a grocery store where customers must choose a checkout lane. In that example, once a customer has chosen her preferred lane, she is not impacted by the service process of any lane but her own. However, in a network with open routing, customers who choose different routes can still end up at the same station at the same time, directly impacting each other’s waiting times. For instance, a customer who finishes service at station 1 and joins the queue at station 2 may encounter another customer there who chose to visit station 2 first and will visit station 1 later. This open routing feature is also prevalent in other service systems such as trade fairs, exhibitions, and festivals.
In this paper, we study how strategic customers choose their routes in an open-routing service environment. Most of the applicable insights can be uncovered by analyzing a two-station service network, in which customers require services from both stations. Each station serves its customers on a first-come, first-serve (FCFS) basis. To model strategic open routing, we initially assume that each customer chooses her own route to acquire service from both stations in the network, with the objective of minimizing her expected total time in the system.

We take Nash equilibrium as our solution concept to describe the strategic behavior of rational customers. Intuitively, one might expect that a rational customer would attempt to avoid the popular route in order to receive service as quickly as possible. However, closer inspection reveals that a rational customer’s decision is driven instead by the need to get into a favorable position at the congested station. This leads to a surprising herding behavior, such that in equilibrium all customers end up choosing exactly the same route. This behavior is related to the notion of “follow the crowd” as discussed in Hassin and Haviv (2003). Specifically, we establish that all of the pure-strategy Nash equilibria of the routing game exhibit herding behavior, and that any mixed-strategy Nash equilibria must be unstable. Moreover, we show that a large and intuitive class of adaptive learning dynamics—which includes both fictitious play and Cournot best-response—converges to herding at one of the two stations. The herding equilibria therefore emerge as the focal equilibria of the routing game.

An early model of herding behavior in an economic context has been studied in Banerjee (1992). More recently, herding behavior similar to what we discover here has been observed in the queueing network of Veeraraghavan and Debo (2009). In that paper, customers have private information regarding the quality of service from different providers, and thus a longer queue may be a signal of higher quality, leading customers to choose the provider with the longer queue. However, the driving force behind the herding behavior is quite different from what we observe; in Veeraraghavan and Debo (2009), customers are motivated by service quality, while in our model customers attempt to minimize their expected total time in the system. In our case, the herding behavior occurs because each customer requires service at both stations. Although by starting with the less-crowded station a customer may glean a shorter wait time before beginning her first service, she will afterwards face a severe penalty upon arriving to the congested station and being near the end of the queue there. Veeraraghavan and Debo (2011) extend their earlier work by incorporating waiting costs, finding a tradeoff between perceived service quality and waiting time. However, again customers are served by only one provider.

To verify the robustness of the herding behavior, we then analyze strategic open routing in several different settings. We first study two-station games with all customers present in the system
when service becomes available. We later relax our modeling assumptions by studying games with non-strategic customers, customers who visit only one of the two stations, and service networks with more than two stations. In all of these settings, we find that strategic customers herd. The sequential version of the original game also displays herding, with the minor twist that the last-moving player alone avoids the crowd.

The effect of strategic behavior in an open-routing service network is important to analyze both for customers and for system planners. Different routing profiles may lead to different congestion levels, and inference about the behavior of others can make a significant difference in an individual customer's expected total system time. Planners who can control whether or not to allow open routing may wish to know whether the behavior of a decentralized system will differ significantly from that of a system with centralized routing. This knowledge will allow them to make more informed planning decisions, incorporating cost considerations and other factors such as customer satisfaction.

The outline of the paper is as follows. We introduce our base model in Section 1.1, and we review the related literature in Section 1.2. Section 2 discusses the open routing game, in which all customers are present in the system when service becomes available and make routing decisions about which station to visit first before learning their relative priority or the decisions of others. In Section 3, we derive the unique subgame perfect equilibrium of the sequential variant of the open routing game; in that setting, customers are aware of their priority and of the routing decisions of those with higher priorities. Section 4 relaxes some assumptions of the open routing game to allow for customers who wish to visit only one of the two stations and systems with more than two stations. Section 5 addresses steady-state systems with arrivals. Finally, Section 6 makes concluding remarks.

1.1. Model Basics: A Two-Station Service Network

Our setting is that of a service network with two stations, station $A$ and station $B$, each with a single server, and with service rates $\mu_A$ and $\mu_B$, respectively. The network serves $N$ customers (or “players”) who are all present in the system when service becomes available. We focus on the case in which the service rates are nonidentical, and thus without loss of generality we assume that the expected service time at station $A$ is greater than that at station $B$ (i.e., we have the service rate relation $\mu_A < \mu_B$). Similar to many service environments, each station operates on a FCFS basis. Figure 1 gives a visual depiction of the network.
Every customer must visit each station exactly once, but the order in which to visit the stations is unrestricted. Customers seek to minimize their expected total time in the system; their action space is the set $\{AB, BA\}$, where the first letter denotes the station visited first. We will at times refer to $AB$ customers to identify those who visit station $A$ first. Similarly, $BA$ customers are those who visit station $B$ first.

We note that the centralized (non-strategic) version of the problem can be viewed as an open-shop scheduling problem with jobs that need to be processed by two servers, where either sequence is permissible for each job. This problem is famously difficult, and the only results available for stochastic systems require that $N$ jobs all arrive at once, that service times all be memoryless, and that the mean service times be the same at both stations for each given job. (See, e.g., Pinedo 2012, Section 13.4.)

1.2. Literature Review

Our work is related to the literature on queues with strategic customers as well as the literature on congestion games. The first stream was started by Naor (1969), who studied the use of tolls to induce desired behavior among customers in a queue. The second was begun by Braess (1968), in his pioneering work demonstrating the now-famous Braess’ Paradox.

Hassin and Haviv (2003) provide a comprehensive survey of the strategic queueing literature. Our model is perhaps most closely related to the work of Parlaktürk and Kumar (2004). Theirs is one of only a few papers incorporating a stochastic network with stations in sequence in which customers choose the order of stations that they visit. The authors demonstrate the existence of unstable Nash equilibria for a two-station network in which every self-interested “job” must have two tasks performed on it, where each station can perform either task on a given job, and the first
task takes on average a shorter time. Each station has a queue for Task 1 and another queue for Task 2, and the system planner may choose which queue to serve next at each station. By contrast, in our model each station can perform only one task and each station has only one queue; hence stations cannot dynamically distinguish between different classes of customers.

Other work in the vicinity includes that of Adiri and Yechiali (1974), who relay a model of multiple queues for a single server, arranged and priced by priority. Their results mirror ours in that arriving customers choosing a queue must take into account the possibility of being preempted (cut in front of) by an arriving customer who chooses a higher priority queue. Naturally, an important distinction is that in our model there are two queues for two servers, as opposed to two or more queues for one server in their work. Most recently, Honnappa and Jain (2015) study what they call the “network concert queueing game.” For feedforward networks of several structures, they use fluid limits to determine symmetric equilibria and the price of anarchy when nonatomic users are allowed to choose both their arrival times and their routes through the network. Cohen and Kelly (1990) present an interesting analog of Braess’ Paradox in a stochastic queueing network. They illustrate that when FCFS nodes are placed in sequence with infinite-server nodes, the mean sojourn time in equilibrium can actually increase if customers are given the ability to switch from one track to the other. The analysis is done in steady state and queue lengths are assumed to be unobservable, simplifying both the queueing and game theoretic portions of the analysis. Other important work in this stream includes Altman and Shimkin (1998), Bell and Stidham Jr (1983), and Glazer and Hassin (1983).

On the empirical side, Pinilla and Prinz (2003) conduct a helpful, mainly simulation-based study of flexible routing schemes for shape deposition manufacturing. They study the standard sequential model and employ simulation to gain insights on a system which allows flexibility. They propose two examples—shape deposition manufacturing and routing in a coffee shop—which can be cast within the open routing framework, and they find from simulation that performance can be significantly improved by dynamically assigning the sequence of tasks to be performed on each item, instead of following a fixed sequence of tasks.

Our work is also related to the broad literature on strategic consumers in operations management. Notably, Cachon and Swinney (2009) introduce a model of fast-fashion retailing in which some consumers are strategic while others are not. They find that the benefit of quick-response, in which inventory can be replenished during the selling season, is greater in the presence of strategic consumers than otherwise. Allon et al. (2011) study a queueing system in which the service provider may choose whether to offer waiting-time information to the consumers, in a setting related to
classical “cheap-talk” games. They characterize multiple equilibria, and show that so-called “babbling” equilibria are always Pareto-dominated by equilibria in which useful information is shared with the customers.

Su and Zenios (2004) model the U.S. kidney allocation system as an $M/M/1$ queueing system in which potential recipients monitor both their position in the queue and the quality of the organ offered to them. They find that the current benchmark of FCFS service results in socially suboptimal allocation of organs because strategic recipients tend to refuse lower quality organs, knowing that they will likely be offered a better organ later. Modifying the queueing discipline to last-come, first-serve leads to the socially optimal outcome, although the authors note that such a discipline will likely be considered too unfair to implement in practice. Also related to organ allocation, Schummer (2016) studies the allocation of randomly arriving objects to customers arranged in a waiting list. He finds that the subgame perfect equilibrium allocation Pareto-dominates all others.

Afêche and Pavlin (2015) take a mechanism design approach to determine the optimal menu of price-lead time pairs to offer to strategic customers with valuations and delay costs jointly determined by a customer-type parameter. Customers with higher types have both an increased valuation for the item and a higher unit waiting cost. The authors identify problem instances in which the revenue-maximizing menu takes one of three distinct structures: first, instances whose solutions involve pooling; second, instances in which customers with intermediate types do not purchase; and third, instances in which the optimal menu involves a strategic delay as in Afêche (2013).

Finally, Veeraraghavan and Debo (2009) investigate competition between two service providers where queues build up and customers have private information regarding the quality of each provider. In such a model, a herding effect may occur, wherein a long queue is a signal of high service quality. In our model customers also herd, but with different motivation, as described earlier. Veeraraghavan and Debo (2011) extend their earlier work to account for waiting costs and find a tradeoff between perceived service quality and waiting time. Additional related work includes Afêche and Mendelson (2004), Debo and Veeraraghavan (2014), and Cui and Veeraraghavan (2015).

Several papers in the congestion literature also warrant discussion. Feldman and Tamir (2012) also consider “jobs” (customers) to be strategic. Their paper focuses mainly on “conflicting congestion effects,” both positive and negative, associated with the level of participation or traffic on a network. In their work, customers are allowed to choose from a set of machines working in parallel. They show that best-response dynamics do not always converge to a Nash equilibrium, but that the schedule generated by the longest processing time heuristic is indeed a Nash equilibrium if
the number of machines is “right.” An important difference from our analysis is that they do not incorporate the ordering of customers on a machine, but they present each player’s cost function as merely the overall load of the machine chosen by that player. In the standard congestion model of which Braess’ Paradox comprises a special case, Roughgarden and Tardos (2002) show that the price of anarchy is at most $4/3$ when the performance measure is the total latency of the system and the latency functions are linear. Finally, in a departure from the convention of treating congestion games as nonatomic, Cominetti et al. (2009) present results for an atomic game on freight transportation networks, and find that the resulting Nash equilibrium may be worse than the corresponding Wardrop equilibrium for the nonatomic game.

2. The Open Routing Game

In this section, we study the open routing game, in which all customers are present in the system when service becomes available and positions in the priority order are drawn uniformly at random. Players are aware that this randomization takes place, but they must make their routing decisions before learning their realized priorities or the routing decisions made by others. The open routing game is therefore equivalent to a symmetric one-shot game in which all players make routing decisions simultaneously. Players seek to minimize their expected total time spent in the system. Service times at stations $A$ and $B$ are first assumed to be deterministic with rates $\mu_A$ and $\mu_B$, respectively, such that $\mu_A < \mu_B$. In Section 2.5, we relax the deterministic service assumption to incorporate stochastic service times. These dynamics resemble service environments in which a large number of customers are present before the service starts.

We first make two observations that help us understand the system time experienced by customers choosing each of the two routes through the network.

PROPERTY 1. If station $B$ ever becomes idle, then it will never build up a queue again.

To understand Property 1, we note that, because $\mu_A < \mu_B$, the service time at station $B$ is shorter than that at station $A$. Hence, after the first service begins, the arrivals to station $B$ occur deterministically with an arrival rate that is smaller than station $B$’s service rate. Therefore, once station $B$ becomes idle, arrivals will never occur close enough together to form a queue. So, the system time for a customer who chooses route $AB$, and who has priority $j$ at station $A$, depends on whether or not station $B$ becomes idle before the customer departs station $A$. If station $B$ idles, then the customer’s system time is the sum of $j$ service times at station $A$ plus her own service time at station $B$. If station $B$ does not idle, then the customer’s system time is the sum of all of the service times at station $B$ that must be completed up to and including herself. This will include all of the $BA$ players, plus the $j−1$ players in front of her at station $A$, as well as herself.
Property 2. Station $A$ never idles from the time it begins its first service until it finishes serving all $N$ customers.

