Discrete State-Space Methods for the Study of Dynamic Economies

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Updated Version

1 Introduction

In this chapter I discuss a number of numerical methods for solving dynamic stochastic general equilibrium models which fall within the common category of discrete state-space methods. These methods can be applied in situations where the state-space of the model in question is given by a finite set of discrete points. In these cases, the methods provide an “exact” solution to the model in question.¹ On the other hand, these methods are frequently applied in situations where the model’s state-space is continuous, in which case the discrete state-space can be viewed as an approximation to the continuous state-space.²

I will discuss discrete state-space methods in the context of two well known examples, a simple one asset version of Lucas’ (1978) consumption-based asset pricing model, and the

¹A simple example in the asset pricing literature is the paper by Mehra and Prescott (1985), in which the only state variable is the exogenous growth rate of the consumer’s endowment income which is assumed to follow a simple two state Markov chain. Many of the papers which have followed from this one have made similar assumptions, a notable exception being Cecchetti, Lam and Mark (1993) who use a mixture distribution for the growth rate of the endowment. Because the distribution mixes a two state Markov chain and an i.i.d. Gaussian random variable, a simple exact solution obtained using discrete state space methods is still available.

²Examples include Baxter, Crucini and Rouwenholt (1990), Christiano (1990), Coleman (1990), Tauchen (1990) and Taylor and Uhlig (1990).

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one sector neoclassical growth model. I will not exhaust the list of possible discrete state-space methods as they are very numerous. Rather, I will describe several examples which illustrate the basic principles involved.

Why use discrete state-space methods? In situations where log-linear approximations to the first-order necessary conditions of a model are close approximations to the exact conditions, there are many advantages to using these approximations. On the other hand, discrete state-space methods are useful when log-linear approximations are not accurate, or when we have no knowledge about their likely accuracy. They are justified on fairly intuitive grounds, but are also supported by a wealth of theoretical results. \(^3\) They are applicable to a wide variety of problems, and are often a good way of making a first pass at solving an unfamiliar model, because they are usually reliable. The main drawback of discrete state-space methods is their computational expense. For simple problems they present no difficulty, but for problems with a large state-space, the expense of computing solutions can be great in terms of both time and computer memory.

In the next section I describe the basic principles of numerical quadrature which underlie most discrete state-space methods. In the third section I show how they can be applied in a very straightforward way to problems in which the state space consists entirely of exogenous variables. In the fourth section I describe methods that can be used when there are endogenous state variables. The fifth section gives some concluding thoughts.

2 The Basics of Quadrature

2.1 How Quadrature Works

Since we are concerned mainly with solving discrete-time dynamic stochastic general equilibrium models, a common feature of all of these models will be that there is at least one first-order necessary condition governing the intertemporal decisions of some agent. Typically, this condition will express the relationship between some variable today and the expected value of some other variable in the future. For example, the marginal cost today, in terms of utility foregone, of any decision must be matched by the sum of all expected future marginal

\(^3\)Bertsekas (1976) contains numerous results pertaining to value function iteration over discrete state spaces. The typical result is that the fixed point of some discretized dynamic programming problem converges pointwise to its continuous equivalent. Atkinson (1976) and Baker (1977) present a wealth of convergence results pertaining to the use of discrete state spaces to solve integral equations. Tauchen and Hussey (1991) and Burnside (1993) present results, respectively, concerning pointwise and absolute convergence of solutions to asset pricing model obtained using discrete state spaces.
benefits. When the state-space of the model is discrete, evaluating the relevant expected value amounts to taking a weighted sum across all the possible values of the state variables in the future, where the weights are the relevant probabilities given the state today. When the state-space is continuous the sum is replaced by an integral taken with respect to the conditional density of the future state given the current state. As one might expect, discrete state-space approximations to continuous state-spaces involve approximating integrals by sums.

