Logarithmic Regret in the Dynamic and Stochastic Knapsack Problem

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We study a dynamic and stochastic knapsack problem in which a decision maker is sequentially presented with \( n \) items with unitary rewards and independent weights that are drawn from a known continuous distribution \( F \). The decision maker seeks to maximize the expected number of items that she includes in the knapsack while satisfying a capacity constraint, and while making terminal decisions as soon as each item weight is revealed. Under mild regularity conditions on the weight distribution \( F \), we prove that the regret—the expected difference between the performance of the best sequential algorithm and that of a prophet who sees all of the weights before making any decision—is, at most, logarithmic in \( n \). Our proof is constructive. We devise a re-optimized heuristic that achieves this regret bound.

Key words: dynamic and stochastic knapsack problem, regret, re-optimization, adaptive online policy.
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1. Introduction

The knapsack problem is one of the classic problems in operations research. It arises in resource allocation, and it counts numerous applications in auctions, logistics, portfolio optimization, scheduling, and transportation among others [Martello and Toth 1990]. In its dynamic and stochastic formulation (see, e.g. Papastavrou et al. 1996, Kleywegt and Papastavrou 1998, 2001) a decision maker (referred to as she) is given a knapsack with finite capacity \( 0 \leq c < \infty \) and is sequentially presented with \( n \) items indexed by \( i \in [n] \equiv \{1, 2, \ldots, n\} \). Each item \( i \) has a weight \( W_i \) that represents the amount of knapsack capacity that item \( i \) occupies if the decision maker chooses to include item \( i \) in the knapsack, and a reward \( R_i \) that the decision maker collects upon inclusion. The pairs \( (W_i, R_i) \), \( i \in [n] \), are independent and with common, known, bivariate distribution supported on the nonnegative orthant. The decision maker sequentially observes the weight-reward pairs \( \{(W_i, R_i) : i \in [n]\} \) and needs to decide whether to include item \( i \) in the knapsack when the pair \( (W_i, R_i) \) is first revealed.
By imposing different assumptions on the weight-reward distribution one recovers well-known related problems and applications. For instance, if one assumes that the weights are all equal to one and that the rewards are random, then one recovers the multi-secretary problem (see, e.g. [Cayley 1875, Moser 1956, Kleinberg 2005]), or the single-resource capacity control revenue management problem (see, e.g. [Talluri and van Ryzin 2004, Section 2.5]). Similarly, if one assumes that the rewards are all equal to one and that the weights are random, then one finds a scheduling problem in which a decision maker seeks to find a maximum-cardinality subset of \( n \) jobs with random durations that are processed by a fixed deadline on a single machine (c.f. [Lipton and Tomkins 1994, Baruah et al. 1994]). When the rewards are all equal, one also recovers the formulation in which the rewards are random but their values are revealed only after the respective items are included in the knapsack.

In this paper, we focus our attention to the formulation in which the rewards are all equal to one, and the weights are independent random variable with common continuous distribution \( F \). We say that a policy \( \pi \) is feasible if the sum of the weights of the items selected by \( \pi \) does not exceed the knapsack capacity \( c \), and we say that the policy is online (or sequential) if the decision to select item \( i \) with weight \( W_i \) depends only on the information available up to and including time \( i \). We then let \( \Pi(n,c) \) be the set of feasible online policies, and we compare the performance of the best online policy to that of a prophet who has full knowledge of the weights \( W_1, W_2, \ldots, W_n \) before making any selection. Under some mild technical conditions on the weight distribution \( F \), we prove that the regret—the expected gap between the performance of the best online policy and its offline counterpart—is bounded by the logarithm of \( n \). Our proof is constructive. We propose a re-optimized heuristic that exhibits logarithmic regret. The heuristic is based on re-solving some related optimization problem at any give time \( i \in [n] \) by using the current—rather than the initial—level of remaining capacity as constraint. The solution of this optimization problem provides us with a state- and time-dependent threshold that mimics that of the optimal online policy.

If all of the weights \( W_1, W_2, \ldots, W_n \) are revealed to the decision maker before she makes any selection, then her choice is obvious. To maximize the total reward she collects, she just selects the smallest \( N^*_n(c) \) values that do not exceed the capacity constraint. Formally, if \( W_{[1,n]} \leq W_{[2,n]} \leq \cdots \leq W_{[n,n]} \) are the order statistics of \( W_1, W_2, \ldots, W_n \), then

\[
N^*_n(c) = \max \left\{ m \in \{0, 1, \ldots, n\} : \sum_{\ell=1}^{m} W_{[\ell,n]} \leq c \right\}.
\]

Here we compare the number of offline selections \( N^*_n(c) \) with the number of selections of an online feasible algorithm \( \hat{\pi}(n,c) \) that is based on a sequence of re-optimized time- and state-dependent threshold functions. That is, if the current level of remaining capacity is \( x \) and the weight of item
We say that a non-negative distribution $F$ with continuous density function $f$ belongs to the typical class if for some $\bar{w} > 0$, the following two conditions hold.
there are 0 < \lambda < 1 and 0 < \gamma < 1 such that
\[
\frac{F(\lambda w)}{F(w)} \leq \gamma < 1 \quad \text{for all } w \in (0, \bar{w}).
\]

(ii) **Monotonicity.** The map \( w \mapsto w^3 f(w) \) is non-decreasing on \((0, \bar{w})\). That is,
\[
w_1^3 f(w_1) \leq w_2^3 f(w_2) \quad \text{for all } 0 < w_1 \leq w_2 < \bar{w}.
\]

The main results of this paper are gathered in the theorem below. First, we provide an upper bound for \( E[N^*_n(c)] \) that holds for any continuous distribution \( F \). Then we turn to distributions that belong to the typical class, and we prove that there is a matching lower bound. As a by-product of our analysis, we establish that the regret is bounded by the logarithm of \( n \).

**Theorem 1 (Logarithmic regret bound).** Given a knapsack with capacity \( 0 \leq c < \infty \) and \( 1 \leq n < \infty \) items with independent weights with continuous distribution \( F \), then
\[
\max_{\pi \in \Pi(n,c)} E[N_n(\pi)] \leq E[N^*_n(c)] \leq nF(\epsilon_n(c)).
\]

Furthermore, there is a feasible online policy \( \hat{\pi}(n,c) \) such that if the weight distribution \( F \) belongs to the typical class then there is a constant \( 1 < M < \infty \) depending only on \( F \) for which
\[
nF(\epsilon_n(c)) - M(1 + \log n) \leq E[N_n(\hat{\pi}(n,c))].
\]

In turn, if \( F \) belongs to the typical class, then we have the regret bound
\[
E[N^*_n(c)] - \max_{\pi \in \Pi(n,c)} E[N_n(\pi)] \leq E[N^*_n(c)] - E[N_n(\hat{\pi}(n,c))] \leq M(1 + \log n).
\]

The upper bound \( E[N^*_n(c)] \leq nF(\epsilon_n(c)) \) was first proved by [Bruss and Robertson (1991)]. Here, we provide an alternative proof that is based on a relaxation of some appropriate optimization problem. The solution to this relaxation is the basis for constructing the re-optimized heuristics \( \hat{\pi}(n,c) \). The lower bound \( E[N_n(\hat{\pi}(n,c))] \geq nF(\epsilon_n(c)) - O(\log n) \) as \( n \to \infty \) is essentially new, and it substantially improves on existing estimates. The best results to date for general weight distribution \( F \) are due to [Rhee and Talagrand (1991)] who studied a time-independent heuristic to prove that
\[
nF(\epsilon_n(c)) \left\{ 1 - \left[ \frac{c\epsilon_n(c)}{c} \right]^{1/2} - \frac{\epsilon_n(c)}{c} \right\} \leq \max_{\pi \in \Pi(n,c)} E[N_n(\pi)] \quad \text{for all } n \geq 1.
\]

For instance, if \( F(x) = \sqrt{x} \) for \( x \in [0,1] \) then the lower bound [5] implies an upper bound for the regret that is \( O(n^{1/3}) \) as \( n \to \infty \). Similarly, if \( F(x) = x^3 \) for \( x \in [0,1] \) then the same lower bound gives us a regret upper bound that behaves like \( O(n^{1/6}) \) as \( n \to \infty \).