Similarly to Property 1, Property 2 follows from the fact that, because $\mu_A < \mu_B$, the service time at station $B$ is shorter than that at station $A$. As soon as station $A$ begins service, it will complete a service every $1/\mu_A$ units of time, but it will receive an arrival every $1/\mu_B < 1/\mu_A$ units of time until the last $BA$ player departs station $B$. Station $A$ will then never become idle until it finishes its workload. The one way in which station $A$ can idle is at the beginning, if all $N$ players choose route $BA$: in that case, station $A$ will idle during the first service time at station $B$, after which it will work continually until it finishes with all $N$ customers.

One might expect that players minimizing their system times would attempt to avoid each other and seek a less congested route. Instead, when the number of players $N$ is large enough, we find that in equilibrium players “herd”—that is, all players take the same route through the network.

Proposition 1 (Herding Equilibria in the Open Routing Game). For $N \geq 2\mu_A/\mu_B + 1$, the open routing game has a Nash equilibrium in which all players “herd” at station $A$, that is, take route $AB$. Furthermore, if $\mu_B < 2\mu_A$ and $N \geq \max\{\mu_B/\mu_A + 1, (2\mu_A + \mu_B)/(2\mu_A - \mu_B)\}$, then the game also has a Nash equilibrium in which all players “herd” at station $B$.

We will discuss the regime in which $\mu_B \geq 2\mu_A$ in Section 2.3, and we give the proof of this proposition and of the results that follow in Appendix A. We next give an intuitive explanation of Proposition 1.

In the first candidate profile, all players visit station $A$ first, so a player will have on average half of the other customers behind her if she visits station $A$ first. However, if she visits station $B$ first while everyone else visits station $A$ first, then she will be the last customer to receive service at station $A$. Therefore, if $N$ is large enough—in fact, $N \geq 3$ is sufficient here—then it is in her best interest not to deviate from the candidate profile. The herding equilibrium at station $B$ can be similarly explained; intuitively, if $N$ is large and the rest of the customers are slated to visit station $B$ first, then a customer is better off being in front of an average of half of the other customers at station $B$. Otherwise, after visiting station $A$ first she will certainly have to wait behind all of the other players at station $B$.

Proposition 1 establishes the existence of Nash equilibria which exhibit herding behavior. However, we must answer several questions to verify that these herding equilibria are indeed plausible: (i) can the herding equilibria be implemented via simple, decentralized learning dynamics? (ii) are there other, non-herding Nash equilibria? and (iii) what happens when some of the players are not
strategic? To address these questions, we establish a key submodularity property for the players' expected system times in the open routing game. This property then allows us to pinpoint the herding profiles as the focal equilibria of the game.

2.1. Submodularity of Expected System Time

Before deriving the submodularity property of the expected system time, we require some additional notation. Let $i$ be an arbitrary player (or customer) index, $s_i$ the strategy of player $i$, and $s_{-i}$ the vector of strategies for all of the other players. We say that $s_i = 1$ if player $i$ chooses route $AB$, and $s_i = 0$ if she chooses route $BA$. Similarly, a value of 1 for a given entry in $s_{-i}$ means that that player has chosen route $AB$, while a value of 0 for a given entry in $s_{-i}$ means that that player has chosen route $BA$. Denote by $T(s_i, s_{-i})$ the expected system time for a player who employs strategy $s_i$ when her opponents play the profile $s_{-i}$. Note that the uniform priority randomization means that the game is symmetric, and thus we require no player index on $T$. Following the definition of Topkis (1998) and letting $\leq$ denote the usual partial order, we will show that the function $T(s_i, s_{-i})$ is submodular, i.e., that it has decreasing differences. Specifically, we will find that

$$T(s_i, s_{-i}) - T(\tilde{s}_i, s_{-i}) \leq T(s_i, \tilde{s}_{-i}) - T(\tilde{s}_i, \tilde{s}_{-i}) \quad \text{for all } \tilde{s}_i \leq s_i \text{ and } \tilde{s}_{-i} \leq s_{-i}. \quad (1)$$

The decreasing differences condition (1) trivially holds if $\tilde{s}_i = s_i$, so we can focus on the case in which $\tilde{s}_i = 0$ and $s_i = 1$. Moreover, because all customers are present when service starts and priorities are drawn uniformly at random, we can replace $s_{-i}$ with the sum of its entries, $m$, and $T(s_i, s_{-i})$ with $T(s_i, m)$. The variable $m$ then simply represents the number of players—excluding player $i$—who have chosen route $AB$. With a slight abuse of notation, we can replace the decreasing differences condition (1) with the equivalent condition

$$T(1, m) - T(0, m) \leq T(1, \tilde{m}) - T(0, \tilde{m}) \quad \text{for all } 0 \leq \tilde{m} \leq m \leq N - 1. \quad (2)$$

We now introduce the shorthand

$$d_m := T(1, m) - T(0, m)$$

and rewrite condition (2) as

$$d_m \leq d_{\tilde{m}} \quad \text{for all } 0 \leq \tilde{m} \leq m \leq N - 1. \quad (3)$$

The difference $d_m$ represents the relative preference of player $i$ for route $AB$ over route $BA$, given that $m$ other players chose route $AB$; a negative value indicates that route $AB$ will result in
a shorter expected system time, and a positive value means that route BA will yield a shorter expected system time. Similarly, a relatively smaller value of $d_m$ indicates a relatively greater preference for route AB. Therefore, if equation (3) holds—and we will show that it does—then the greater the number of other customers who have chosen route AB, the greater relative preference each customer will have for route AB.

**Proposition 2 (Submodularity of Expected System Time).** If

$$N > N_{sub} := \frac{2\mu_B}{\mu_A} + 1,$$

(4)

then each player’s expected system time in the open routing game is submodular. Moreover, we have

$$d_m < d_{m-1} \quad \text{for all } 1 \leq m < N - \frac{\mu_B}{\mu_A}.$$

(5)

**Proof.** As discussed, submodularity is equivalent to the decreasing differences condition (3), i.e.,

$$d_m \leq d_{\tilde{m}} \quad \text{for all } 0 \leq \tilde{m} \leq m \leq N - 1.$$

For our proof, we divide $T(1, m)$, the expected system time for a fixed player choosing route AB when $m$ other players are also choosing the same route, into two components. We first define $\tilde{T}(1, m)$ to be the expected system time for the player choosing route AB, not counting any waiting time that she may experience in the queue at station B. Because priorities are drawn uniformly at random, we have

$$\tilde{T}(1, m) := \sum_{j=1}^{m+1} \frac{1}{m+1} \left( \frac{j}{\mu_A} + \frac{1}{\mu_B} \right) = \frac{1}{\mu_B} + \frac{m+2}{2\mu_A}.$$

(6)

Next, we define $\delta_m$ as the difference between the total expected system time and the expression in equation (6), i.e., $\delta_m := T(1, m) - \tilde{T}(1, m)$. By definition, then, $\delta_m$ is equal to the expected time spent waiting in the queue at station B.

Station A never idles until it finishes as long as at least one player chooses route AB. Therefore, for $m \geq 1$, the expected system time for a player choosing route BA when $m$ other players choose route AB is given by

$$T(0, m) = \sum_{\ell=1}^{N-m} \frac{1}{N-m} \left( \frac{m+\ell}{\mu_A} \right) = \frac{N+m+1}{2\mu_A}.$$

(7)

By equations (6) and (7), we have that for all $m \geq 1$,

$$d_m := T(1, m) - T(0, m) = \delta_m + \tilde{T}(1, m) - T(0, m) = \delta_m + \frac{1}{\mu_B} - \frac{N-1}{2\mu_A},$$

(8)

which implies that for all $1 \leq \tilde{m} \leq m \leq N - 1$,

$$d_m \leq d_{\tilde{m}} \quad \text{if and only if } \delta_m \leq \delta_{\tilde{m}}.$$

(9)
For $m \geq 1$, define $\delta_m^{(j)}$ to be the wait time in queue at station $B$ experienced by the fixed player when $m$ other players choose route $AB$, given that she chooses route $AB$ and receives priority $j$ at station $A$. Then we have the representations

$$\delta_m^{(1)} = \left( \frac{N - m - 1}{\mu_B} - \frac{1}{\mu_A} \right) +,$$

and $\delta_m^{(j+1)} = \left( \delta_m^{(j)} + \frac{1}{\mu_B} - \frac{1}{\mu_A} \right) +$ for all $j = 1, \ldots m$. Because $1/\mu_B - 1/\mu_A < 0$, we have that

$$\delta_m^{(j+1)} \leq \delta_m^{(j)} \text{ for all } j = 1, \ldots, m. \quad (10)$$

Moreover, because the function $f(x) := (x - 1/\mu_A)^+$ is non-decreasing, we also have

$$\delta_m^{(1)} \leq \delta_{m-1}^{(1)},$$

and therefore, for $j = 2, \ldots, m + 1$,

$$\delta_m^{(j)} = \left( \delta_m^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A} \right) + \leq \left( \delta_{m-1}^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A} \right) + = \delta_{m-1}^{(j)}. \quad (11)$$

We have thus established that $\delta_m^{(j)}$ is monotonically decreasing both in $j$ for any given $m$ and in $m$ for any given $j$. Next, because priorities are drawn uniformly at random, we have that

$$\delta_m = \frac{1}{m + 1} \sum_{j=1}^{m+1} \delta_m^{(j)} \text{ for all } m \geq 1. \quad (12)$$

Equation (12) and the monotonicity of $\delta_m^{(j)}$ then give us

$$\delta_m = \frac{1}{m + 1} \sum_{j=1}^{m+1} \delta_m^{(j)} \leq \frac{1}{m} \sum_{j=1}^{m} \delta_m^{(j)} \leq \frac{1}{m} \sum_{j=1}^{m} \delta_{m-1}^{(j)} = \delta_{m-1} \text{ for all } 2 \leq m \leq N - 1. \quad (13)$$

The first inequality comes from the monotonicity in $j$ from equation (10); the average of $m + 1$ real numbers is smaller than the average of the largest $m$ such numbers. The second inequality comes from the monotonicity in $m$ in equation (11). Equations (9) and (13) then imply that

$$d_m \leq d_{m-1} \text{ for all } 2 \leq m \leq N - 1, \quad (14)$$

which satisfies the decreasing differences condition. Moreover, when $2 \leq m < N - \mu_B/\mu_A$, we have that

$$\delta_m^{(1)} = \left( \frac{N - m - 1}{\mu_B} - \frac{1}{\mu_A} \right) + < \left( \frac{N - m}{\mu_B} - \frac{1}{\mu_A} \right) + = \delta_{m-1}^{(1)}, \quad (15)$$

so we conclude that the inequalities in equations (13) and (14) hold strictly in this range.
Lastly, we directly evaluate and compare $d_1$ and $d_0$. Observe that $N > N_{\text{sub}}$ implies $2/\mu_A < (N - 1)/\mu_B$, so when $m = 0$ or $m = 1$, if the fixed player chooses route $AB$, she always faces a queue at station $B$. Therefore,

$$T(1, 1) = \frac{1}{2} \left( \frac{N - 1}{\mu_B} + \frac{N}{\mu_B} \right) = \frac{2N - 1}{2\mu_B} \quad \text{and} \quad T(1, 0) = \frac{N}{\mu_B}. $$

When the player chooses route $BA$, we note that station $A$ will not idle if $m = 1$, but it idles for $1/\mu_B$ units of time if $m = 0$. Therefore,

$$T(0, 1) = \frac{1}{N - 1} \sum_{j=1}^{N-1} \frac{j + 1}{\mu_A} = \frac{N + 2}{2\mu_B} \quad \text{and} \quad T(0, 0) = \sum_{j=1}^{N} \frac{1}{N} \left( \frac{j}{\mu_A} + \frac{1}{\mu_B} \right) = \frac{1}{\mu_B} + \frac{N + 1}{2\mu_A}. $$

Applying the equations above, we have the relation

$$d_1 = \frac{N - 1}{\mu_B} - \frac{N + 1}{2\mu_A} + \left( \frac{1}{2\mu_B} - \frac{1}{2\mu_A} \right) < \frac{N - 1}{\mu_B} - \frac{N + 1}{2\mu_A} = T(1, 0) - T(0, 0) = d_0. \quad (16)$$

Equations (14) and (16) imply that the expected system time has decreasing differences and is therefore submodular, and equations (15) and (16) imply equation (5). □

Observe that $d_{\tilde{m}} < 0$ implies that, if a player were aware that exactly $\tilde{m}$ others were choosing route $AB$, then her expected system time for route $AB$ would be less than that for route $BA$, and thus she would prefer route $AB$. Furthermore, the decreasing differences property gives us that if $d_{\tilde{m}} < 0$, then $d_m < 0$ also for $\tilde{m} \leq m$, and therefore for any $\tilde{m} \leq m$, a player who knew that $m$ others had chosen route $AB$ would also want to choose route $AB$. Intuitively, a critical mass of players choosing a given route tends to attract the remaining players to the same route; if many players go to station $A$ first, then the others should join them, and similarly for station $B$. This idea will be formalized in the next subsection when we discuss convergence of adaptive dynamics to the herding equilibria. The strict inequality in equation (5) of Proposition 2 plays a key role in establishing this convergence.