The methods for approximating integrals by sums that I will focus on here are formally referred to as quadrature methods. Abstracting, for the moment, from the Euler equations of some dynamic model, suppose that one had to compute the integral \( \int_Y \psi(y)f(y)dy \) where I will refer to \( \psi \) as the kernel function and \( f \) is some density function defined over the set \( Y \).\(^4\) This would be, of course, the expected value of \( \psi(y) \). Quadrature methods are based on the notion that one can use the approximation

\[
\sum_{i=1}^{N} \psi(y_{i,N})w_{i,N} \approx \int_Y \psi(y)f(y)dy, \tag{1}
\]

where the points \( y_{i,N} \in Y, i = 1, \ldots, N \), are chosen, presumably, according to some rule, while the weight given to each point, \( w_{i,N} \), presumably, relates to the density function \( f \) in the neighborhood of those points. In general, a quadrature method requires a rule for choosing the points, \( y_{i,N} \), and a rule for choosing the weights, \( w_{i,N} \).

There is a large mathematical literature on numerical integration which describes numerous techniques for implementing the left-hand side of (1) on a computer to obtain an approximation to the right hand side.\(^5\) The simplest method, at least when the density \( f \) has compact support, is to used an equally spaced grid of points and simply take the average of the function \( \psi(y)f(y) \) evaluated at these points. The method I will focus on here is Gaussian quadrature, as advocated by Tauchen and Hussey (1991). This method is defined by a specific set of rules for choosing the points and weights which are related to the density function \( f \). It has many attractive properties, not least of which is the fact that the approximation in (1) is exact for \( \psi \) functions which are polynomials of degree \( 2N - 1 \) or less.

In the next two subsections I will outline (i) the rules for choosing points and weights

\(^4\)The reader may be familiar with a more basic problem in numerical quadrature, which is to compute the integral \( \int g(y)dy \) over some interval. We use the form \( \int_\psi(y)f(y)dy \) to emphasize the stochastic nature of the variable \( y \).

\(^5\)A good starting point for someone with an interest in numerical integration in general is the chapter on this subject in Press, et. al. (1992). They provide further detailed references.
under Gaussian quadrature and (ii) some of the useful properties of Gaussian quadrature
rules than enable us to interpret them. The reader is who is less interested in the details of
quadrature may wish to skip ahead to section 3.

2.2 Selection of Points and Weights under Gaussian Quadrature

Since arbitrary accuracy of a Gaussian quadrature rule will typically require us to let $N$
be arbitrarily large, it is a somewhat restrictive method, because this implies that all non-
negative integer moments of $y$ must exist. Assuming that this is the case, the method provides
straightforward rules for determining the points and weights for numerical integration. These
rules are based on the properties of the orthogonal polynomials corresponding to the density
function, $f$.\(^6\) The existence of these polynomials is guaranteed by the assumption made
above concerning the moments of $y$.\(^7\) The set of orthogonal polynomials, $\{\phi_N(y)\}_{N=0}^{\infty}$ for
the density $f(y)$ are determined according to the following rules

$$\phi_N(y) = \lambda_{N0} + \lambda_{N1}y + \lambda_{N2}y^2 + \cdots \lambda_{NN}y^N, \quad \lambda_{NN} > 0$$  \hspace{1cm}(2)

$$\int_Y \phi_N(y)\phi_M(y)f(y)dy = \delta_{NM},$$  \hspace{1cm}(3)

where $\delta_{NM} = 1$ if $N = M$ and 0 otherwise. Generally, the sequence is

Notice that for $N = 0$, (3) implies that

$$\int_Y \phi_0^2(y)f(y)dy = \int_Y \lambda_{N0}^2f(y)dy = \lambda_{00}^2 = 1,$$  \hspace{1cm}(4)

or, given the sign restriction on $\lambda_{00}$ implied by (2), $\lambda_0 = 1$. But, then (3) implies that

$$\int_Y \phi_N(y)f(y)dy = 0 \text{ for } N \geq 1.$$  \hspace{1cm}(5)

In other words, each of the nontrivial polynomials is a mean zero function of $y$. Furthermore, this provides (3) with the following natural interpretation: it requires that the nontrivial polynomials be mutually uncorrelated, and that they have unit
variance.