A case that deserves special attention is when \( F \) is the uniform distribution on the unit interval and \( c = 1 \). In this context, the [Rhee and Talagrand (1991)] lower bound provides us with a regret
upper bound that behaves like $O(n^{1/4})$ as $n \to \infty$, but better bounds are available in the literature. This special dynamic and stochastic knapsack problem is in fact equivalent to the problem of the sequential selection of a monotone increasing subsequence from a sample of $n$ independent observation with continuous distribution (see Samuels and Steele 1981, Coffman et al. 1987, pp. 457–458). For this subsequence-selection problem, Arlotto et al. (2015, 2018) proved that the expected performance $\nu_n^*$ of the best online policy satisfies the estimate $\nu_n^* = \sqrt{2n} - O(\log n)$ as $n \to \infty$. The equivalence between the two problems, however, holds only for uniform weights.

As Theorem 1 suggests, the weight distribution $F$ plays a crucial role in the estimates for the dynamic and stochastic knapsack problem with equal rewards, while the monotone subsequence problem is distribution invariant as it only needs to account for relative orders. More importantly, Seksenbayev (2018) characterized the second order asymptotic expansion of $\nu_n^*$ and established that $\nu_n^* = \sqrt{2n} - \frac{1}{12} \log n + O(1)$ as $n \to \infty$. This remarkable result offers an important suggestion that could strengthen the lower bound we prove in Theorem 1 for the dynamic and stochastic knapsack problem. No online algorithm can be within $O(1)$ for offline sort, and the regret bound we prove is of the correct order.

This last observation highlights the difference between this dynamic and stochastic knapsack problem and its dual formulation in which one takes unitary weights and independent random rewards. For the dual formulation, Arlotto and Gurvich (2018) (see also Wu et al. 2015, Bumpensanti and Wang 2018, Vera and Banerjee 2018) proved that if the reward distribution is discrete, then the regret is uniformly bounded in the number of items $n$ and the knapsack capacity $c$. In particular, in that context, the best online algorithm works just as well as the offline sort. Instead, when item weights are random and with a continuous distribution, the presence of arbitrarily small items makes the prophet advantage much more substantial.

Organization of the paper

The paper is organized as follows. Section 2 proves the prophet upper bound $\mathbb{E}[N_n^*(c)] \leq n F_0(c)$ by showing that the offline sort algorithm can be reinterpreted as a parsimonious threshold policy and by solving a relaxation of some related optimization problem. This solution then guides us in the construction of policy $\hat{\pi}(n,c)$ that is presented in Section 3. In Section 4, we discuss the generality of the typical class of distributions, and we derive some properties. In Section 5, we use such properties to prove that the re-optimized policy $\hat{\pi}(n,c)$ exhibits logarithmic regret. Finally, in Section 6 we make closing remarks and underscore some open problems.

2. A prophet upper bound

The performance of any online algorithm is bounded above by the full-information (or offline) sort. If the decision maker knows all of the weights $W_1, W_2, \ldots, W_n$ before making any decision,
then the number of items she includes in the knapsack is equal to the largest number \( k \) such that the sum of the smallest \( k \) realizations does not exceed the capacity constraint. That is, if \( W_{(1,n)} \leq W_{(2,n)} \leq \cdots \leq W_{(n,n)} \) are the order statistics of \( W \equiv \{ W_1, W_2, \ldots, W_n \} \), then the number \( N^*_n(c) \) of offline selections when the initial knapsack capacity is \( c \) is given by

\[
N^*_n(c) = \max \left\{ m \in \{0,1,\ldots,n\} : \sum_{\ell=1}^{m} W_{(\ell,n)} \leq c \text{ and } W_{(\ell,n)} \in W \text{ for all } \ell \in [n] \right\}.
\]

The random variable \( N^*_n(c) \) has been studied extensively in the related literature. Coffman et al. (1987) showed that

\[
N^*_n(c) \sim nF(\epsilon_n(c)) \text{ in probability as } n \to \infty,
\]

provided that the weight distribution \( F \) is continuous, strictly increasing in \( w \) when \( F(w) < 1 \), and \( F(w) \sim Aw^\alpha \) as \( w \to 0 \) for some \( A, \alpha > 0 \). Four years later, Bruss and Robertson (1991) proved that the same result holds under more general conditions, and Boshuizen and Kertz (1999) established the asymptotic normality of \( N^*_n(c) \) after the usual centering and scaling for different classes of weight distribution \( F \). Lemma 4.1 in Bruss and Robertson (1991) is particularly relevant to our discussion here since it tells us that

\[
\mathbb{E}[N^*_n(c)] \leq nF(\epsilon_n(c)) \quad \text{ for all } n \geq 1.
\]

Here we provide an alternative proof of the same upper bound. The proof relies on the observation that the offline sort algorithm can be equivalently described as an algorithm that selects items with weight that is below some threshold. For any given realization \( W_1, W_2, \ldots, W_n \) one can in fact compute the value \( W_{(N^*_n(c),n)} \) of the largest weight that is selected for inclusion, and one can then select all of the items \( i \in [n] \) that have weight \( W_i \leq W_{(N^*_n(c),n)} \). A shortcoming of this interpretation is that one needs to know the realization of the weight \( W_i \) (as well as the realizations of all of the other weights) to compute the threshold \( W_{(N^*_n(c),n)} \). As it turns out, this is not needed in general. The next lemma shows that there is a thresholding algorithm that makes the same selections of offline sort, but in which the threshold used to decide whether to select an item is computed without using the information about that item’s weight.