Proposition 2 implies that if, as we henceforth assume, players’ utility functions decrease linearly with their expected system times, then their utility functions are supermodular, and the open routing game is a supermodular game in the sense of Topkis (1998, Section 4.1). Supermodular games have received much attention in the literature. For example, it is well documented that if a supermodular game has a unique Nash equilibrium, then a wide range of learning rules will converge to it (see Milgrom and Roberts 1990). However, because the game that we study has multiple equilibria, we cannot directly apply the classical convergence result. Nevertheless, in the subsequent subsection, we apply supermodularity and the strict inequalities in equation (5) to prove that in our game, a large class of learning rules converges to one of the herding equilibria. Next, we state a simple corollary of Proposition 2, which will be used in the derivation of the subsequent convergence result.
Corollary 1. If $N > N_{\text{sub}}$, then for any $m = 1, \ldots, N-1$, either $d_m < 0$ or $d_m < d_{m-1}$.

Proof. If $m < N - \mu_B/\mu_A$, then the statement is immediately verified by Proposition 2. If $m \geq N - \mu_B/\mu_A$, we then must have

$$\frac{N - m}{\mu_B} \leq \frac{1}{\mu_A},$$

which implies that the queue at station $B$ clears before the first $AB$ customer arrives. Recall that $\delta_m = T(1,m) - \tilde{T}(1,m)$ is equal to the expected time spent waiting in the queue at station $B$.

Because the queue at station $B$ clears before the first $AB$ customer arrives, we have $\delta_m = 0$, and equation (8) implies that

$$d_m = \frac{1}{\mu_B} - \frac{N - 1}{2\mu_A} < \frac{1}{\mu_B} - \frac{1}{\mu_A} < 0,$$

(17)

where the first inequality follows because $N > N_{\text{sub}} > 3$. □

2.2. Adaptive Dynamics Converge to Herding

As we have seen, when the service rates are close together and $N$ is large, the herding strategy profiles are Nash equilibria. We next show that in addition, a general class of decentralized learning processes will converge to one of these herding equilibria.

We propose a model of learning which allows customers to update their beliefs in each period based on the play observed and also incorporates a “memory” of past actions. First, we assume that in each period players choose their routes to minimize their expected system times based on their current beliefs. When a player faces a tie, we assume that the player always chooses route $AB$. Let $\beta^{(t)}_i = (\beta^{(t)}_{i,0}, \ldots, \beta^{(t)}_{i,N-1})$ be the vector of player $i$’s beliefs, i.e., $\beta^{(t)}_{i,j}$ is player $i$’s probability assessment, at the beginning of period $t$, that exactly $j$ players (not including herself) will take route $AB$ in period $t$, for $j = 0, 1, \ldots, N - 1$. We allow the initial beliefs, $\beta^{(1)}_i$, to be an arbitrary vector of probabilities summing to one. Next, let $x^{(t)}$ be the realized total number of players who take route $AB$ in period $t$, and let $x^{(t)}_i$ be the decision of player $i$ (with $x^{(t)}_i = 1$ for choosing route $AB$ and 0 otherwise). We also let $x^{(t)}_{-i}$ denote the realized number of players who take route $AB$ in period $t$, excluding player $i$; that is, $x^{(t)}_{-i} = x^{(t)} - x^{(t)}_i$.

For $m \in \{0, 1, \ldots, N - 1\}$, let $e(m) \in \mathbb{R}^N$ be the vector with a one in the $(m+1)$-st entry, and zeros in all of the remaining entries. Given a sequence of real numbers $\{\alpha_t\}$ with $0 \leq \alpha_t \leq 1$ for all $t = 1, 2, \ldots$, the beliefs in our model satisfy the recursion

$$\beta^{(t+1)}_i = (1 - \alpha_t)\beta^{(t)}_i + \alpha_t e(x^{(t)}_{-i}) \quad t = 1, 2, \ldots.$$  

(18)

We give the name $\{\alpha_t\}$-learning to the process in which beliefs evolve according to equation (18). We also note that choosing $\alpha_t = 1/t$ results in the familiar learning rule known as fictitious play introduced in Brown (1951). Within fictitious play learning, players best-respond to the empirical
frequency of past moves. Similarly, by letting $\alpha_t = 1$ we recover the Cournot best-response model, in which players best-respond to the path of play realized in the prior period. Under a mild regularity condition on the sequence $\{\alpha_t\}$ and on the initial beliefs $\beta_i^{(t)}$, we find that if all players update their beliefs according to the learning model (18), then play will converge to one of the herding equilibria.

**Proposition 3 (Convergence of $\{\alpha_t\}$-Learning to Herding Equilibria).** Define $m^*$ by

$$m^* = \min\{m \in \{1, \ldots, N\} : d_m \leq 0\},$$

and consider the $\{\alpha_t\}$-learning process in equation (18). If (i) $N > N_{\text{sub}}$, (ii) the sequence $\{\alpha_t\}$ is such that

$$\lim_{t \to \infty} \prod_{t = t'} (1 - \alpha_t) = 0 \quad \text{for all } t' \geq 1,$$

and (iii) there exists some $\ell \geq 1$ such that $x^{(\ell)} \neq m^*$, then players will converge to one of the herding equilibria in finitely many periods. That is, there exists $t_0 < \infty$ such that either

$$x^{(t+1)} = N \quad \text{or} \quad x^{(t+1)} = 0 \quad \text{for all } t \geq t_0.$$

**Proof.** Let $\langle \cdot, \cdot \rangle$ denote the standard vector product, and let $d$ be the vector of differences $d_m = T(1, m) - T(0, m)$ of waiting times for choosing routes $AB$ and $BA$, i.e., $d = (d_0, d_1, \ldots, d_{N-1})$. Define $\pi_i^{(t)}$ as the expected difference in system times for player $i$, given her belief at time $t$, which is given by

$$\pi_i^{(t)} := \langle \beta_i^{(t)}, d \rangle.$$

We proceed by cases.

**Case 1:** suppose that $x^{(\ell)} \geq m^* + 1$. Then for any player $i$ who chose route $AB$ in period $\ell$, we must have $x_i^{(\ell)} \geq m^*$, and also $\pi_i^{(\ell)} \leq 0$ because otherwise player $i$ would have chosen route $BA$ in period $\ell$. This implies that

$$\pi_i^{(\ell+1)} = (1 - \alpha_{\ell}) \pi_i^{(\ell)} + \alpha_{\ell} \langle e(x_i^{(\ell)}), d \rangle \leq (1 - \alpha_{\ell}) \pi_i^{(\ell)} + \alpha_{\ell} d_{m^*} \leq 0,$$

where the first inequality follows from the submodularity of the expected system time, and the second inequality follows by the definition of $m^*$. Equation (20) implies that all of the customers who chose route $AB$ in period $\ell$ will do so again in periods $\ell + 1$, $\ell + 2$, $\ldots$. Therefore, for any $t \geq \ell$, we have that $x_i^{(t)} = 1$ for each player $i$ that chose route $AB$ in period $\ell$, and this implies that

$$m^* + 1 \leq x^{(t)} \leq x^{(t)} \quad \text{for all } t \geq \ell.$$
For $\bar{t} \geq \ell$ and for any customer $i$ who chose route BA in period $\ell$, we have that
\[
\pi_i^{(\bar{t}+1)} = \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \sum_{t=\ell}^{\bar{t}} \alpha_t\langle e(x_i^{(t)}), d \rangle \prod_{s=t+1}^{\bar{t}} (1 - \alpha_s)
\leq \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \sum_{t=\ell}^{\bar{t}} \alpha_t d_{m^*+1} \prod_{s=t+1}^{\bar{t}} (1 - \alpha_s)
\overset{(22)}{=} \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\right) d_{m^*+1},
\]
where the inequality follows by the submodularity of the expected system time. Note that we always have $d_{m^*+1} < 0$, because if $d_{m^*} < 0$, then $d_{m^*+1} \leq d_{m^*} < 0$; and if $d_{m^*} = 0$, then by Corollary 1, we have $d_{m^*+1} < 0$. Combining this with the assumption in equation (19), there must exist $t_0$ such that for all $\bar{t} \geq t_0$,
\[
\pi_i^{(\bar{t}+1)} \leq \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\right) d_{m^*+1} < 0.
\]
This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 1$ for every customer $i$ that chose route BA in period $\ell$.

**Case 2:** suppose that $x^{(n)} \leq m^* - 1$. Then for any customer $i$ who chose route BA in period $\ell$, we must have $\pi_i^{(n)} > 0$ and $x_i^{(n)} \leq m^* - 1$. This implies that
\[
\pi_i^{(n+1)} = (1 - \alpha_{\ell})\pi_i^{(n)} + \alpha_{\ell}\langle e(x_i^{(n)}), d \rangle \geq (1 - \alpha_{\ell})\pi_i^{(n)} + \alpha_{\ell} d_{m^* - 1} > 0,
\]
where the first inequality follows from the submodularity of the expected system time, and the second inequality follows from the definition of $m^*$. Therefore, for any $t \geq \ell$, we have that $x_i^{(n)} = 0$ for every customer $i$ that chose route BA in period $\ell$. Finally, for $\bar{t} \geq \ell$ and for any customer $i$ that chose route AB in period $\ell$, we have
\[
\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\right) d_{m^* - 2}.
\]
The assumption in equation (19) and the fact that $d_{m^* - 2} > 0$ together imply that there must exist $t_0$ such that for all $\bar{t} \geq t_0$,
\[
\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\pi_i^{(t)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1 - \alpha_t)\right) d_{m^* - 2} > 0.
\]
This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 0$ for each customer $i$ that chose route AB in period $\ell$. $\square$
In Proposition 3, condition (19) essentially enforces that players’ earlier beliefs and actions must eventually fade so that play does not get stuck on strategy profiles that are anchored to initial beliefs. For example, natural learning rules such as fictitious play and Cournot best-response satisfy this condition. This result reinforces the intuition that many players choosing a given route exerts a pull on others to do the same. The only circumstance which can possibly avoid herding is that in which \( x(t) = m^* \) for all \( t = 1, 2, \ldots \). For this stagnation to occur, we must have \( x(1) = m^* \), an unlikely event if \( N \) is large and, say, initial beliefs are drawn as independent random vectors uniformly distributed on the \( N \)-dimensional probability simplex. Even if this first-period event is realized, route switching in later periods is inevitable because the beliefs of \( AB \) customers will be moving to favor route \( BA \), and those of \( BA \) customers will be moving to favor route \( AB \). When this route switching occurs, it must always be perfectly symmetric to maintain \( x(t) = m^* \), which is also unlikely.

Next, we state a corollary which shows that for Cournot best-response, the requirement of \( x(t) \neq m^* \) is not required when \( N \) is large enough.

**Corollary 2 (Convergence of Cournot best-response).** Under Cournot best-response, if \( N > \max\{N_{\text{sub}}, 2\mu_A/(\mu_B - \mu_A)\} \), then players will converge to one of the herding equilibria in finitely many periods.

### 2.3. Equilibrium Refinement

We have established that in the open routing game, players following intuitive learning rules such as Cournot best-response will converge to one of the herding equilibria. An immediate consequence of the results on learning is that the herding profiles are the only pure-strategy Nash equilibria for this game.

**Corollary 3 (Equilibrium Refinement).** If \( N > N_{\text{sub}} \), then the only pure-strategy Nash equilibria of the open routing game are the symmetric herding equilibria of Proposition 1.

Exploiting the decreasing differences property of the expected system time, we also find that if there is significant service rate disparity—specifically, if service at station \( B \) is more than twice as fast as service at station \( A \)—then route \( AB \) is a strictly dominant strategy for all players.