To take a specific example, suppose, for the moment, that $f(y)$ is the standard normal
density. We have seen that $\phi_0(y) = 1$ for any $f$. For $N = 1$, $\phi_1(y) = \lambda_{10} + \lambda_{11}y$, and (3)
implies that

$$0 = \int_Y \phi_1(y)\phi_0(y)f(y)dy = \int_Y (\lambda_{10} + \lambda_{11}y)f(y)dy = \lambda_{10}$$  \hspace{1cm}(5)

$$1 = \int_Y \phi_1^2(y)f(y)dy = \int_Y (\lambda_{10} + \lambda_{11}y)^2f(y)dy = \lambda_{10}^2 + \lambda_{11}^2.$$  \hspace{1cm}(6)

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\(^6\)A classic reference on the properties of orthogonal polynomials is Szegö (1939).

\(^7\)Uniqueness of the sequence of orthogonal polynomials follows if and only if

$$\begin{vmatrix}
E\left( 1 \ y \ \cdots \ y^N \right) & \begin{pmatrix} 1 & y & \cdots & y^N \end{pmatrix}
\end{vmatrix}
$$

is nonzero for all $N$.  

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Thus \( \lambda_{10} = 0 \) while the sign restriction from (2) implies that \( \lambda_{11} = 1 \). Therefore \( \phi_1(y) = y \).

For \( N = 2 \), we have \( \phi_2(y) = \lambda_{20} + \lambda_{21} y + \lambda_{22} y^2 \). It follows from (3) that

\[
0 = \int_Y \phi_2(y) \phi_0(y) f(y) dy = \int_Y (\lambda_{20} + \lambda_{21} y + \lambda_{22} y^2) f(y) dy = \lambda_{20} + \lambda_{22} \tag{7}
\]

\[
0 = \int_Y \phi_2(y) \phi_1(y) f(y) dy = \int_Y (\lambda_{20} + \lambda_{21} y + \lambda_{22} y^2) y f(y) dy = \lambda_{21} \tag{8}
\]

\[
1 = \int_Y \phi_2^2(y) f(y) dy = \int_Y (\lambda_{20} + \lambda_{21} y + \lambda_{22} y^2)^2 f(y) dy = \lambda_{20}^2 + 2\lambda_{20} \lambda_{22} + \lambda_{21}^2 + 3\lambda_{22}^2. \tag{9}
\]

Thus \( \lambda_{20} = -1/\sqrt{2} \), \( \lambda_{21} = 0 \), and \( \lambda_{22} = 1/\sqrt{2} \), so that \( \phi_2(y) = (y^2 - 1)/\sqrt{2} \). Continuing this process iteratively, one obtains the \( N \)th orthogonal polynomial for the standard normal from the recursive formula

\[
\phi_N(y) = \sqrt{\frac{1}{N}} \phi_{N-1}(y) - \sqrt{\frac{N-1}{N}} \phi_{N-2}(y). \tag{10}
\]

Once we have the set of orthogonal polynomials corresponding to the density \( f \), obtaining the points and weights for quadrature is straightforward. If we are using an \( N \)-point Gaussian quadrature rule,\(^8\) the \( N \) points are located at the roots of the \( N \)th orthogonal polynomial. For the standard normal example this would mean that for \( N = 1 \) there is one point, \( y_{1,1} = 0 \), for \( N = 2 \) there are points \( y_{1,2} = -1 \) and \( y_{2,2} = 1 \), for \( N = 3 \) the points are \( y_{1,3} = -\sqrt{3} \), \( y_{2,3} = 0 \) and \( y_{3,3} = \sqrt{3} \), and so on.