**Lemma 2 (Threshold policy equivalence).** Let \( W_{(1,n)} \leq W_{(2,n)} \leq \cdots \leq W_{(n,n)} \) be the order statistics of \( W \equiv \{ W_1, W_2, \ldots, W_n \} \) and, for \( i \in [n] \), let \( W_{(1,n-i)} \leq W_{(2,n-i)} \leq \cdots \leq W_{(n-1,n-i)} \) be the order statistics of \( W_i = W \setminus \{ W_i \} \). Then, for

\[
\tau^i_{n-1}(c) = \max \left\{ m \in \{0,1,\ldots,n-1\} : \sum_{\ell=1}^{m} W_{(\ell,n-i)} \leq c \text{ and } W_{(\ell,n-i)} \in W_i \text{ for all } \ell \in [n-1] \right\}
\]

(7)
we have that

\[ W_i \leq W_{(N^*_n(c), n)} \quad \text{if and only if} \quad W_i \leq h(W_i) = \max \left\{ W_{(\tau^c_{n-1}(c), n-1)}, c - \sum_{\ell=1}^{\tau^c_{n-1}(c)} W_{(\ell, n-1)} \right\}. \quad (8) \]

In turn, it follows that

\[ N^*_n(c) = \sum_{i=1}^{n} \mathbb{1} \{ W_i \leq h(W_i) \}. \quad (9) \]

**Proof of Lemma** \[ \square \] The equivalence \[ \square \] is an obvious consequence of \[ \square \], so we focus on proving the latter. If \( N^*_n(c) = n \) we have that \( \tau^c_{n-1}(c) = n - 1 \) and \( W_i \leq c - \sum_{\ell=1}^{\tau^c_{n-1}(c)} W_{(\ell, n-1)} \) for all \( i \in [n] \), so equivalence \[ \square \] immediately follows. Instead, if \( N^*_n(c) < n \) the proof of \[ \square \] requires more work. As a warm-up we note that since the sets \( W \) and \( W_i \) differ only in one element, then

\[ W_{(\ell, n)} \leq W_{(\ell, n-1)} \leq W_{(\ell+1, n)} \quad \text{for all} \quad \ell \in [n-1]. \quad (10) \]

If we now recall the definitions of \( \tau^c_{n-1}(c) \) and \( N^*_n(c) \) and use the inequalities above we obtain that

\[ \sum_{\ell=1}^{\tau^c_{n-1}(c)} W_{(\ell, n)} \leq \sum_{\ell=1}^{\tau^c_{n-1}(c)} W_{(\ell, n-1)} \leq c \quad \text{and} \quad \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n-1)} \leq \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell+1, n)} \leq \sum_{\ell=1}^{N^*_n(c)} W_{(\ell, n)} \leq c. \]

These two bounds respectively tell us that the offline sort algorithm on \( W \) selects at least \( \tau^c_{n-1}(c) \) observations, and that the same algorithm on \( W_i \) selects at least \( N^*_n(c) - 1 \) items. Thus, it follows that

\[ N^*_n(c) - 1 \leq \tau^c_{n-1}(c) \leq N^*_n(c), \]

and we use these bounds to prove the equivalence \[ \square \].

If. We now suppose that \( W_i \leq h(W_i) = \max \left\{ W_{(\tau^c_{n-1}(c), n-1)}, c - \sum_{\ell=1}^{\tau^c_{n-1}(c)} W_{(\ell, n-1)} \right\} \), and we seek to show that \( W_i \leq W_{(N^*_n(c), n-1)} \). We consider two cases, one per each possible realization of \( \tau^c_{n-1}(c) \).

**CASE 1:** \( \tau^c_{n-1}(c) = N^*_n(c) - 1 \). If \( \tau^c_{n-1}(c) = N^*_n(c) - 1 \) then the definition of \( \tau^c_{n-1}(c) \) in \[ \square \] tells us that

\[ c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n-1)} < W_{(N^*_n(c), n-1)}, \]

so if we apply the right inequality of \[ \square \] to \( \ell = N^*_n(c) - 1 \) and \( \ell = N^*_n(c) \), we obtain that

\[ W_{(N^*_n(c)-1, n-1)} \leq W_{(N^*_n(c), n)} \quad \text{and} \quad c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n-1)} < W_{(N^*_n(c)+1, n)}, \quad (11) \]

If \( W_{(N^*_n(c), n)} = W_{(N^*_n(c)+1, n)} \) then the two inequalities in \[ \square \] give us that \( h(W_i) = \max \left\{ W_{(N^*_n(c)-1, n-1)}, c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n-1)} \right\} \leq W_{(N^*_n(c), n)} \), so we also have that \( W_i \leq W_{(N^*_n(c), n)} \).

On the other hand, if \( W_{(N^*_n(c), n)} < W_{(N^*_n(c)+1, n)} \) then the bounds in \[ \square \] imply that \( h(W_i) < W_{(N^*_n(c)+1, n)} \), so we obtain from \( W_i \leq h(W_i) \) that \( W_i \leq W_{(N^*_n(c), n)} \).
Case 2: $\tau_{n-1}^i(c) = N^*_n(c)$. The left inequality of (10) with $\ell = N^*_n(c)$ tells us that we have two sub-cases to consider here: (i) when $W_{(N^*_n(c), n)}$ is equal to $W_{(N^*_n(c), n-1)}$, and (ii) when $W_{(N^*_n(c), n)}$ is strictly smaller than $W_{(N^*_n(c), n-1)}$. In the first sub-case, if $\tau_{n-1}^i(c) = N^*_n(c)$ and $W_{(N^*_n(c), n)} = W_{(N^*_n(c), n-1)}$, then the first $N^*_n(c)$ order statistics of $W$ and of $W_i$ agree and $c - \sum_{\ell=1}^{N^*_n(c)} W_{(\ell, n)} = c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n)} < W_{(N^*_n(c)+1, n)}$. Thus, if $W_{(N^*_n(c), n)} = W_{(N^*_n(c)+1, n)}$ then $h(W_i) = \max\{W_{(N^*_n(c), n)}, c - \sum_{\ell=1}^{N^*_n(c)} W_{(\ell, n)}\} = W_{(N^*_n(c), n)}$, and we are done. Otherwise, if $W_{(N^*_n(c), n)} < W_{(N^*_n(c)+1, n)}$ then $h(W_i) < W_{(N^*_n(c)+1, n)}$ so that $W_i \leq h(W_i) < W_{(N^*_n(c)+1, n)}$ implies that $W_i \leq W_{(N^*_n(c), n)}$. In the second sub-case, if $\tau_{n-1}^i(c) = N^*_n(c)$ and $W_{(N^*_n(c), n)} < W_{(N^*_n(c), n-1)}$ then we have that $W_i = W_{(N^*_n(c), n)}$ and the result follows.

Only If. We now suppose that $W_i \leq W_{(N^*_n(c), n)}$, and we show that $W_i \leq h(W_i) = \max\{W_{(\tau_{n-1}^i(c), n-1)}, c - \sum_{\ell=1}^{\tau_{n-1}^i(c)} W_{(\ell, n)}\}$ by proving that $W_{(N^*_n(c), n)} \leq h(W_i)$. Just as before, we consider separately the two possible realizations of $\tau_{n-1}^i(c)$.

Case 1: $\tau_{n-1}^i(c) = N^*_n(c) - 1$. We have two sub-cases to consider here. First, if $W_{(N^*_n(c), n)} \leq W_{(N^*_n(c) - 1, n-1)}$ then the lower bound $W_{(N^*_n(c), n)} \leq h(W_i)$ is trivial. Second, if $W_{(N^*_n(c), n)} < W_{(N^*_n(c) - 1, n-1)}$ we show that the right maximand is bounded below by $W_{(N^*_n(c), n)}$. In this instance, the first $N^*_n(c) - 1$ order statistic of $W$ and $W_i$ agree so the definition of $N^*_n(c)$ gives us that $W_{(N^*_n(c), n)} \leq c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n)} = c - \sum_{\ell=1}^{N^*_n(c)-1} W_{(\ell, n-1)}$, and we are done.