**Corollary 4 (Dominant Strategy).** If \( 2\mu_A \leq \mu_B \) and \( N > N_{\text{sub}} \), then route \( AB \) is a strictly dominant strategy for all players.

So, if the service rates are close together, then both herding profiles are Nash equilibria, while if they are far apart, then it is a dominant strategy to visit the slower station \( A \) first.

However, Corollary 3 does not completely rule out the existence of Nash equilibria in which players adopt mixed strategies. Our next result rules out any Nash equilibria in which some but
not all players use properly mixed strategies, and shows that any mixed-strategy Nash equilibrium is not stable enough to survive even the slightest perturbation in another player’s strategy.

**Proposition 4 (Elimination and Instability of Mixed Equilibria).** If $N > N_{\text{sub}}$, then the open routing game has no Nash equilibria in which some players mix and others use pure strategies. Moreover, any Nash equilibrium in which all $N$ players use mixed strategies is unstable; that is, a small perturbation in the strategy of any player will cause other players to strictly prefer a pure strategy.

Therefore, the herding equilibria are the only pure-strategy Nash equilibria of the open routing game, and no mixed-strategy Nash equilibria exist unless all of the players employ properly mixed strategies. Such mixed-strategy equilibria, however, are quite unstable and unlikely to be implemented. With these results in favor of the herding equilibria and the fact that an intuitive class of learning rules converges to them, we see that the herding effect exerts a strong influence on the behavior of rational customers in the open routing game.

### 2.4. The Open Routing Game with Non-Strategic Customers

In practice some (perhaps many) customers may not attempt to minimize their overall expected system times. For example, in an amusement park a customer may have a preference for a certain ride which causes her to visit that ride first without even considering the possibility that her decision may result in a longer expected total system time. Accordingly, we now examine the effect on the system of customers who are not strategic, that is, customers who must visit both stations but who have a pre-determined route which they will follow without contemplating any alternative.

To this end, consider a system as described in Section 1.1 with $N$ customers, and again assume that priorities are drawn uniformly at random. Suppose now that $N = N_{AB} + N_{BA} + N_S$, where $N_{AB}$ is the number of non-strategic customers who will take route $AB$ no matter what, $N_{BA}$ is the number of non-strategic customers who will take route $BA$ no matter what, and $N_S$ is the number of strategic customers. In this system, a strategic customer could feasibly encounter $N_{AB}, \ldots, N_{AB} + N_S - 1$ other customers choosing route $AB$. Therefore, we can focus on

$$d_m = T(1, m) - T(0, m), \text{ for } m = N_{AB}, \ldots, N_{AB} + N_S - 1.$$ 

Recall that $d_m$ represents the relative preference for route $AB$ over route $BA$ for a strategic customer, when there are $m$ other customers choosing route $AB$.

By Proposition 2, we immediately have that $d_m$ is decreasing in $m$ in the range of $N_{AB}, \ldots, N_{AB} + N_S - 1$ as long as $N > N_{\text{sub}}$. Therefore, as in Section 2.1, the expected system times for the customers that are strategic are submodular. The submodularity in turn implies that the herding equilibria (among the strategic customers) prevail as the only pure-strategy Nash equilibria. This is summarized in the next corollary.
Corollary 5 (Equilibria with Non-Strategic Customers). If \( N > N_{\text{sub}} \), then exactly one of the following holds:

(i) route \( AB \) is a dominant strategy for all strategic customers;
(ii) route \( BA \) is a dominant strategy for all strategic customers;
(iii) both herding profiles are Nash equilibria, and there are no other pure-strategy Nash equilibria.

We note that Proposition 3 also holds in this setting with non-strategic customers, in the sense that strategic customers will converge to one of the herding equilibria under a general class of learning dynamics. These results suggest that customers who are aware that some others are not rational can still implement an equilibrium profile which involves herding. Moreover, depending on the system parameters, herding at one station can be a dominant strategy.

2.5. The Stochastic Open Routing Game

We now discuss how the herding behavior that we observe in the open routing game continues to emerge when the service times are stochastic. We refer to this new setting as the stochastic open routing game. Specifically, for \( \mu_A < \mu_B \) we now suppose that the service times at stations \( A \) and \( B \) are independent and identically distributed random variables with means \( 1/\mu_A \) and \( 1/\mu_B \) and variances \( \sigma^2_A \) and \( \sigma^2_B \), respectively.

Properties 1 and 2 do not extend to this environment because the uncertainty in service times may induce idle and busy periods that did not arise in the deterministic setting. As a result, we can no longer express customers’ expected system times in closed form. Nevertheless, we can bound the expected system times when all customers herd by applying existing results from queueing theory. More specifically, when all customers choose route \( AB \), the waiting times at station \( B \) behave as in an underloaded \( GI/GI/1 \) queue (single-server queue with general independent arrival and service time distributions) that started empty, with arrival rate \( \mu_A \) and service rate \( \mu_B \). Then, a stochastic dominance argument tells us that we can bound the expected waiting time at station \( B \) for any customer by the expected steady-state waiting time. A classical result of Kingman (1962) then gives us a bound for the steady-state waiting time, so we can prove that the profile in which all customers choose route \( AB \) is a Nash equilibrium provided that \( N \) is large enough.

Proposition 5 (Nash Equilibrium with Route \( AB \)—Stochastic Service). If

\[
N > 1 + \frac{2\mu_A}{\mu_B} + \frac{\mu^2_A(\sigma^2_A + \sigma^2_B)}{1 - \mu_A/\mu_B},
\]

then it is a symmetric Nash equilibrium for all customers to choose route \( AB \).

Next, we consider the case in which all customers are choosing route \( BA \). In this case, the waiting times at station \( A \) behave as in an overloaded \( GI/GI/1 \) queue with arrival rate \( \mu_B \) and service
rate $\mu_A$, which does not have a steady-state distribution. To bound the expected waiting time in an overloaded $GI/GI/1$ queue, we apply Lindley’s equation to show that the total idling time is represented by the “dual” process of the overloaded system, and that this “dual” process has the same distribution as the waiting time in an underloaded $GI/GI/1$ queue (see Grimmett and Stirzaker 2001, Section 11.5). This allows us to bound the total expected idling time, and to show that under suitable conditions it is a Nash equilibrium for all customers to choose route $BA$. The result is summarized in the next proposition.

**Proposition 6 (Nash Equilibrium with Route $BA$—Stochastic Service).** If we have $\mu_A < \mu_B < 2\mu_A$ and

$$N \geq \frac{\mu_B + 2\mu_A}{2\mu_A - \mu_B} + \frac{\mu_A^2 \mu_B^2 (\sigma_A^2 + \sigma_B^2)}{(2\mu_A - \mu_B)(\mu_B - \mu_A)},$$

(26)

then it is a symmetric Nash equilibrium for all customers to choose route $BA$.

### 3. The Sequential Open Routing Game

We have thus far assumed that customers had no visibility into the state of the network when making routing decisions, i.e., that they were not aware of their relative priorities or of the decisions made by others. In many applications, however, customers may have some knowledge about their position in the queue and about the routes that others have chosen. For example, in service systems such as amusement parks, at times customers may pre-queue, allowing them at the preset opening time to choose their route conditioned on the actions of those preceding them. To study such a system, we consider the sequential open routing game, in which customers make decisions according to their relative position among the $N$ customers and can observe the routes chosen by those who move before them. This setting facilitates a natural representation as an extensive-form game, and we will find its unique subgame perfect equilibrium using backward induction.

We remark that the open routing game of Section 2 represents one extreme, in which customers have no information, while the sequential open routing game that we study here corresponds to the other extreme, in which customers have perfect information regarding their position and the state of the network prior to their decision. A real-world service system might fall somewhere in between: customers likely have a sense of their position and of the decisions made by others, though perhaps not perfect knowledge. Interestingly, in this sequential environment we observe a phenomenon that is similar to the herding behavior observed in the open routing game: in equilibrium all customers but one visit the slower station $A$ first.

#### 3.1. Subgame Perfect Equilibrium

As in the open routing game of Section 2, we consider a two-station service network with deterministic service times and $N$ customers present at the beginning of service availability. We note,
however, that unlike the open routing game of Section 2, the sequential open routing game is not symmetric, and accordingly the customer index \( i \) will no longer be arbitrary. We index customers \( i = 1, 2, \ldots, N \), by the order in which they make routing decisions, so customer 1 is the first to move, customer 2 is the second, etc. We also assume that a customer’s position in the order corresponds to her priority, so customer 1 will be served first at whichever station she chooses, and customer \( i \geq 2 \) will wait behind any customers in the set \( \{1, \ldots, i-1\} \) who have chosen the same route as her. Next, for \( i = 1, 2, \ldots, N \), we define \( y_i^A \in \{0, 1, \ldots, i\} \) to be the number of players that have chosen route \( AB \), up to and including customer \( i \). To derive the subgame perfect equilibrium via backward induction, we need to analyze the total time that player \( N \), the last customer to make her routing decision, spends in the system under different strategy profiles. The state observed by player \( N \) depends only on \( y_{N-1}^A \), the number of the first \( N-1 \) customers that chose route \( AB \).

**System Time for Customer \( N \).** Given a value of \( y_{N-1}^A \), we use \( S_N^A \) to denote the system time that customer \( N \), the last customer to move, experiences if she takes route \( AB \), and \( S_N^B \) to denote the system time that she experiences if she takes route \( BA \).

Because \( \mu_A < \mu_B \) and the service times are deterministic, we see that if customer \( N \) first visits the faster station \( B \), followed by the slower station \( A \) (i.e., she chooses route \( BA \)), then she experiences exactly the same system time as if she merely joined a queue of \( N-1 \) customers all waiting at station \( A \), as long as at least one customer before her chose route \( AB \) (i.e., as long as \( y_{N-1}^A \geq 1 \)). If no one else chose route \( AB \), that is, if \( y_{N-1}^A = 0 \), then the above is still true except that station \( A \) sits idle until the first departure from station \( B \), so customer \( N \)’s system time is increased by \( 1/\mu_B \). Therefore, we can express \( S_N^B \), the system time for the case in which customer \( N \) chooses route \( BA \), as

\[
S_N^B = \begin{cases} 
\frac{N}{\mu_A} + \frac{1}{\mu_B} & \text{if } y_{N-1}^A = 0, \\
\frac{N}{\mu_A} & \text{if } y_{N-1}^A \geq 1.
\end{cases}
\]  

(27)

Now, suppose that customer \( N \) chooses route \( AB \), visiting the slower station \( A \) first, followed by the faster station \( B \). If player \( N \) arrives at station \( B \) and finds it idle, then \( S_N^A \), the system time that she experiences if she visits station \( A \) first, is simply equal to \( (y_{N-1}^A + 1)/\mu_A \), the time that she spends at station \( A \), plus the time it takes for her to be served at station \( B \), which is equal to \( 1/\mu_B \). If player \( N \) arrives at station \( B \) and finds it busy, then by Property 1, station \( B \) must never have been idle since it started service. As player \( N \) will be the last customer served at station \( B \), we have then that \( S_N^A \) is equal to \( N/\mu_B \). To summarize, the system time that customer \( N \) experiences from visiting station \( A \) first, \( S_N^A \), is given by

\[
S_N^A = \begin{cases} 
\frac{y_{N-1}^A + 1}{\mu_A} + \frac{1}{\mu_B} & \text{if } \frac{y_{N-1}^A + 1}{\mu_A} \geq \frac{N-1}{\mu_B}, \\
\frac{N}{\mu_B} & \text{if } \frac{y_{N-1}^A + 1}{\mu_A} < \frac{N-1}{\mu_B}.
\end{cases}
\]  

(28)
where \((y_{N-1}^A + 1)/\mu_A \geq (N - 1)/\mu_B\) is the condition that ensures that station \(B\) is idle upon customer \(N\)’s departure from station \(A\).

**Equilibrium Strategy Profile.** With the evaluation of customer \(N\)’s system times we deduce her strategy in the set of subgame perfect equilibria. Because \(\mu_A < \mu_B\), whenever the second condition in equation (28) is met we have that \(S_A^N < S_B^N\), so customer \(N\)’s system time from choosing route \(AB\) is the shorter.

If instead the first condition in equation (28) holds, then we have two cases. First, if the number of preceding customers choosing route \(AB\) satisfies the inequality \(y_{N-1}^A \leq N - 2\), then \(S_A^N\) is bounded above by \((N - 1)/\mu_A + 1/\mu_B\), so we again have that \(S_A^N < S_B^N\).