The next step is to choose the weights. The weights for an \( N \)-point Gaussian quadrature rule are chosen so that if the kernel is a \( 2N-1 \)th-or-lower ordered polynomial the quadrature approximation is exact. Formally, define the set

\[
P_j = \{ P(y) = p_0 + p_1 y + p_2 y^2 + \cdots | p_i = 0 \text{ for } i > j, \ p_i \in \mathbb{R} \text{ otherwise} \}. \tag{11}
\]

Then the weights are chosen so that

\[
\sum_{i=1}^N P(y_{i,N}) w_{i,N} = \int_Y P(y) f(y) dy, \quad \forall P(y) \in P_{2N-1}. \tag{12}
\]

\(^8\)That is, the number of terms in the sum on the left-hand side of (1) is \( N \).
That weights with this property exist is proven in Szegö (1939), Theorem 3.4.1. Although (12) represents $2N$ restrictions (because the rule must be exact for all polynomials of $0$th to $2N - 1$th orders) on the $N$ values of the weights, only $N$ of the restrictions are unique.

To return to the standard normal example, a 1-point rule must be exact for all polynomials up to first order. Therefore

$$ p_0 w_{1,1} = \int_Y p_0 f(y) dy = p_0 \quad (13) $$

$$ (p_0 + p_1 y_1) w_{1,1} = p_0 w_{1,1} = \int_Y (p_0 + p_1 y) f(y) dy = p_0 \quad (14) $$

which both imply that $w_{1,1} = 1$ (notice that the second equation is redundant). For a two point rule, all third-order or higher polynomials must be exactly integrated. The two unique conditions implied by (12) are: $w_{1,2} = w_{2,2}$ and $w_{1,2} + w_{2,2} = 1$. Therefore, the weights are $w_{1,2} = w_{2,2} = 1/2$.

### 2.3 Properties of Gaussian Quadrature Rules

One important general feature of Gaussian quadrature is that the weights generated by (12) will sum to one for any $N$. That is, $\sum_{i=1}^{N} w_{i,N} = 1$ for all $N$.\footnote{Szegö (1939), Theorem 3.4.2.} This provides an important interpretation of the quadrature approximation (12). While the right hand side of the equation is the expected value of $\psi(y)$ given the continuous density $f$, the left hand side is an approximation to this which can be interpreted as the expected value of $\psi(y)$ when $y$ has a discrete distribution over the set $\{y_{1,N}, y_{2,N}, \ldots, y_{N,N}\}$ with associated probabilities $\{w_{1,N}, w_{2,N}, \ldots, w_{N,N}\}$.

Another interesting feature of Gaussian quadrature rules is that there exist points, $z_{i,N}$, $i = 0, \ldots, N$, which satisfy

$$ z_{0,N} < y_{1,N} < z_{1,N} < y_{2,N} < \cdots $$

$$ < y_{N-1,N} < z_{N-1,N} < y_{N,N} < z_{N,N} \quad (15) $$

such that

$$ w_{i,N} = \int_{z_{i-1,N}}^{z_{i,N}} f(y) dy. \quad (16) $$

The end points $z_{0,N}$ and $z_{N,N}$ represent the lower and upper limits of the set $Y$ and can be
infinite as in the case of the normal distribution. This means that

\[
\sum_{i=1}^{N} \psi(y_{i,N}) w_{i,N} = \sum_{i=1}^{N} \psi(y_{i,N}) \int_{z_{i-1,N}}^{z_{i,N}} f(y)dy = \int_{Y} \sum_{i=1}^{N} \psi(y_{i,N}) 1_{(z_{i-1,N}, z_{i,N})}(y) f(y)dy
\]