Case 2: $\tau_{n-1}^i(c) = N^*_n(c)$. If $\tau_{n-1}^i(c) = N^*_n(c)$ the left inequality of (10) tells us that $W_{(N^*_n(c), n)} \leq W_{(N^*_n(c), n-1)}$, so the lower bound $W_{(N^*_n(c), n)} \leq h(W_i)$ immediately follows.

The representation (9) for $N^*_n(c)$ provides us with an easy way for proving that $\mathbb{E}[N^*_n(c)] \leq nF(\epsilon_n(c))$. We just need to note that the expected number of offline selections is bounded above by the solution of some appropriate optimization problem.

**Proposition 3 (Prophet upper bound).** Given $1 \leq n < \infty$ independent item weights with continuous distribution $F$ and a knapsack with capacity $0 \leq c < \infty$, then for $\epsilon_n(c) = \sup\{\epsilon \in [0, \infty) : \int_0^\epsilon w f(w) \, dw \leq n^{-1}c\}$ we have that

$$\mathbb{E}[N^*_n(c)] \leq nF(\epsilon_n(c)).$$  

(12)

**Proof.** To prove inequality (12), we begin with two easy cases. If $c = 0$ then $N^*_n(0) = 0$, and the bound (12) is trivial. Similarly, if $\mu = \int_0^\infty w f(w) \, dw$ and $n\mu < c < \infty$ then the definition of the function $\epsilon_n(c)$ tells us that $\epsilon_n(c) = \infty$ so $F(\epsilon_n(c)) = 1$ and the bound (12) is again trivial because $N^*_n(c) \leq n$ for all $c \in [0, \infty)$.

Next, we consider the case in which $0 < c \leq n\mu$. If $G_i = \sigma\{W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_n\}$ is the $\sigma$-field generated by the sample $W_i$, then we obtain from Lemma 2 and from the definition (6) that
for each \( i \in [n] \) there is a \( \mathcal{G}_i \)-measurable threshold \( h(W_i) \) such that one has the representation as well as the capacity constraint

\[
N_n^*(c) = \sum_{i=1}^{n} 1 \{ W_i \leq h(W_i) \} \quad \text{and} \quad \sum_{i=1}^{n} W_i 1 \{ W_i \leq h(W_i) \} \leq c.
\]

In turn, we can obtain an upper bound for \( \mathbb{E}[N_n^*(c)] \) by maximizing the sum \( \sum_{i=1}^{n} \mathbb{E}[1 \{ W_i \leq h_i \}] \) over all thresholds \((h_1, h_2, \ldots, h_n)\) that satisfy an analogous capacity constraint and that have the same measurability property. Formally, we have the inequality

\[
\mathbb{E}[N_n^*(c)] \leq \max_{(h_1, \ldots, h_n)} \sum_{i=1}^{n} \mathbb{E}[1 \{ W_i \leq h_i \}] \tag{13}
\]

\[\text{s.t.} \quad \sum_{i=1}^{n} W_i 1 \{ W_i \leq h_i \} \leq c \quad (\text{almost surely}) \quad h_i \in \mathcal{G}_i \quad \text{for all } i \in [n].\]

Since \( \epsilon_n(c) > 0 \) and because the capacity constraint holds almost surely (and thus also in expectation), we have the further upper bound

\[
\mathbb{E}[N_n^*(c)] \leq \max_{(h_1, \ldots, h_n)} \sum_{i=1}^{n} \mathbb{E}[1 \{ W_i \leq h_i \} \{ 1 - \epsilon_n^{-1}(c) W_i \}] + c \epsilon_n^{-1}(c) \tag{14}
\]

\[\text{s.t.} \quad \sum_{i=1}^{n} W_i 1 \{ W_i \leq h_i \} \leq c \quad (\text{almost surely}) \quad h_i \in \mathcal{G}_i \quad \text{for all } i \in [n].\]

Because \( h_i \) is \( \mathcal{G}_i \)-measurable, an application of the tower property gives us that

\[
\mathbb{E} \mathbb{E}[1 \{ W_i \leq h_i \} \{ 1 - \epsilon_n^{-1}(c) W_i \} | \mathcal{G}_i] = \mathbb{E} \left[ \int_{0}^{h_i} \{ 1 - \epsilon_n^{-1}(c) w \} f(w) \, dw \right],
\]

so, after we drop the two constraints in \( (14) \) we obtain that

\[
\mathbb{E}[N_n^*(c)] \leq p^* = \max_{(h_1, \ldots, h_n)} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{h_i} \{ 1 - \epsilon_n^{-1}(c) w \} f(w) \, dw \right] + c \epsilon_n^{-1}(c). \tag{15}
\]

The maximization problem on the right hand side is separable, and the quantity

\[
\mathbb{E} \left[ \int_{0}^{h_i} \{ 1 - \epsilon_n^{-1}(c) w \} f(w) \, dw \right]
\]

is maximized by setting \( h_i = \epsilon_n(c) \) almost surely and for all \( i \in [n] \). Thus, it follows that

\[
p^* = \sum_{i=1}^{n} \max_{h_i} \mathbb{E} \left[ \int_{0}^{h_i} \{ 1 - \epsilon_n^{-1}(c) w \} f(w) \, dw \right] + c \epsilon_n^{-1}(c)
\]

\[= n \left\{ F(\epsilon_n(c)) - \epsilon_n^{-1}(c) \left[ \int_{0}^{\epsilon_n(c)} w f(w) \, dw - n^{-1} c \right] \right\}.\]

The integral representation \((2)\) then tells us that the second summand is equal to zero, so after we recall \((15)\) we obtain that

\[
\mathbb{E}[N_n^*(c)] \leq p^* = n F(\epsilon_n(c)) \quad \text{for all } 0 < c \leq n \mu,
\]

completing the proof of \((12)\). \(\square\)
3. The re-optimized policy $\hat{\pi}(n, c)$ and its value function

The proof of Proposition 3 tells us that when $0 \leq c \leq n\mu$ then the consumption function $\epsilon_n(c)$ is also the solution of a relaxation to the optimization problem (13) that provides us with an upper bound for the expected number of items selected by the offline sort algorithm. Here, we use the consumption function $\epsilon_k(x)$ in (1) to construct the online feasible threshold policy $\hat{\pi}(n, c)$. Specifically, since $\epsilon_k(x)$ may exceed $x$, we set

$$\hat{h}_k(x) = \min\{x, \epsilon_k(x)\} \quad \text{for all } x \in [0, \infty) \text{ and all } 1 \leq k < \infty,$$  (16)

and we define the re-optimized policy $\hat{\pi}(n, c)$ through the threshold $\{\hat{h}_n, \hat{h}_{n-1}, \ldots, \hat{h}_1\}$. Thus, if the remaining capacity is $x$ when item $i$ is first presented, then item $i$ is selected if and only if its weight $W_i \leq \hat{h}_{n-i+1}(x)$.