Second, if \(y_{N-1}^A = N - 1\) and customer \(N\) chooses route \(AB\), then she will be served last at station \(A\) and finish there after \(N/\mu_A\) units of time, but she will still require a service time of \(1/\mu_B\) at station \(B\). On the other hand, if she takes route \(BA\), then she will immediately be served at station \(B\) and leave the system exactly when station \(A\) completes its workload, after \(N/\mu_A\) time units. The extra \(1/\mu_B\) from route \(AB\) causes \(S_A^N\) to be greater than \(S_B^N\) for \(y_{N-1}^A = N - 1\).

Summarizing, we have that \(S_B^N < S_A^N\) if and only if \(y_{N-1}^A = N - 1\), and otherwise \(S_A^N < S_B^N\). Therefore, customer \(N\)’s optimal strategy is to choose route \(BA\) if and only if \(y_{N-1}^A = N - 1\), and route \(AB\) otherwise.

With the optimal strategy profile for customer \(N\), we now inductively determine the subgame perfect equilibrium strategy profile for customers \(N-n\), for every \(n\) in the set \(\{1,\ldots,N-1\}\).

**Proposition 7 (Unique Subgame Perfect Equilibrium).** The following strategies form the unique subgame perfect equilibrium of the sequential open routing game:

1. Customers \(1,\ldots,N-1\) visit station \(A\) first in every subhistory.
2. Customer \(N\) visits station \(B\) first if and only if she observes \(y_{N-1}^A = N - 1\); otherwise, she visits station \(A\) first.

Moreover, along the equilibrium path the first \(N-1\) customers to move visit station \(A\) first, and the final customer visits station \(B\) first.

Intuitively, one might expect that customers should join the queue with the shortest wait time because that minimizes their waiting time before getting served at one of the two stations. However, Proposition 7 states that if customers are rational, then all except the last will first visit station \(A\), which is slower and which each customer except the first observes to have more customers in its queue than station \(B\). This behavior is best explained by what we call delayed overtaking. If customers later in the order will visit the slower station \(A\) first, then it is in the best interest of a customer who moves earlier to immediately join the queue at station \(A\) because otherwise, during her time at the faster station \(B\), others will overtake her at station \(A\).
Interestingly, we observe that the optimal actions of the first $N - 1$ customers are completely independent of the state that they observe and are essentially driven by the strategies of those later in the order. Note also that the equilibrium actions—i.e., the herding profiles—from the open routing game of Section 2 cannot be supported as equilibria (Nash or subgame perfect) for the sequential game. Namely, if the last customer to move is aware of her position and the decisions of the other customers, then her best response to all of the earlier customers visiting the same station (either $A$ or $B$) is always to visit the other station, breaking both of the equilibria of Proposition 1. However, in equilibrium only one customer visits station $B$ first here, so the subgame perfect equilibrium of this section is quite similar to the herding equilibrium at station $A$ discussed in Section 2. Therefore, in an environment in which decisions are made sequentially, the pull to the slower station has the strongest impact on behavior, and the delayed overtaking effect dominates.

On the whole, we find that behavior in the sequential open routing game parallels that of the one-shot version. The most important departures are (i) that the delayed overtaking effect always dominates in the sequential environment, supporting a form of herding only at the slower station and (ii) that in equilibrium the last customer, recognizing that she is the last to move, takes the opposite route of all of the other players. When customers are allowed to observe the moves of others, the ability to condition actions on others’ routes and the lack of uncertainty cause the space of equilibria to shrink.

4. The Prevalence of Herding
We have seen that the herding behavior predominates in the open routing game. With enough players, the herding equilibria are always the only pure-strategy Nash equilibria of the game. Furthermore, when the service rates are far apart, customers have a strictly dominant strategy which implements the herding equilibrium at station $A$. The same result continues to hold even when some customers are not strategic and have their routes fixed in advance, or when we allow for stochastic service times. In the sequential variant of the open routing game, we similarly observe a form of herding, albeit only at the slower station $A$. In this section, we overview additional settings in which the herding behavior continues to prevail. The technical details associated with this overview are discussed in Appendix B.

Customers Who Visit Only One Station
First, we discuss the impact of “dedicated” customers, who require service only at one of the two stations. In an amusement park, for example, some customers may like one ride better than the other and decide to visit only their favorite. Similarly, in a shopping mall some customers may only need products from one store.
In the context of the open routing game of Section 2, we now relax the assumption that all customers must visit both stations. Specifically, we decompose $N$ as $N = N_A + N_B + N_S$, where $N_A$ customers visit only station $A$, $N_B$ customers visit only station $B$, and the remaining $N_S$ customers visit both stations but may choose the order (these last we will call “strategic” customers because they must make a routing decision, just as in Section 2.4). When there are enough strategic players (i.e., when $N_S$ is large enough), both the herding profile at station $A$ and that at station $B$ are Nash equilibria of this game. The intuition behind the proof is similar to that of the open routing game. If all of the strategic customers herd at station $A$, then a customer who deviates will again be the last customer to be served at station $A$, and therefore she prefers to follow the crowd. We also observe in several numerical experiments that the expected system time appears to be submodular as long as there is not too much imbalance in the fractions of customer types. The herding behavior is robust to the presence of “dedicated” customers who require service only at one of the two stations.

**S-Station Open Routing Game**

We now discuss systems with more than two stations. Consider a generalized version of the open routing game of Section 2 for a system with $S$ stations and $N$ players. For stations $\xi = 1, 2, \ldots, S$, let $\mu_\xi$ be the service rate at station $\xi$, and assume that $\mu_1 < \mu_2 < \cdots < \mu_S$.

If players make routing decisions simultaneously and if priorities are drawn uniformly at random, then players must choose from among the $S!$ possible routing vectors. In this case, we have a Nash equilibrium which is analogous to that in the open routing game with two stations. Specifically, it is a Nash equilibrium for players to choose a route that visits stations in order of increasing service rate, from the slowest to the fastest. Thus once again we have a herding effect, where most players follow the crowd and congregate at a single station.

**5. Queueing Networks in Steady State**

Lastly, we investigate equilibrium behavior when customers arrive according to a Poisson process with rate $\lambda$, rather than all being present in the system when service becomes available. Again every customer must be processed at both station $A$ and station $B$. Service times at stations $A$ and $B$ are independent and exponentially distributed with rates $\mu_A$ and $\mu_B$, respectively. Here we take $\lambda < \mu_A < \mu_B$, and we assume that arriving customers find the queueing network in steady state and are served in order of arrival at each station. Note that $\lambda < \mu_A$ implies that, in contrast to earlier settings, with positive probability station $A$ will have idle time in between services. The state of the system is not observable to the customers, i.e., customers may not condition their actions on the system state which includes the queue lengths and types of customers at both stations.
The distribution of the queue length encountered by a customer at the first station in her route is the steady-state distribution of the queue length at that station. In addition, classical results from Kelly (1979) give us that the distribution for the queue length observed by a customer at the second station in her route also matches the steady-state distribution for the queue length at that station. Next, we state a lemma which is a version of Corollary 3.5 from Kelly (1979) modified to our queueing network, which has customer types $AB$ and $BA$, and queues $A$ and $B$.

**Lemma 1 (Kelly 1979, Corollary 3.5).** When a customer of type $\gamma \in \{AB, BA\}$ reaches station $\xi \in \{A, B\}$, the probability that she finds $\kappa$ customers at station $\xi$ is equal to the steady-state probability that there are $\kappa$ customers at station $\xi$.

We use “type” to refer to a realized route through the network ($AB$ or $BA$) and “class” to refer to an ex-ante routing probability. Specifically, let $p = (p_1, p_2, \ldots)$ be a (possibly infinite-dimensional) vector of routing probabilities corresponding to strategy profiles chosen by a positive fraction of customers, e.g., $p_1$ is the probability that a customer of class 1 chooses route $AB$. Let $z = (z_1, z_2, \ldots)$ be the corresponding fractions, i.e., the entries of $z$ are strictly positive numbers summing to 1 representing the proportions of the population belonging to the different classes. Then, we have that external arrivals of each type occur via independent Poisson streams with rate $\lambda p_{AB}$ to route $AB$ and $\lambda p_{BA}$ to route $BA$, where $p_{AB}$ and $p_{BA}$ are given by

$$p_{AB} = \langle p, z \rangle \quad \text{and} \quad p_{BA} = 1 - p_{AB}.$$ 

The following result implies that the expected system time is the same for both routes.

**Corollary 6.** For any pair of $p$ and $z$, the expected system time is the same for customers taking route $AB$ and $BA$.

**Proof.** Let $\mathbb{E}[S^A]$ be the expected system time for a customer who takes route $AB$, and similarly let $\mathbb{E}[S^B]$ be the expected system time for a customer who takes route $BA$. Also, take $L_A(p, z)$ and $L_B(p, z)$ to be the steady-state expected number of customers at stations $A$ and $B$, respectively, when the population chooses routes according to $p$ and $z$. By Lemma 1, we have

$$\mathbb{E}[S^A] = \frac{L_A(p, z)}{\mu_A} + \frac{L_B(p, z)}{\mu_B} = \mathbb{E}[S^B]. \qed$$

In this setting, we define our notion of equilibrium to be that customers of every class must be best-responding to the population proportions defined by $p$ and $z$. As the expected system time is the same for either route, customers of every class are playing a best-response to all other classes, and no class has incentive to change its strategy. Therefore, any $p$ and $z$ form an equilibrium.

Observe that when the system is allowed to reach steady state, the herding profiles in which all customers choose the same route are indeed equilibria. However, as any other feasible routing
profiles also form an equilibrium, there is no reason to give special preference to the herding profiles in this setting. Consequently, the herding behavior which prevails in Section 2 no longer predominates in underloaded memoryless systems.

6. Conclusion

We model customer behavior for service networks in which self-interested customers require service at each station and are permitted to determine their routes through the network. In our base two-station model, customers are present in the system when service becomes available and make decisions about which station to visit first. We find that the expected system time for each customer is a submodular function, and we exploit this property throughout.

In equilibrium, customers herd at one station; that is, all of the customers take the same route through the network. This behavior is motivated by the need to avoid arriving late to the congested station. If all of the other customers are visiting the same station first, then a customer who visits the other station guarantees herself to be served last at the busy one, and she is thus better off following the crowd. However, if the service rates are far apart, then it is a dominant strategy for all players to visit the slower station first. We see also that the herding behavior is stable enough that a large class of learning rules converges to one of the herding equilibria, even if players play both strategies early on.

The herding behavior of the base two-station network is prevalent in several other settings. For example, we find that herding extends to systems with stochastic service times, systems in which some customers visit only one station, and systems with more than two stations. When we study a sequential form of the open routing game, we find that the knowledge of their own priority and of the moves of others causes customers to visit the slower station first to avoid being overtaken there by later-moving customers, ruling out herding at the faster station. Still, a form of herding occurs, as in equilibrium all but the last customer to play visit the slower station first. In contrast, when we look at a system with memoryless arrivals and service times that experiences substantial idle time, we no longer observe the herding effect. For a system in steady state, different routes through the network have the same expected system time, and customers have no incentive to choose one route over the other.

In summary, our results suggest that herding in service networks with strategic open routing predominates mainly when a large number of customers are present in the system. The management of congestion and customer experience is vital in service systems. As more customers behave strategically, system planners might need to account for herding in the design of their service networks. Our analysis establishes the prevalence of herding under various assumptions, coalescing around busy systems with many customers present. Simply put, “if you can’t beat ’em, join ’em.”
Appendix A: Proofs of Results

Proof of Proposition 1. The fact that priorities are drawn uniformly at random implies that the game is symmetric, and we thus consider player (or customer) \( i \), where \( i \) is an arbitrary player index. Suppose that all customers take route \( AB \) and that \( N \geq 2\mu_A/\mu_B + 1 \). The system time experienced by player \( i \) depends on which priority she is assigned. Because all customers are taking route \( AB \), by Property 1, customer \( i \) will always find station \( B \) idle when she finishes service at station \( A \). When assigned priority \( j \), customer \( i \) will wait for \( j-1 \) players to be served at station \( A \), be served herself, and then immediately be served at station \( B \). Thus, as a function of her priority \( j \), customer \( i \)’s total system time \( S^A(j) \) is given by

\[
S^A(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}, \quad j = 1, \ldots, N.
\]  

Let \( T(1, m) \) denote customer \( i \)’s expected system time if she chooses route \( AB \) and \( m \) other customers also choose route \( AB \), and let \( T(0, m) \) denote customer \( i \)’s expected system time if she chooses route \( BA \) and \( m \) other customers choose route \( AB \). As priorities are drawn uniformly at random, customer \( i \)’s expected system time when following the candidate equilibrium strategy is given by

\[
T(1, N-1) = \sum_{j=1}^{N} \frac{1}{N} S^A(j) = \sum_{j=1}^{N} \frac{1}{N} \left( \frac{j}{\mu_A} + \frac{1}{\mu_B} \right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.
\]  

If she takes route \( BA \), then player \( i \) will be behind all \( N-1 \) other players when she gets to station \( A \). So, her total system time \( T(0, N-1) \) is deterministic and is given by

\[
T(0, N-1) = \frac{1}{\mu_B} + \left( N \frac{1}{\mu_A} - \frac{1}{\mu_B} \right) = \frac{N}{\mu_A}.
\]  

Now, our assumption that \( N \geq 2\mu_A/\mu_B + 1 \) implies that

\[
\frac{1}{\mu_B} + \frac{1}{2\mu_A} \leq \frac{N}{2\mu_A} \Rightarrow \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_A}.
\]

Therefore, customer \( i \) has no incentive to deviate as \( T(1, N-1) \leq T(0, N-1) \), and we have a Nash equilibrium.