(17)

where \(1_{(z_{i-1,N}, z_{i,N})}(y)\) is the indicator function which is 1 if \(y \in (z_{i-1,N}, z_{i,N})\), and is zero otherwise. Notice that if we define \(\psi_N(y) = \sum_{i=1}^{N} \psi(y_{i,N}) 1_{(z_{i-1,N}, z_{i,N})}(y)\), an equivalent expression for (1) is

\[
\int_{Y} \psi_N(y) f(y)dy \approx \int_{Y} \psi(y) f(y)dy
\]

(18)

where we are approximating the expectation on the right hand side by replacing the kernel \(\psi(y)\) by a function that approximates it, in this case a step-function. This provides an alternative interpretation of quadrature rules as a numerical method in which arbitrary integrands are replaced by functions which lie in a narrower class of functions, in this case step-functions. Thus, there is a natural relationship between quadrature and the weighted residual methods discussed in this volume by McGrattan.

Finally, in cases where \(Y\) is a compact set \([a, b]\), \(f(y)\) is an arbitrary density on \(Y\) and \(\psi(y)\) is any function for which the Riemann-Stieltjes integral on the right hand side of (1) exists, it follows that

\[
\lim_{N \to \infty} \sum_{i=1}^{N} \psi(y_{i,N}) w_{i,N} = \int_{Y} \psi(y) f(y)dy.
\]

(19)

To conclude we have seen that Gaussian quadrature is a very natural method for several reasons. The orthogonal polynomials on which quadrature is based form an orthonormal basis with respect to the density function for \(y\). The points and weights are selected in such a way that finite ordered polynomials can be exactly integrated using quadrature formulas. And, finally, the weights have a natural interpretation as the probabilities associated with intervals around the quadrature points.

### 3 Solving Models with Exogenous State Variables

#### 3.1 An Asset Pricing Example

To illustrate the manner in which models with strictly exogenous state variables can be solved using discrete state space methods, I will use an asset pricing example based on
Lucas (1978). Suppose there is an economy populated by \( N \) identical agents, each with instantaneous utility function

\[
U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}, \tag{20}
\]

where \( C_t \) is the agent’s consumption at time \( t \) and \( \gamma \) is the coefficient of relative risk aversion. Suppose that all output in this economy is obtained from \( K \) assets which produce stochastic endowments of a single perishable consumption good for each unit the agent owns at the beginning of time \( t \). That is, if the agent owns \( S_{kt} \) units of asset \( k \) at the beginning of time \( t \), he receives an endowment of \( S_{kt}D_{kt} \) units of the consumption good, where \( D_{kt} \) is identical for each unit of the \( k \)th asset held by an agent and is an exogenous stochastic process. These consumption goods can be consumed or traded for shares of the assets. If the price at date \( t \) of the \( k \)th asset in units of consumption is \( P_{kt} \), each agent’s budget constraint is given by

\[
C_t + \sum_{k=1}^{K} P_{kt}S_{kt+1} \leq \sum_{k=1}^{K} (P_{kt} + D_{kt})S_{kt}. \tag{21}
\]

Assuming that agents discount their expected streams of utility with the factor \( \beta \), at time 0 the agent maximizes

\[
E_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{22}
\]

by choosing contingency plans for \( C_t \) and \( \{S_{kt+1}\}_{k=1}^{K} \) subject to (21) for \( t = 0, 1, \ldots \). Substituting the budget constraint into the objective function we obtain

\[
E_0 \sum_{t=0}^{\infty} \beta^t U \left[ \sum_{k=1}^{K} (P_{kt} + D_{kt})S_{kt} - \sum_{k=1}^{K} P_{kt}S_{kt+1} \right]. \tag{23}
\]

The first order conditions for this problem are

\[
P_{kt}U'(C_t) = \beta E_t U'(C_{t+1})(P_{kt+1} + D_{kt+1}), \quad k = 1, \ldots, K. \tag{24}
\]