In turn, the threshold functions $\{\hat{h}_k : 1 \leq k < \infty\}$ induce a sequence of value functions $\{\hat{v}_k : [0, \infty) \to \mathbb{R}_+ : 0 \leq k < \infty\}$ that are defined recursively. We set $\hat{v}_0(x) = 0$ for all $x \in [0, \infty)$ and we let $\hat{v}_k(x)$ be given by the recursion

$$\hat{v}_k(x) = \left(1 - F(\hat{h}_k(x))\right) \hat{v}_{k-1}(x) + \int_0^{\hat{h}_k(x)} \{1 + \hat{v}_{k-1}(x-w)\} dF(w).$$  (17)

To verify the validity of this recursion, we condition on the weight of the $k$th-to-last item, $W = w$. On the one hand, if $w > \hat{h}_k(x)$ then the item is rejected, the level of remaining capacity does not change, and the number of items that are yet to be seen decreases by one. That is, if the item is rejected, the expected reward to-go is given by $\hat{v}_{k-1}(x)$ and, since rejections happen with probability $1 - F(\hat{h}_k(x))$, we recover the first summand of (17). On the other hand, if $w \leq \hat{h}_k(x)$ the $k$th-to-last item is included in the knapsack. Such a decision produces an immediate reward of one, and it depletes $w$ units of capacity so that the new remaining capacity becomes $x - w$. The number of items that are yet to be seen also decreases to $k - 1$. The decision maker’s payoff for including an item is then given by $1 + \hat{v}_{k-1}(x-w)$ and, by integrating this payoff for $w \in [0, \hat{h}_k(x)]$, we find the second summand of the recursion (17). The value function $\hat{v}_k(x)$ then represents the expected number of items selected by the re-optimized policy $\hat{\pi}(k, x)$ when the number of items that are yet to be revealed is $k$ and the current level of remaining knapsack capacity is $x$. Thus, we also have that

$$\mathbb{E}[N_n(\hat{\pi}(n, c))] = \hat{v}_n(c) \quad \text{for all } n \geq 1 \text{ and all } c \in [0, \infty).$$

We conclude this section by collecting two properties of the consumption function $\epsilon_k(\cdot)$. First, the definition (1) immediately implies that the consumption functions are non-increasing in $k$. That is, one has the monotonicity

$$\epsilon_{k+1}(x) \leq \epsilon_k(x) \quad \text{for all } x \in [0, \infty) \text{ and all } k \geq 1.$$  (18)
Second, provided that the weight distribution $F$ has continuous density $f$, then the implicit function theorem tells us that the function $\epsilon_k(x)$ is differentiable on $(0, k\mu)$, and that its first derivative $\epsilon'_k(x)$ is given by

$$
\epsilon'_k(x) = \frac{1}{k \epsilon_k(x) f(\epsilon_k(x))}
$$

if $0 < x < k\mu$. \hfill (19)

4. On the typical class

The dynamic and stochastic knapsack problem with equal rewards is quite sensitive to the weight distribution $F$. Because the weights are not equal, the remaining capacity process exhibits substantial randomness, and this may lead to unexpected behavior. As such, regularity conditions on the weight distribution $F$ are commonplace in the related literature. For instance, Coffman et al. (1987) only consider distributions $F$ such that $F(w) \sim Aw^\alpha$ as $w \to 0$ for some $A, \alpha > 0$, while Bruss and Robertson (1991) expand this class to include all of the weight distributions $F$ such that $\limsup_{w \to 0^+} F(\lambda w)/F(w) < 1$. Furthermore, Papastavrou et al. (1996, Section 5) show that one must require concavity of $F$ to obtain structural properties such as monotonicity of the optimal threshold functions and concavity of the optimal value functions.

Here, we consider distributions that belong to the typical class characterized in Definition 1. The class is rich enough to include most well-know continuous distributions. The condition (3) regarding the behavior of $F$ at zero is equivalent to the condition required by Bruss and Robertson (1991), while the mild monotonicity condition (4) is required by our analysis. The first implication of condition (3) is given in the next lemma.

**Lemma 4 (Equivalence of CDF Conditions).** There are constants $0 < \lambda < 1$ and $0 < \gamma < 1$ and a value $\bar{w} > 0$ such that

$$
\frac{F(\lambda w)}{F(w)} \leq \gamma < 1 \text{ for all } w \in (0, \bar{w})
$$

if and only if there is a constant $1 < M < \infty$ such that

$$
\frac{w F(w)}{\int_0^w u dF(u)} \leq M < \infty \text{ for all } w \in (0, \bar{w}).
$$

**Proof of Lemma 4.** If. Suppose there is a constant $1 < M < \infty$ such that condition (21) holds. Next, note that for any $\lambda \in (0, 1)$ and any $w \in (0, \bar{w})$ one has the bounds

$$
0 \leq \int_0^w u dF(u) \leq \lambda w \int_0^\lambda dF(u) + w \int_{\lambda w}^w dF(u) = wF(w) - wF(\lambda w)(1 - \lambda),
$$

so it follows that

$$
\frac{F(w)}{F(w) - F(\lambda w)(1 - \lambda)} \leq \frac{w F(w)}{\int_0^w u dF(u)}.
$$
In turn, condition (21) tells us that there is $1 < M < \infty$ such that the right-hand side above is bounded by $M$ so, after rearranging, we obtain that

$$\frac{F(\lambda w)}{F(w)} \leq \frac{M - 1}{M(1 - \lambda)} \quad \text{for all } w \in (0, \bar{w}).$$

Condition (20) then follows after one chooses any $\lambda > M^{-1}$ and sets $\gamma = (M - 1)/[M(1 - \lambda)] < 1$.

*Only if.* Suppose that there are constants $0 < \lambda < 1$ and $0 < \gamma < 1$ such that condition (20) holds for some $\bar{w} > 0$. Then we have that

$$0 < 1 - \gamma \leq 1 - \frac{F(\lambda w)}{F(w)} = \int_{\lambda w}^{w} \frac{dF(u)}{F(w)} \quad \text{for all } w \in (0, \bar{w}).$$

Moreover, if we multiply both sides by $\lambda w$ and use the fact that $\lambda w \leq u$ for all $u \in (\lambda w, w)$ we also have that

$$\lambda w (1 - \gamma) \leq \int_{\lambda w}^{w} \frac{\lambda w}{F(w)} dF(u) \leq \int_{0}^{w} \frac{u}{F(w)} dF(u).$$

Next, we divide both sides by $w$ and rearrange to obtain that

$$\frac{wF(w)}{\int_{0}^{w} u dF(u)} \leq \frac{1}{\lambda (1 - \gamma)} \quad \text{for all } w \in (0, \bar{w}),$$

so condition (21) follows by setting $M = [\lambda(1 - \gamma)]^{-1}$, and the proof is now complete. □

**Distributions that do and do not belong to the typical class.**

As we mentioned earlier, the typical class of distributions is large enough to include most well-known non-negative continuous distributions. Here we provide several specific examples.

1. *Power distributions.* Distributions such that $F(w) = A w^\alpha$ for some $A, \alpha > 0$ on $(0, \bar{w})$ are typical. Condition (3) is immediately verified. We also have that the function $w^3 f(w) = A \alpha w^{\alpha + 2}$ is increasing because $A, \alpha > 0$, so (4) holds as well.

2. *Convex distributions.* Distributions $F$ that are convex in a neighborhood of 0 and that have continuous density $f$ are typical. Convexity tells us that $F(\lambda w) \leq F(w) \lambda$ so (3) follows. Furthermore, convexity also gives us that the density $f$ is non-decreasing, so (4) is verified.