With the herding equilibrium at station \( A \) established, assume now that \( \mu_B < 2\mu_A \) and \( N \geq \max\{\mu_B/\mu_A + 1, 2\mu_A + \mu_B/(2\mu_A - \mu_B)\} \), and suppose that all customers take route \( BA \). We will evaluate whether any customer has incentive to deviate. The condition \( N \geq (\mu_B/\mu_A) + 1 \) ensures that, if player \( i \) deviates and takes route \( AB \), then station \( B \) will not finish serving all \( N-1 \) other players before player \( i \) finishes at station \( A \). Applying the same notation as the previous case, customer \( i \)’s total system time \( T(1, 0) \) from taking route \( AB \) is deterministic and is given by

\[
T(1, 0) = \frac{N}{\mu_B}.
\]  

If customer \( i \) takes route \( BA \), then her priority at station \( B \) is drawn uniformly at random, and with probability \( 1/N \) she will be in position \( j \), for \( j = 1, \ldots, N \). Suppose that she draws priority \( j \); she will wait for \( j-1 \) customers to be served at station \( B \), be served herself there, and then wait in a queue at station \( A \). Because all customers take route \( BA \), station \( A \) will idle for the first \( 1/\mu_B \) units of time, and then will work continuously until it has processed all \( N \) customers. Player \( i \)’s system time corresponds to the time when station \( A \) finishes with the \( j \)-th job. Thus, we have that customer \( i \)’s system time \( S^B(j) \) is

\[
S^B(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}.
\]
We can now calculate her expected total system time \( T(0, 0) \) as
\[
T(0, 0) = \sum_{j=1}^{N} \frac{1}{N} S^B(j) = \sum_{j=1}^{N} \frac{1}{N} \left( \frac{j}{\mu_A} + \frac{1}{\mu_B} \right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.
\]

By our assumption that \( \mu_B < 2\mu_A \) and \( N \geq (2\mu_A + \mu_B)/(2\mu_A - \mu_B) \), we have
\[
2\mu_A + \mu_B \leq (2\mu_A - \mu_B)N \implies \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_B}.
\]

Therefore we have \( T(0, 0) \leq T(1, 0) \), implying that no customer has incentive to deviate, and we have a Nash equilibrium. \( \square \)

**Proof of Corollary 2.** By Proposition 3, we know that if there exists some \( \ell \geq 0 \) such that \( x^{(\ell)} \neq m^* \), then Cournot best-response will converge to herding because it is a special case of \( \{\alpha_t\} \)-learning. We proceed to show that with Cournot best-response, there always exists some such \( \ell \). Consider an arbitrary path of play in which customers play Cournot best-response, that is, \( \{\alpha_t\} \)-learning with \( \alpha_t = 1 \) for all \( t \geq 1 \). In period \( t \), \( x^{(t)} \) players choose route \( AB \). If \( x^{(t)} \neq m^* \), then Proposition 3 implies that play will converge to herding in finitely many periods. If instead \( x^{(t)} = m^* \), then we have \( d_{x^{(t)}} \leq 0 \) and \( d_{x^{(t-1)}} > 0 \) by the definition of \( m^* \).

Clearly, if \( N \) is odd, then \( m^* \neq N/2 \), and next, we show that \( m^* \neq N/2 \) when \( N \) is even. Define \( Q_\frac{N}{2} \) by
\[
Q_\frac{N}{2} := \min\left\{ \frac{N}{2}, \left\lfloor \frac{\mu_A(N/2 - 1)}{\mu_B - \mu_A} \right\rfloor \right\}.
\]

The quantity \( Q_\frac{N}{2} \) represents the number of \( AB \) customers who face a queue at station \( B \) when they depart station \( A \), given that a total of \( N/2 \) customers chose route \( AB \). We now have
\[
T(1, \frac{N}{2} - 1) = \sum_{k=1}^{Q_\frac{N}{2}} \frac{1}{(\frac{N}{2})} \left( \frac{k + \frac{N}{2}}{\mu_B} \right) + \sum_{k=Q_\frac{N}{2}+1}^{\frac{N}{2}} \frac{1}{(\frac{N}{2})} \left( \frac{k}{\mu_A} + \frac{1}{\mu_B} \right) \quad (33)
\]
and
\[
T(0, \frac{N}{2} - 1) = \sum_{j=1}^{\frac{N}{2}+1} \left( \frac{1}{(\frac{N}{2})+1} \right) \left( \frac{j + \frac{N}{2} - 1}{\mu_A} \right) \
\geq \sum_{k=1}^{\frac{N}{2}} \frac{1}{(\frac{N}{2})} \left( \frac{j + \frac{N}{2} - 1}{\mu_A} \right), \quad (34)
\]
where the inequality in equation (34) comes from the fact that the average of a set of \( (N/2)+1 \) real numbers is larger than the average of the smallest \( N/2 \) numbers in the set. We can then write
\[
d_{\frac{N}{2} - 1} = T(1, \frac{N}{2} - 1) - T(0, \frac{N}{2} - 1) 
\leq \sum_{k=1}^{Q_\frac{N}{2}} \left( \frac{1}{(\frac{N}{2})} \left( \frac{k + \frac{N}{2}}{\mu_B} - \frac{k + \frac{N}{2} - 1}{\mu_A} \right) \right) + \sum_{k=Q_\frac{N}{2}+1}^{\frac{N}{2}} \left( \frac{1}{(\frac{N}{2})} \left( \frac{k}{\mu_A} + \frac{1}{\mu_B} - \frac{k + \frac{N}{2} - 1}{\mu_A} \right) \right).
\]

From the assumption that \( N > 2\mu_A/(\mu_B - \mu_A) \), we get
\[
\frac{k + \frac{N}{2}}{\mu_B} - \frac{k + \frac{N}{2} - 1}{\mu_A} = -\frac{N(\mu_B - \mu_A)}{2\mu_A \mu_B} + \frac{k}{\mu_B} - \frac{k-1}{\mu_A} < \frac{k-1}{\mu_B} - \frac{k-1}{\mu_A} \leq 0;
\]
which implies that every term in the first summation above is strictly negative. Also, by the assumption that 
\( N \geq N_{\text{sub}} + 1 = 2\mu_B/\mu_A + 2 \), we have

\[
\frac{1}{\mu_B} - \frac{N}{2\mu_A} + \frac{1}{\mu_A} \leq \frac{1}{\mu_B} - \frac{\mu_B}{\mu_A^2} \leq \frac{1}{\mu_B} - \frac{1}{\mu_A} < 0;
\]

which implies that every term in the second summation is strictly negative. By this reasoning and the

decreasing differences condition, we conclude that

\[
d_{N/2} \leq d_{N/2-1} < 0,
\]

contradicting the definition of \( m^* \), which requires that \( d_{m^*-1} > 0 \). Therefore \( m^* \neq N/2 \). If \( x^{(t)} = m^* \), then in

period \( t+1 \) the customers who played route \( AB \) in period \( t \) will switch to route \( BA \), and the customers who

played route \( BA \) will switch to route \( AB \). The fact that \( m^* \neq N/2 \) implies that \( x^{(t+1)} \neq m^* \), and therefore

play will converge to herding by Proposition 3. □

**Proof of Corollary 3.** Assume by contradiction that there exists a pure-strategy Nash equilibrium in

which \( 0 < N_{AB} < N \) players choose route \( AB \), and \( N_{BA} = N - N_{AB} \) players choose route \( BA \). These assumptions imply that \( d_{N_{AB}} \geq 0 \) and \( d_{N_{AB}-1} \leq 0 \). But by Proposition 2, we must also have \( d_{N_{AB}} \leq d_{N_{AB}-1} \). This implies that \( d_{N_{AB}} = d_{N_{AB}-1} = 0 \), which contradicts Corollary 1. Therefore, there exist no pure-strategy Nash equilibria besides the herding equilibria of Proposition 1. □

**Proof of Corollary 4.** Proposition 2 tells us that if \( N > N_{\text{sub}} \), then the game has decreasing differences, i.e., \( d_{N-1} \leq d_{N-2} \leq \cdots \leq d_1 \leq d_0 \). The relation \( 2\mu_A \leq \mu_B \) then implies that

\[
\frac{\mu_B}{2\mu_A} > \frac{N-1}{N+1} \implies \frac{N-1}{\mu_B} < \frac{N+1}{2\mu_A} \implies d_0 = T(1,0) - T(0,0) = \frac{N-1}{\mu_B} - \frac{N+1}{2\mu_A} < 0.
\]

Decreasing differences and the fact that \( d_0 < 0 \) gives us that \( d_m = T(1,m) - T(0,m) < 0 \) for all \( m \in \{0,1,\ldots,N-1\} \). Therefore a player’s expected system time is always smaller for route \( AB \) than route \( BA \), no matter how many other players choose route \( BA \). □

**Proof of Proposition 4.** For players \( i = 1,2,\ldots,N \), let \( s_i \in [0,1] \) denote player \( i \)'s strategy—specifically, the probability that player \( i \) chooses route \( AB \). Assume by way of contradiction that there exists a Nash equilibrium with some players adopting mixed strategies and other players adopting pure strategies. Let \( N_{AB} \) be the number of customers playing the pure strategy of choosing route \( AB \), \( N_{BA} \) be the number of players playing the pure strategy of choosing route \( BA \), and \( N_M := N - N_{AB} - N_{BA} \) be the number of customers playing “properly” mixed strategies, i.e., placing strictly positive probability on both routes. Let the index \( i \) be defined such that \( s_i = 1 \) for \( i = 1,\ldots,N_{AB} \); \( s_i = 0 \) for \( i = N_{AB} + 1,\ldots,N_{AB} + N_{BA} \); and \( 0 < s_i < 1 \) for \( i = N_{AB} + N_{BA} + 1,\ldots,N \). By our assumption, \( N_{AB} + N_{BA} < N \) and either \( N_{AB} \geq 1 \) or \( N_{BA} \geq 1 \).

Let \( \Gamma_i \) denote the difference between player \( i \)'s expected system time from choosing route \( AB \) and the expected system time from choosing route \( BA \). Let \( \eta_k \) be the probability, given their strategies, that exactly \( k \) players choose route \( AB \) among players in the set \( \{N_{AB} + N_{BA} + 2,\ldots,N\} \). Player \( N_{AB} + N_{BA} + 1 \)'s difference between her expected system times from taking routes \( AB \) and \( BA \) can be expressed as

\[
0 = \Gamma_{N_{AB} + N_{BA} + 1} = \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB} + k}, \quad (35)
\]
where the fact that \( \Gamma_{N_{AB}+N_{BA}+1} = 0 \) is true by assumption; if player \( N_{AB} + N_{BA} + 1 \) is employing a mixed strategy in this Nash equilibrium, then she must be indifferent between route \( AB \) and route \( BA \).

Assume first that \( N_{AB} \geq 1 \). Then player 1 is choosing route \( AB \). In this case, the difference between her expected system times from taking routes \( AB \) and \( BA \) can be expressed as

\[
\Gamma_1 = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k-1}
\]

\[= (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k-1},\]

where the last equality holds by equation (35). Moreover, because player 1 is choosing route \( AB \), that route must be weakly better for her, implying that

\[
\sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k-1} \leq 0. \tag{36}
\]

Now, combining equations (35), (36), and the decreasing differences property, we get

\[
\sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k} = \sum_{k=0}^{N_{M}-1} \eta_k d_{N_{AB}+k-1} = 0 \tag{37}
\]

If the decreasing differences condition holds strictly (that is, if \( d_{N_{AB}+k} < d_{N_{AB}+k-1} \) for some \( k \in \{0, 1, \ldots, N_{M}-1\} \), then equation (37) cannot hold. Therefore we must have

\[
d_{N_{AB}-1} = d_{N_{AB}} = \ldots = d_{N_{AB}+N_{M}-1} = 0. \tag{38}
\]

Equation (38) contradicts Corollary 1, and thus \( N_{AB} \) must be equal to zero and therefore \( N_{BA} \geq 1 \). In that case, player 1 is choosing route \( BA \), and the difference between her expected times from choosing routes \( AB \) and \( BA \) can be expressed as

\[
\Gamma_1 = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_{M}-1} \eta_k d_{1+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_{M}-1} \eta_k d_{1+k} = \sum_{k=0}^{N_{M}-1} \eta_k d_{1+k},
\]

where the last equality holds by equation (35). Moreover, because player 1 is choosing route \( BA \), that route must be weakly better for her, implying that

\[
\sum_{k=0}^{N_{M}-1} \eta_k d_{1+k} \geq 0. \tag{39}
\]

Now, applying a similar argument as in the case with \( N_{AB} \geq 1 \), combining equations (35), (39), and the decreasing differences property gives us

\[
d_0 = d_1 = \ldots = d_{N_{M}} = 0. \tag{40}
\]

Equation (40) also contradicts Corollary 1. Therefore, there are no Nash equilibria of the open routing game in which some players adopt mixed strategies and other players adopt pure strategies.