3. *Convex combinations of typical distributions.* The class of typical distributions is closed under convex combinations. If $F$ and $G$ are two typical distributions and $\beta \in [0, 1]$ then the distribution $\beta F + (1 - \beta)G$ is typical.

It is important to note, however, that one can construct examples of distributions that do not belong to the typical class. For instance, the distribution $F(w) = \frac{\log w}{\log \bar{w}}$ for $\bar{w} < 1$ and $w \in (0, \bar{w})$ is an example that satisfies condition (4) but violates condition (3). For a fixed $0 < \lambda < 1$, one can easily check that

$$\limsup_{w \to 0^+} \frac{F(\lambda w)}{F(w)} = \limsup_{w \to 0^+} \frac{\log w}{\log \lambda + \log w} = 1,$$
so condition \(3\) fails to hold. On the other hand, the function \(w^3 f(w) = \frac{w^2 \log \bar{w}}{(\log w)^2}\) is increasing on \((0, \bar{w})\) and condition \(4\) is satisfied.

The distribution \(F(w) = A \int_0^w \{\sin(1/u)\}^2 du\) for \(w \in (0, \bar{w})\) and \(A = (\int_0^w \{\sin(1/u)\}^2 du)^{-1} > 0\) is an example that satisfies condition \(3\) while violating condition \(4\). In fact, one has that the limit

\[
\limsup_{w \to 0^+} \frac{F(\lambda w)}{F(w)} = \lambda < 1,
\]

and one also has that the function \(w^3 f(w) = A w^3 \{\sin(1/w)\}^2\) oscillates infinitely many times in a (positive) neighborhood of zero, so the monotonicity \(4\) fails to hold.

5. A logarithmic regret bound

To prove that the regret grows at most logarithmically, we let

\[
K = \left\lfloor \frac{c}{\int_0^w w f(w) dw} \right\rfloor
\]

and focus on dynamic and stochastic knapsack problems that have more than \(K\) items. Of course, this is without loss of generality because the quantity \(K\) defined in \(22\) is a constant that does not depend on the number of items \(n\), so we can ignore the last \(K\) decisions without affecting our regret bound. When \(k \geq K\) we have (i) that \(\epsilon_k(x) \leq \bar{w}\) for all \(x \in [0, c]\), and (ii) that the integral representation \(2\) always holds. Thus, we are focusing on problem instances in which we can use the properties of the typical class in full.

The proof of the regret bound then comes in two parts. In the next section we derive several estimates that have to do with the weight distribution belonging to the typical class and with \(k \geq K\), while in Section 5.2 we estimate the gap \(k F(\epsilon_k(x)) - \hat{v}_k(x)\).

5.1. Preliminary observations

When \(k \geq K\) the properties that characterize typical weight distributions can be used to obtain general estimates that are crucial to our analysis. As a warm-up we obtain the following estimate on the mismatch between the probability of an item weight being smaller than the feasible threshold \(\hat{h}_k\) and the probability of the same weight being smaller than the consumption function \(\epsilon_k\).

**Lemma 5.** If the weight distribution \(F\) belongs to the typical class then there is \(1 < M < \infty\) such that

\[
\frac{k \epsilon_k(x) F(\epsilon_k(x))}{x} \leq M \quad \text{for all } x \in (0, c) \text{ and all } k \geq K = \left\lfloor \frac{c}{\int_0^w w f(w) dw} \right\rfloor.
\]

In turn, we also have that

\[
F(\epsilon_k(x)) - F(\hat{h}_k(x)) \leq \frac{M}{k} \quad \text{for all } x \in [0, c] \text{ and all } k \geq K.
\]
Proof. The uniform bound (23) is essentially a restatement of inequality (21) in Lemma 4. If \( x \in (0, c] \) and \( k \geq K \), then we have that
\[
\frac{x}{k} \leq \frac{x}{K} \leq \frac{c}{K} \leq \int_0^\bar{w} w f(w) \, dw \leq \mu,
\]
so the definition (1) of the consumption function \( \epsilon_k(x) \) and the equality (2) give us that
\[
\epsilon_k(x) \leq \bar{w} \quad \text{and} \quad \int_{\epsilon_k(x)}^{\epsilon_k(x)+1} w f(w) \, dw = \frac{x}{k} \quad \text{for all} \quad k \geq K \quad \text{and all} \quad x \in (0, c].
\]
(25)
The two observations in (25) together with the bound (21) in which we replace \( w \) with \( \epsilon_k(x) \) then imply that
\[
\frac{k \epsilon_k(x) F(\epsilon_k(x))}{x} = \frac{\epsilon_k(x) F(\epsilon_k(x))}{\int_0^{\epsilon_k(x)} w f(u) \, du} \leq M \quad \text{for all} \quad k \geq K \quad \text{and} \quad x \in (0, c],
\]
concluding the proof of the uniform bound (23).

We now turn to inequality (24). If \( x = 0 \) then inequality (24) is obvious. Otherwise, if \( x > 0 \) we recall from (16) that \( \hat{h}_k(x) = \min\{x, \epsilon_k(x)\} \), so the left-hand side of (24) is equal to 0 when \( \epsilon_k(x) \leq x < \infty \), and inequality (24) is again trivial. Instead, if \( 0 < x < \epsilon_k(x) \), we obtain from (23) that
\[
F(\epsilon_k(x)) \leq \frac{M x}{k \epsilon_k(x)} \leq \frac{M}{k} \quad \text{for all} \quad k \geq K \quad \text{and} \quad 0 < x < \epsilon_k(x),
\]
concluding the proof of the lemma. □

In the same spirit of Lemma 5, we can also estimate the difference in the probability of selecting an upcoming item as a function of the number of items that are yet to be seen.

LEMMA 6. For all \( x \in [0, c] \) and all \( k \geq K \) we have that
\[
F(\epsilon_{k+1}(x)) - F(\epsilon_k(x)) \leq -\frac{x}{k(k+1)\epsilon_k(x)}.
\]

Proof. For \( k \geq K \) the equality (2) and the monotonicity (18) give us the representation
\[
\frac{x}{k} - \frac{x}{k+1} = \int_{\epsilon_k(x)}^{\epsilon_{k+1}(x)} w dF(w), \quad \text{for all} \quad x \in [0, c].
\]
If we now replace the integrand \( w f(w) \) with the upper bound \( \epsilon_k(x) f(w) \) and rearrange, we obtain
\[
\frac{x}{k(k+1)} \leq \epsilon_k(x) [F(\epsilon_k(x)) - F(\epsilon_{k+1}(x))],
\]
completing the proof of the lemma. □

Typical weight distributions are also nice because one can tightly approximate the difference \( F(\epsilon_k(x)) - F(\epsilon_k(x-w)) \) that accounts for the sensitivity in the remaining capacity of the probability of selecting the \( k \)th-to-last item. A formal estimate is given in the next proposition, and it constitutes a key step in our argument.
Proposition 7. If the weight distribution $F$ belongs to the typical class, then there is a constant $1 < M < \infty$ such that one has the inequality
\[
1 - \frac{F(\epsilon_k(x - w))}{F(\epsilon_k(x))} \leq \frac{w^2}{x^2} (1 - M^{-1}) + \frac{w}{k\epsilon_k(x) F(\epsilon_k(x))}
\] (26)
for all $w \in [0,x]$, $x \in (0,c]$, and all $k \geq K = \left[ \frac{c}{\int_0^x w f(w) dw} \right]$.