Consider now an equilibrium in which all \( N \) players mix, that is, \( N_{M} = N \) and \( N_{AB} = N_{BA} = 0 \). Let \( \nu_k \) be the probability, given their strategies, that exactly \( k \) players choose route \( AB \) among players in the set \( \{3, \ldots, N\} \). Player 1’s expected savings from taking route \( BA \) instead of route \( AB \) is given by

\[
\Gamma_1 = s_2 \sum_{k=0}^{N-2} \nu_k d_{1+k} + (1 - s_2) \sum_{k=0}^{N-2} \nu_k d_{k} = 0,
\]
by assumption since player 1 must be indifferent if she plays a mixed strategy in equilibrium. Denote by \( \tilde{\Gamma}_1 \) the perturbed expected savings for player 1 if player 2 increases her probability of route \( AB \) to \( s_2 + \epsilon \). We have
\[
\tilde{\Gamma}_1 = (s_1 + \epsilon) \sum_{k=0}^{N-2} d_{1+k} \nu_k + (1 - s_1 - \epsilon) \sum_{k=0}^{N-2} d_k \nu_k
\]
\[
= \epsilon \left( \sum_{k=0}^{N-2} \nu_k (d_{1+k} - d_k) \right)
\]
\[
< 0,
\]
where the strict inequality follows from decreasing differences and the fact that \( d_1 < d_0 \), from the proof of Proposition 2. Therefore, for any \( \epsilon > 0 \), if player 2 perturbs her strategy by increasing her probability of route \( AB \) to \( s_2 + \epsilon \), then \( s_1 \) is no longer a best response for player 1, and thus this mixed-strategy Nash equilibrium is not “\( \epsilon \)-stable.” The same argument applies to the perturbation of any one player’s strategy. It should be observed that this notion of \( \epsilon \)-stability is related to but much stronger than the notion of an “evolutionarily stable strategy” as defined in Hassin and Haviv (2003).

**Proof of Corollary 5.** Similar to the argument of Corollary 3, suppose that there is a pure-strategy Nash equilibrium besides the herding equilibria. Then there must exist some \( N_{AB} < m < N_{AB} + N_S - 1 \) such that \( d_m \geq 0 \) and \( d_{m-1} \leq 0 \). Combining this with Proposition 2, we must have \( d_m = d_{m-1} = 0 \), but this contradicts Corollary 1. Therefore, there cannot be non-herding pure-strategy Nash equilibria. Next, we identify three regimes corresponding to the three conclusions in the statement of the proposition.

**Case 1:** \( d_{N_{AB}} < 0 \).

If \( d_{N_{AB}} < 0 \), then even if only non-strategic customers take route \( AB \), it is in the interest of each strategic customer to deviate to route \( AB \). Decreasing differences implies that it is a dominant strategy for all of the strategic players to choose route \( AB \).

**Case 2:** \( d_{N_{AB} + N_S - 1} > 0 \).

If \( d_{N_{AB} + N_S - 1} > 0 \), then even if all of the other strategic customers are taking route \( AB \), a strategic customer would rather take route \( BA \). Decreasing differences gives that it is a dominant strategy for all of the strategic players to choose route \( BA \).

**Case 3:** \( d_{N_{AB}} \geq 0 \) and \( d_{N_{AB} + N_S - 1} \leq 0 \).

In this case, both herding profiles are Nash equilibria.

**Proof of Proposition 5.** Assume that all \( N \) players visit station \( A \) first, so station \( B \) is initially empty until the first departure from station \( A \). Observe that if all \( N \) customers visit station \( A \) first, then station \( B \) behaves like a \( GI/GI/1 \) queueing system with arrival rate \( \mu_A \) and service rate \( \mu_B \) (recall that \( \mu_A < \mu_B \), so such a system would be stable). Let \( F_{W_0^B} \) be the distribution function for a random variable which is independent of the arrival and service processes and which may modify the initial state of the queueing system. Let \( W_k^B \), \( k \geq 1 \), be the waiting time that the \( k \)-th departure from station \( A \) experiences at station \( B \), and let \( F_{W_{k}^B} \) be the distribution function of \( W_k^B \). Note that with probability 1 we have \( W_0^B = 0 \) and \( W_1^B = 0 \) because station \( B \) is initially empty. Therefore, \( F_{W_0^B} \) stochastically dominates \( F_{W_1^B} \), which we denote as
\[
F_{W_0^B} \leq_{st} F_{W_1^B}.
\]
Define $F_{W^B_k}$ as the stationary waiting-time distribution function for a $GI/GI/1$ queueing system with arrival rate $\mu_A$ and service rate $\mu_B$. By Theorem 6.2.1 in Müller and Stoyan (2002), we then have that

$$F_{W^B_k} \leq_{st} F_{W^B_\infty}$$

for all $k = 1, 2, \ldots$. \hfill (41)

Armed with the stochastic dominance relation (41), we consider the candidate equilibrium profile in which all customers visit station $A$ first and evaluate the prospect of deviating to route $BA$. If the customer follows her current strategy, she will receive priority $k$ at station $A$, for $k = 1, \ldots, N$, with probability $1/N$. Conditional on her priority $k$, her expected total system time $E[S^A|k]$ is given by

$$E[S^A|k] = \frac{k}{\mu_A} + E[W^B_k] + \frac{1}{\mu_B}.$$ 

Equation (41) implies that $E[W^B_k] \leq E[W^B_\infty]$, where $E[W^B_\infty]$ is the steady-state expected waiting time from the distribution function $F_{W^B_\infty}$. We then have the bound

$$E[S^A|k] \leq \frac{k}{\mu_A} + E[W^B_\infty] + \frac{1}{\mu_B} \leq \frac{k}{\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B},$$

where the last inequality follows from bounds for the steady-state expected waiting time in queue for a $GI/GI/1$ queue found in Kingman (1962). Taking expectation over the priority in equation (42), we have

$$E[S^A] \leq \frac{1}{N} \sum_{k=1}^{N} \left( \frac{k}{\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B} \right) = \frac{N+1}{2\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B}.$$ 

If a customer deviates and visits station $B$ first, then she will be the last to be served at station $A$, so her expected system time $E[S^B]$ satisfies $E[S^B] \geq N/\mu_A$. Finally, by Equation (25), we have

$$\frac{2\mu_A}{\mu_B} + \frac{\mu_A^2(\sigma_A^2 + \sigma_B^2)}{1 - \mu_A/\mu_B} < N - 1$$

$$\Rightarrow \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B} < \frac{N - 1}{2\mu_A}$$

$$\Rightarrow E[S^A] \leq \frac{N+1}{2\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B} < \frac{N}{\mu_A} \leq E[S^B].$$

As a customer’s expected total system time is shorter if she follows the candidate profile and visits station $A$ first, she has no incentive to deviate. We conclude that it is a Nash equilibrium for all customers to visit station $A$ first. \hfill \square

**Proof of Proposition 6.** Let $X^A_k$ and $X^B_k$ be the service time experienced by the $k$-th customer to be served at stations $A$ and $B$, respectively. Suppose that all players visit station $B$ first, and let $W^A_k$ be the waiting time at station $A$ experienced by the $k$-th departure from station $B$. Define $U_k$ by

$$U_k := X^A_k - X^B_{k+1}.$$ 

By Lindley’s equation, then, we have

$$W^A_1 = 0$$

$$W^A_2 = \max\{0, W^A_1 + U_1\} = \max\{0, U_1\}$$

$$\cdots$$

$$W^A_{k+1} = \max\{0, U_k, U_k + U_{k-1}, \ldots, U_k + U_{k-1} + \cdots + U_1\}.$$ 

(45)
Now, define $I_k^A$ to be the cumulative idle time experienced by station $A$ before the arrival of the $k$-th customer, when all customers visit station $A$ first. We can relate the waiting time $W_k^A$ of the $k$-th customer to the idle time and the excess workload by

$$I_k^A = W_k^A - \left(\sum_{i=1}^{k-1} X_i^A - \sum_{j=1}^k X_j^B\right)$$

$$= W_k^A - \sum_{i=1}^{k-1} U_i + X_1^B$$

$$= X_1^B + \max\left\{-\sum_{i=1}^{k-1} U_i, -\sum_{i=1}^{k-2} U_i, \ldots, -U_1, 0\right\}.$$

Note then that $-U_i$ has the same distribution as $X_{k-i}^B - X_{k-i+1}^A$, and therefore, $I_k^A - X_1^B$ has the same distribution as

$$W_k^B := \max\left\{\sum_{i=1}^{k-1} (X_i^B - X_{k-i}^A), \sum_{i=2}^{k-1} (X_i^B - X_{i+1}^A), \ldots, (X_{k-1}^B - X_k^A), 0\right\},$$

which is the “dual process” of $W_k^A$. Moreover, by Lindley’s equation, we can view the equation (46) as the wait time of a single server queue with interarrival time distribution corresponding to station $A$’s service time distribution, and service time distribution corresponding to station $A$’s interarrival time distribution.

From equation (41) in the proof of Proposition 5 and Kingman (1962), we have that

$$E[W_k^B] \leq E[W_k^B] \leq \frac{\mu_A (\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)}.$$  

Let $S^A$ be the system time associated with choosing route $AB$, and $S^B$ be the system time associated with choosing route $BA$. Now, given a priority $k$ at station $B$, we have

$$E[S^B|k] = \frac{k}{\mu_A} + E[I_k^A]$$

$$= \frac{k}{\mu_A} + \frac{1}{\mu_B} + E[W_k^B]$$

$$\leq \frac{k}{\mu_A} + \frac{1}{\mu_B} + E[W_k^B],$$

where the equality follows from the fact that $I_k^A - X_1^B$ has the same distribution as $W_k^A$, while the inequality follows from equation (47). Applying this inequality, we get that

$$E[S^B] \leq E[W_k^B] + \frac{1}{\mu_B} + \sum_{k=1}^N \frac{1}{N} \left(\frac{k}{\mu_A}\right)$$

$$= E[W_k^B] + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}$$

$$\leq \frac{\mu_A (\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A},$$

with the last inequality follows from equation (47). Finally, because a player deviating to route $AB$ will be the last customer served at station $B$, we clearly have

$$E[S^A] \geq \frac{N}{\mu_B}.$$  

(48)
Equation (26) then gives
\[
\frac{1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A\mu_B(\sigma_A^2 + \sigma_B^2)}{2(\mu_B - \mu_A)} \leq N \left( \frac{2\mu_A - \mu_B}{2\mu_A\mu_B} \right)
\]
\[\implies E[S^B] \leq \frac{N+1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1 - \mu_A/\mu_B)} \leq \frac{N}{\mu_B} \leq E[S^A].\]

As it is, the expected system time from choosing route BA is less than that from choosing route AB. Therefore, no customer has incentive to deviate, and we have that it is a Nash equilibrium for every player to choose route BA. □

**Proof of Proposition 7.** We establish the subgame perfect equilibrium using backward induction. First, equations (27) and (28) tell us that the unique optimal strategy for customer N is to take route BA if \(y^N_{N-1} = N - 1\) and route AB otherwise. Note that this strategy ensures that in equilibrium always at least one customer will take route AB. Consequently, being the slower station, in equilibrium station A will never idle until it has processed all N customers. Next, we use induction to identify the optimal strategy for all other players. For the induction hypothesis, assume for some integer \(2 \leq n \leq N - 1\) that the strategy of subsequent customers \(N - n'\), where \(1 \leq n' < n\), is to always take route AB, and that the final customer to move, customer N, follows the optimal strategy derived above.

To serve as the base case, we first verify the induction hypothesis for \(n = 2\) by deriving the equilibrium strategy for customer \(N - 1\). Player N, the last to move, is the only customer that follows player \(N - 1\), and in equilibrium customer N will follow the strategy of visiting station B first if \(y^A_{N-1} = N - 1\), and station A otherwise. So, if customer \(N - 1\) visits station B first, then customer N will visit station A first, and customer \(N - 1\) will be the last person served at station A. She will then experience system time given by
\[
S^{B}_{N-1} = \frac{N}{\mu_A}
\]
(49)
because, as noted, station A never idles given player N’s equilibrium strategy. If customer \(N - 1\) takes route AB, then her system time will depend on how many players before her made the same choice. If a small enough number of them chose route AB that when player \(N - 1\) departs from station A she will find station B busy, then her system time \(S^A_{N-1}\) is given by
\[
S^A_{N-1} = \frac{N - 1}{\mu_B} < \frac{N - 1}{\mu_A} < \frac{N - 1}{\mu_A} + \frac{1}{\mu_A} = S^{B}_{N-1}.
\]
(50)

On the other hand, if enough customers before customer \(N - 1\) chose route AB that she will find station B idle when she departs from station A, then her system time \(S^A_{N-1}\) is given by
\[
S^A_{N-1} = \frac{y^A_{N-2} + 1}{\mu_A} + \frac{1}{\mu_B} < \frac{N - 1}{\mu_A} + \frac{1}{\mu_A} = S^{B}_{N-1}
\]
(51)
Thus, the equilibrium strategy for customer \(N - 1\) is to visit station A first in every subhistory. Combined with the equilibrium strategy for customer \(N\), equations (49)-(51) verify the induction hypothesis for \(n = 2\).

Now, assume that the induction hypothesis holds for some integer \(2 \leq n \leq N - 1\). Let \(S^{B}_{N-n}\) be the system time that customer \(N - n\) experiences if she chooses to join station B first. Similarly, let \(S^{A}_{N-n}\) be the system time that customer \(N - n\) experiences if she chooses to join station A first. If customer \(N - n\) chooses route AB, then everyone after her will join station A first, and customer \(N - n\) will be the last to be served at
station A. Because station A never idles, we then must have $S^B_{N-n} = N/\mu_A$. To study $S^A_{N-n}$, we need to consider the following two cases.

**Case 1:** The number of AB customers before customer $N - n$, denoted by $y^A_{N-n-1}$, is small enough that customer $N - n$ finds station B busy when she finishes at station A. Then by Property 2, station B has never idled since starting service and we must have $S^A_{N-n} = (N-n)/\mu_B$. Moreover,

$$S^A_{N-n} = \frac{N-n}{\mu_B} < \frac{N-n}{\mu_A} = S^B_{N-n}. \quad (52)$$

**Case 2:** The number of AB customers before customer $N - n$, denoted by $y^A_{N-n-1}$, is big enough that customer $N - n$ finds station B idle when she finishes at station A. In this case, we have $S^A_{N-n} = (y^A_{N-n-1} + 1)/\mu_A + (1/\mu_B)$. Therefore, we have that

$$S^A_{N-n} = \frac{y^A_{N-n-1} + 1}{\mu_A} + \frac{1}{\mu_B} < \frac{N-1}{\mu_A} + \frac{1}{\mu_A} = S^B_{N-n}. \quad (53)$$

In either case, choosing route AB results in a strictly shorter system time for customer $N - n$. Thus, if the induction hypothesis holds for some $n \geq 2$, we now have that for any $n \leq \bar{n} \leq N - 1$, the unique optimal strategy for customer $N - n$ is to take route AB regardless of the subhistory, given that all subsequent customers act optimally. Having verified the induction hypothesis for $n = 2$, we obtain the unique subgame perfect equilibrium of our game, comprised of the strategies stated in the proposition. Furthermore, inspection of the strategies reveals that the resulting equilibrium path entails the first $N - 1$ players taking route AB, and the final player $N$ taking route BA. □

Appendix B: Results from Section 4

Customers Who Visit Only One Station

**Proposition 8 (Nash Equilibrium with Route AB—Dedicated Customers).** If

$$N_S \geq (N_B + 1)\left(\frac{2\mu_A}{\mu_B}\right) + 1 - N_A, \quad (54)$$

then it is a Nash equilibrium for all $N_S$ strategic players to visit station A first.

**Proof.** Suppose that all of the strategic players choose route AB. If so, then a player who deviates to take route BA will be served at station A after all $N_A$ of the “A-only” players and after the other $N_S - 1$ strategic players. We therefore have the following lower bound on the expected system time $E[S^B]$ from deviation given by

$$E[S^B] \geq \frac{N_A + N_S}{\mu_A}. \quad (55)$$

Next we calculate an upper bound on the expected system time from following the profile and taking route AB. Here we note that Property 1 still applies to this system. Namely, if station B ever idles, then it will never build up a queue again. First, consider the case with no $B$-only players (that is, $N_B = 0$). In this case, there will never be a queue at station B, and strategic customers departing from station A will immediately enter service at station B. Therefore, with no $B$-only customers we have an exact expression for $E[S^A]$ given by

$$E[S^A] = \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S} \left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right).$$
Now, for a given priority $j$, if $N_B$ is greater than 0, then the maximum increase in the system time $S^A$ from the case with no $B$-only players is equal to $N_B/\mu_B$, the increased initial workload at station $B$. Thus, for any value of $N_B$ we have the bound

$$E[S^A] \leq \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S} \left( \frac{j}{\mu_A} + \frac{1}{\mu_B} + \frac{N_B}{\mu_B} \right) = \frac{N_A+N_S+1}{2\mu_A} + \frac{N_A+1}{\mu_B}. \quad (56)$$

Equations (54), (55), and (56) give

$$E[S^A] \leq \frac{N_B+1}{\mu_B} + \frac{N_A+N_S+1}{2\mu_A} \leq \frac{N_A+N_S}{\mu_A} \leq E[S^B]. \quad (57)$$

We conclude that strategic players have no incentive to deviate, and therefore it is a symmetric Nash equilibrium for all strategic players to visit station $A$ first. \qed

**Proposition 9** (Nash Equilibrium with Route $BA$—Dedicated Customers). If $\mu_B < 2\mu_A$ and

$$N_S \geq \frac{\mu_B(2N_A+1) - \mu_A(N_B-2)}{2\mu_A - \mu_B}, \quad (58)$$

then it is a Nash equilibrium for all $N_S$ strategic players to visit station $B$ first.

**Proof.** Suppose that all of the strategic customers are following the profile of visiting station $B$ first, and consider a player who contemplates deviating and visiting station $A$ first. If she deviates, then she will certainly not enter service at station $B$ until after station $B$ processes all $N_B$ $B$-only players as well as the other $N_S-1$ strategic customers. Therefore, we can bound her expected system time $E[S^A]$ from deviating by

$$E[S^A] \geq N_B + N_S. \quad (59)$$

Next, suppose that she follows the profile and visits station $B$ first, and further suppose that she receives priority $k$ at station $B$. Let $Z_k$ be the random variable representing the number of strategic customers among the first $k-1$ players at station $B$. Given that there are $N_A+N_S-1$ other players at station $B$, $N_S-1$ of which are strategic, the random variable $Z_k$ has a hypergeometric distribution with $k-1$ trials from a population of size $N_B+N_S-1$ containing $N_S-1$ successes. Therefore, its mean is

$$E[Z_k] = (k-1) \frac{N_S-1}{N_B+N_S-1}. \quad (60)$$

With a large number of $B$-only players it is possible that station $A$ will become idle while some strategic customers have not yet been served at station $B$. Consider a strategic customer who is assigned priority $k$ at station $B$. The greatest amount of idle time that could possibly be introduced at station $A$ is the sum of the service times at station $B$ of the $B$-only customers with a higher priority. There are exactly $k-1-Z_k$ such players. Define $I_k$ as the amount of time that station $A$ spends idle before the customer with priority $k$ at station $B$ finishes her service at station $B$. If $N_A = 0$, then station $A$ would also idle for $1/\mu_B$ before it receives its first customer. An upper bound on the mean of $E[I_k]$ is then

$$E[I_k] \leq \frac{1}{\mu_B} + E\left[ \frac{k-1-Z_k}{\mu_B} \right] = \frac{1}{\mu_B} + \frac{k-1-E[Z_k]}{\mu_B}. \quad (61)$$
We can now express the expected system time from choosing route $BA$ by

$$\mathbb{E}[S_B] = \sum_{k=1}^{N_B+N_S} \frac{1}{N_B+N_S} \left( \frac{N_A+1+E[Z_k]}{\mu_A} + E[I_k] \right) \leq \sum_{k=1}^{N_B+N_S} \frac{1}{N_B+N_S} \left( \frac{N_A+1+E[Z_k]}{\mu_A} + \frac{k-1-E[Z_k]}{\mu_B} + \frac{1}{\mu_B} \right) = \frac{1}{\mu_B} + \frac{N_A+1}{\mu_A} + \frac{N_S-1}{2\mu_A} + \frac{N_B}{2\mu_B},$$

(62)

where the inequality follows from equation (61) and the last equality follows from substituting the expression in equation (60) for $E[Z_k]$ and then evaluating the summation. Because $\mu_B < 2\mu_A$, equations (58), (59), and (62) then give

$$\frac{\mu_B(2N_A+1) - \mu_A(N_B-2)}{2\mu_A - \mu_B} \leq N_S \leq \frac{2N_A+1}{2\mu_A} - \frac{N_B-2}{2\mu_B} \leq N_S \left( \frac{1}{\mu_B} - \frac{1}{2\mu_A} \right) \Rightarrow \mathbb{E}[S_B] \leq \frac{1}{\mu_B} + \frac{N_A+1}{\mu_A} + \frac{N_S-1}{2\mu_A} + \frac{N_B}{2\mu_B} \leq \frac{N_B+N_S}{\mu_B} \leq \mathbb{E}[S_A].$$

(63)

Because $\mathbb{E}[S_B] \leq \mathbb{E}[S_A]$, we conclude that no player has incentive to deviate, and thus it is a Nash equilibrium for all $N_S$ strategic players to visit station $B$ first. □

**S-Station Open Routing Game**

**Proposition 10 (Nash Equilibrium for Unobservable S-Station System).** If we have

$$N \geq 1 + 2\mu_1 \sum_{\xi=2}^{S} \frac{1}{\mu_\xi},$$

(64)

then the unobservable S-station open routing game has a Nash equilibrium in which all players choose the routing vector $(1, 2, \ldots, S)$.

**Proof.** Suppose that all players are following the routing vector $(1, 2, \ldots, S)$, and consider a player who contemplates deviation. On path (that is, if everyone follows the prescribed profile), the only time any customer will face a queue is at station 1. All of the other stations start empty and then receive arrivals only when the station immediately before them completes a service. Because the routing vector is in order of increasing service rate, a queue will never build up at any station other than station 1. Thus, if the customer follows the prescribed profile, then her expected system time $\mathbb{E}[S_{EQ}]$ is given by

$$\mathbb{E}[S_{EQ}] = \sum_{j=1}^{N} \frac{1}{N} \left( \frac{j}{\mu_1} \right) + \sum_{\xi=2}^{S} \frac{1}{\mu_\xi} = \frac{N+1}{2\mu_1} + \sum_{\xi=2}^{S} \frac{1}{\mu_\xi}.$$

Because the player will not face a queue at any station besides station 1, she cannot possibly improve her system time by changing the order of stations that she visits after station 1. Thus the only deviations that we must consider are those which involve a vector that starts at a station other than station 1. If the player
starts at station $\xi \geq 2$, then she will necessarily be the last customer to be served at station 1. Thus, we easily have a lower bound on her expected system time $E[S^D]$ from deviating given by
\[
E[S^D] \geq \frac{N}{\mu_1}.
\]
Equation (64) then gives
\[
N \geq 1 + 2\mu_1 \sum_{\xi=2}^{S} \frac{1}{\mu_\xi}
\]
\[
\implies \sum_{\xi=2}^{S} \frac{1}{\mu_\xi} \leq \frac{N - 1}{2\mu_1}
\]
\[
\implies E[S^{EQ}] = \frac{N + 1}{2\mu_1} + \sum_{\xi=2}^{S} \frac{1}{\mu_\xi} \leq \frac{N}{\mu_1} \leq E[S^D].
\]
The player’s expected system time is less if she follows the prescribed profile, and we therefore have a symmetric Nash equilibrium where all customers herd at station 1 and follow the routing vector $(1, 2, \ldots, S)$. \qed

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References


Pinilla, J. M. and Prinz, F. B. (2003). Lead-time reduction through flexible routing: application to shape deposition manufac-