The proof of Proposition 7 requires the following intermediate estimate.

Lemma 8 (Convexity upper bound). If the weight distribution $F$ has continuous density $f$ then for all $k \geq K$, $x \in [0,c]$ and $y \in [0,1]$ we have the integral representation
\[
kF(\epsilon_k(x)) - kF(\epsilon_k(x(1-y))) = \int_{x(1-y)}^x \frac{1}{\epsilon_k(u)} du.
\]
Moreover, if the distribution $F$ belongs to the typical class the map $x \mapsto \epsilon_k(x)^{-1}$ is convex on $(0,c)$, so we also have the upper bound
\[
kF(\epsilon_k(x)) - kF(\epsilon_k(x(1-y))) \leq \frac{xy}{2} \left[ \frac{1}{\epsilon_k(x)} + \frac{1}{\epsilon_k(x(1-y))} \right].
\] (27)

Proof. Since the weight distribution $F$ has continuous density and $c/\mu \leq K \leq k$ we have from (19) that the first derivative
\[
\epsilon_k'(x) = \frac{1}{k\epsilon_k(x) f(\epsilon_k(x))} \quad \text{for all } x \in (0,c).
\]
Thus, it follows that the map $x \mapsto F(\epsilon_k(x))$ is differentiable on $(0,c)$ and
\[
(kF(\epsilon_k(x)))' = k\epsilon_k'(x) f(\epsilon_k(x)) = \frac{1}{\epsilon_k(x)} \quad \text{for all } x \in (0,c).
\]
In turn, the fundamental theorem of calculus tells us that for $y \in [0,1]$ we have the integral representation
\[
\int_{x(1-y)}^x \frac{1}{\epsilon_k(u)} du = kF(\epsilon_k(x)) - kF(\epsilon_k(x(1-y))),
\]
proving the first assertion of the lemma.

To check the convexity of the map $x \mapsto \epsilon_k(x)^{-1}$, we use the expression of the first derivative (19) one more time to obtain for $k \geq K$ that
\[
\left( \frac{1}{\epsilon_k(x)} \right)' = -\frac{\epsilon_k'(x)}{\epsilon_k^2(x)} = -\frac{1}{k\epsilon_k^2(x) f(\epsilon_k(x))}.
\]
In turn, if $F$ belongs to the typical class and $k \geq K$ then the monotonicity condition (4) tells us that the first derivative $(1/\epsilon_k(x))'$ is non-decreasing on $(0,c)$, so the map $x \mapsto \epsilon_k(x)^{-1}$ is convex.
This convexity property then provides us with a linear majorant
\[ m_k(u) = \frac{u - x}{yx} \left( \frac{1}{\epsilon_k(x)} - \frac{1}{\epsilon_k((1 - y)x)} \right) + \frac{1}{\epsilon_k(x)} \]
such that
\[ \frac{1}{\epsilon_k(u)} \leq m_k(u) \quad \text{for all } u \in [(1 - y)x, x]. \]
Integration of the majorant \( m_k(u) \) over \( [(1 - y)x, x] \) gives us the upper bound \([27]\), and the proof of the lemma follows. 

We now have all of the estimates we need to complete the proof of Proposition \([7]\).

**Proof of Proposition \([7]\).** If \( w = 0 \) then inequality \([26]\) is trivial. Otherwise, for \( K \leq k < \infty \) we consider the function \( g_k : (0, c] \times (0, 1] \to \mathbb{R} \) given by
\[ g_k(x, y) = \frac{1}{y^2} \left\{ 1 - \frac{F(\epsilon_k(x(1 - y)))}{F(\epsilon_k(x))} - \frac{xy}{k\epsilon_k(x)F(\epsilon_k(x))} \right\}, \]
and we note that inequality \([26]\) follows by setting \( y = w/x \leq 1 \) and rearranging, provided that one has the uniform bound
\[ g_k(x, y) \leq 1 - M^{-1} \quad \text{for all } x \in (0, c], \ y \in (0, 1], \text{ and } k \geq K. \] \([28]\)
The function \( g_k(x, y) \) is differentiable with respect to \( y \) for any given \( x \in (0, c] \), and if
\[ B = \frac{2}{y^3kF(\epsilon_k(x))} \geq 0, \]
then the \( y \)-derivative of \( g_k(x, y) \) can be written as
\[ \frac{\partial}{\partial y} g_k(x, y) = B \left\{ \frac{xy}{2} \left[ \frac{1}{\epsilon_k(x)} + \frac{1}{\epsilon_k(x(1 - y))} \right] - kF(\epsilon_k(x)) + kF(\epsilon_k(x(1 - y))) \right\}. \]
Inequality \([27]\) of Lemma \([8]\) then tells us that the \( y \)-derivative of \( g_k(x, y) \) is non-negative so that the map \( y \mapsto g_k(x, y) \) is non-decreasing in \( y \) for any given \( x \in (0, c] \). In turn, we have that
\[ g_k(x, y) \leq g_k(x, 1) = 1 - \frac{x}{k\epsilon_k(x)F(\epsilon_k(x))}, \]
so inequality \([28]\) follows from the uniform bound \([23]\), and the proof of the proposition is now complete. 
\( \square \)
5.2. Analysis of residuals

To estimate the gap between the expected number of items selected by policy $\hat{\pi}(n,c)$ and the prophet upper bound $nF(\epsilon_n(c))$, we study appropriate residual functions. Specifically, we let

$$r_k(x) = kF(\epsilon_k(x)) - \hat{\nu}_k(x) \quad \text{for } x \in [0,c] \text{ and } 1 \leq k \leq n$$

be the residual function at time $k$ when the level of remaining capacity is $x$. The residual function $r_k(x)$ is continuous and defined on a compact interval, so if we maximize with respect to $x$ we obtain the maximal residual

$$\rho_k = \max_{0 \leq x \leq c} r_k(x) \quad \text{for } k \in [n].$$

The second half of Theorem 1 is just a corollary of the following proposition, which verifies that the maximal residual $\rho_n = O(\log n)$ as $n \to \infty$.

**Proposition 9.** If the weight distribution $F$ belongs to the typical class, then there is a constant $1 < M < \infty$ such that the maximal residual

$$\rho_n = \max_{0 \leq x \leq c} \{nF(\epsilon_n(x)) - \hat{\nu}_n(x)\} \leq M + M \log n \quad \text{for all } n \geq 1.$$

For the proof of this proposition we write the maximal residual $\rho_n$ as a telescoping sum, and we obtain an appropriate upper bound for each summand. The upper bound follows from the following lemma.

**Lemma 10.** If the weight distribution $F$ belongs to the typical class, then there is a constant $1 < M < \infty$ such that the difference

$$r_{k+1}(x) - \rho_k \leq \frac{M}{k} \quad \text{for all } x \in [0,c] \text{ and all } k \geq K.$$

**Proof.** The residual function $r_k(x)$ defined in (29) provides us with an alternative representation for the value function $\hat{\nu}_{k+1}(x)$ which gives us the expected number of items selected by policy $\hat{\pi}(k+1,x)$. Specifically, if we substitute $\hat{\nu}_k(x)$ with $kF(\epsilon_k(x)) - r_k(x)$ in the recursion (17), we then obtain that

$$\hat{\nu}_{k+1}(x) = \{1 - F(\hat{h}_k(x))\}\{kF(\epsilon_k(x)) - r_k(x)\} + \int_0^{\hat{h}_k(x)} \{1 + kF(\epsilon_k(x-w)) - r_k(x-w)\} f(w) \, dw.$$

Next, if we replace the residuals $r_k(\cdot)$ with their maximal value $\rho_k$ and rearrange, we obtain the lower bound

$$kF(\epsilon_k(x)) + F(\hat{h}_k(x)) + \int_0^{\hat{h}_k(x)} \{kF(\epsilon_k(x-w)) - kF(\epsilon_k(x))\} f(w) \, dw \leq \hat{\nu}_{k+1}(x) + \rho_k.$$  

(31)
In turn, the definition \( (29) \) of the residual function tells us that

\[
r_{k+1}(x) - \rho_k = (k + 1) F(\epsilon_{k+1}(x)) - (\widehat{h}_{k+1}(x) + \rho_k),
\]

so if we replace the sum \( \widehat{h}_{k+1}(x) + \rho_k \) with its lower bound \( (31) \) and rearrange, we obtain the upper bound

\[
r_{k+1}(x) - \rho_k \leq (k + 1) F(\epsilon_{k+1}(x)) - k F(\epsilon_k(x)) - F(\widehat{h}_k(x)) + k F(\epsilon_k(x)) \int_0^{\widehat{h}_k(x)} \left\{ 1 - \frac{F(\epsilon_k(x-w))}{F(\epsilon_k(x))} \right\} f(w) \, dw.
\]

Next, we introduce the shorthand

\[
\mathcal{I}_k(x) = \int_0^{\widehat{h}_k(x)} \left\{ 1 - \frac{F(\epsilon_k(x-w))}{F(\epsilon_k(x))} \right\} f(w) \, dw
\]

for the integral that appears on the right-hand side of \( (32) \). For \( w \in [0, \widehat{h}_k(x)] \) we have the trivial bound \( w^2 \leq w \widehat{h}_k(x) \) so if we replace \( w^2 \) with its upper bound \( \widehat{h}_k(x) \) on the right-hand side of \( (26) \) and integrate we obtain that there is \( 1 < M < \infty \) such that

\[
\mathcal{I}_k(x) \leq \left[ (1 - M^{-1}) \frac{\widehat{h}_k(x)}{x^2} + \frac{1}{k \epsilon_k(x) F(\epsilon_k(x))} \right] \int_0^{\widehat{h}_k(x)} w f(w) \, dw.
\]

We now multiply both sides by \( k F(\epsilon_k(x)) \) and simplify to obtain that

\[
k F(\epsilon_k(x)) \mathcal{I}_k(x) \leq \left[ k F(\epsilon_k(x)) (1 - M^{-1}) \frac{\widehat{h}_k(x)}{x^2} + \frac{1}{\epsilon_k(x)} \right] \int_0^{\widehat{h}_k(x)} w f(w) \, dw.
\]

The definition of \( \widehat{h}_k(x) = \min\{x, \epsilon_k(x)\} \) tells us that we can obtain a further upper bound if we replace \( \widehat{h}_k(x) \) with \( \epsilon_k(x) \) on the last right-hand side. When we perform this replacement and recall the equality \( (2) \), we find that

\[
k F(\epsilon_k(x)) \mathcal{I}_k(x) \leq (1 - M^{-1}) \frac{\epsilon_k(x) F(\epsilon_k(x))}{x} + \frac{x}{k \epsilon_k(x)}.
\]

If we now apply the uniform upper bound \( (23) \) to the first summand on the right-hand side, and rearrange, we obtain that

\[
k F(\epsilon_k(x)) \mathcal{I}_k(x) \leq \frac{M - 1}{k} + \frac{x}{k \epsilon_k(x)}.
\]

We now replace the last summand of \( (32) \) with the upper bound above and rearrange to obtain that

\[
r_{k+1}(x) - \rho_k \leq \frac{M - 1}{k} + (k + 1) \left\{ F(\epsilon_{k+1}(x)) - F(\epsilon_k(x)) + \frac{x}{k(k+1) \epsilon_k(x)} \right\} + F(\epsilon_k(x)) - F(\widehat{h}_k(x)).
\]
Here, Lemma 6 tells us that the second summand on the right-hand side is non-positive, and inequality (24) tells us that there is $1 < M < \infty$ such that the difference $F(\epsilon_k(x)) - F(\hat{h}_k(x))$ is bounded above by $M/k$. When we assemble these observations, we finally find that
\[ r_{k+1}(x) - \rho_k \leq \frac{2M - 1}{k} \quad \text{for all } x \in [0, c] \text{ and all } k \geq K, \]
concluding the proof of the lemma. \[ \square \]

We now have all of the tools we need to complete the proof of Proposition 9 that follows next.

**Proof of Proposition 9.** We write the maximal residual $\rho_n$ in (30) as a telescoping sum and use the definition (29) of the residual function to obtain that
\[ \rho_n = \rho_K + \sum_{k=K}^{n-1} \{\rho_{k+1} - \rho_k\} \leq K + \sum_{k=K}^{n-1} \{\rho_{k+1} - \rho_k\}. \]
Lemma 10 then tells us that
\[ \rho_{k+1} - \rho_k \leq \frac{M}{k} \quad \text{for all } K \leq k \leq n, \]
so when we combine the last two observations we obtain that there is a constant $1 < M < \infty$ such that
\[ \rho_n \leq M + M \log n, \]
just as needed. \[ \square \]

6. Conclusions and future direction

In this paper we studied the dynamic and stochastic knapsack problem with unitary rewards and independent random weights with common continuous distribution $F$. We proved that—under some mild regularity conditions on the weight distribution—the regret is, at most, logarithmic in $n$. In particular, we showed that this regret bound is attained by a re-optimized heuristic that can be expressed in closed-form.

Two questions stem naturally from our analysis. The first one entails the difference in performance between the re-optimized heuristic and the optimal online policy. Based on an extensive numerical analysis, we conjecture that
\[ \max_{\pi \in \Pi(n,c)} \mathbb{E}[N_n(\pi)] = \mathbb{E}[N_n(\widehat{\pi}(n,c))] + O(1) \tag{33} \]
for all $n \geq 1$ and for a large class of weight distributions. However, it is well-known that the optimal policy often lacks of desirable structural properties, so proving (33) is unlikely to be easy. The
second question has to do with the performance of the offline sort algorithm. Here, numerical evidence suggests that
\[ E[N_n^p(c)] = nF(\epsilon_n(c)) + O(1) \]
for all \( n \geq 1 \) and most continuous weight distributions \( F \).

Resolving the two conjectures above would imply that our regret bound is of the correct order, and that no online algorithm can get within a constant of offline sort. This is in contrast with some other dynamic and stochastic knapsack problems in which the sequential decision maker does essentially as well as the prophet. It also suggests that when items have random weights, then the design of near-optimal heuristics requires more care than usual.

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