Since the agents are identical they will make the same decisions given the state of the world. As a result it is convenient to assume that the total supply of each asset is \( N \), so that \( S_{kt} = 1 \), for all \( k \) and \( t \), for every agent, in equilibrium. Then, the budget constraint implies that \( C_t = \sum_{k=1}^{K} D_{kt} \) for all \( t \). So we have the Euler equations

\[
P_{kt}U'(\sum_{k=1}^{K} D_{kt}) = \beta E_t U'(\sum_{k=1}^{K} D_{kt+1})(P_{kt+1} + D_{kt+1}), \quad k = 1, \ldots, K. \tag{25}
\]
3.2 The Case of a Single I.I.D. Shock

If \( K = 1 \), so that there is only one asset then we have one Euler equation for that asset which, dropping the \( k \) subscript, is

\[
P_t U'(D_t) = \beta E_t U'(D_{t+1})(P_{t+1} + D_{t+1}).
\]

With the assumption that utility is isoelastic, as in (20), this becomes

\[
P_t D_t^{-\gamma} = \beta E_t D_{t+1}^{-\gamma}(P_{t+1} + D_{t+1}). \tag{26}
\]

It is convenient to express this equation in terms of the price-dividend ratio \( V_t = P_t/D_t \).

\[
V_t D_t^{1-\gamma} = \beta E_t D_{t+1}^{1-\gamma}(V_{t+1} + 1), \tag{27}
\]

or

\[
V_t = \beta E_t X_{t+1}^{1-\gamma}(V_{t+1} + 1), \tag{28}
\]

where \( X_{t+1} = D_{t+1}/D_t \). Equation (28) implicitly defines a solution for the price-dividend ratio at time \( t \), as a function of those variables known at time \( t \) which are useful in forecasting functions of future values of dividend growth.

The special case we will focus on for the moment is when the logarithm of \( X_t \) is an i.i.d. normal random variable with mean \( \mu \) and variance \( \sigma^2 \). Altug and Labadie (1994, p.83) show that the solution for \( V_t \) in this case is

\[
V_t = \frac{\beta \exp\left(\alpha \mu + \frac{1}{2} \alpha^2 \sigma^2\right)}{1 - \beta \exp\left(\alpha \mu + \frac{1}{2} \alpha^2 \sigma^2\right)} \tag{29}
\]

as long as

\[
\beta \exp\left(\alpha \mu + \frac{1}{2} \alpha^2 \sigma^2\right) < 1, \tag{30}
\]

where \( \alpha = 1 - \gamma \).

Suppose we were unaware of this solution and tried to determine an approximate solution for \( V_t \), as a function of \( x_t = \ln (X_t) \), using a discrete state-space method. In other words, suppose we tried to find some function \( V \) such that

\[
V(x_t) = \beta E_t \exp(\alpha x_{t+1}) [V(x_{t+1}) + 1]
= \int \beta \exp(\alpha x_{t+1}) [V(x_{t+1}) + 1] f(x_{t+1}) dx_{t+1} \tag{31}
\]
where \( f \) is the density function for \( x_{t+1} \). Imagine that we tried to approximate the integral on the right-hand side using an \( N \)-point Gaussian quadrature rule. We would then have the equation

\[
V(x_t) \approx \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] w_{i,N}
\]

where the \( y_{i,N} \), and \( w_{i,N} \) are the points and weights corresponding to an \( N \)-point rule for a normal random variable with mean \( \mu \) and variance \( \sigma^2 \). Suppose we consider (32) for all \( x_t \in \{ y_{1,N}, y_{2,N}, \ldots, y_{N,N} \} \). We then have \( N \) equations

\[
V(y_{j,N}) \approx \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] w_{i,N}, \quad j = 1, \ldots, N.
\]

Now suppose we convert (33) into \( N \) strict equalities in the \( N \) unknowns \( V(y_{j,N}) \). Then we have

\[
V(y_{j,N}) = \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] w_{i,N}, \quad j = 1, \ldots, N.
\]

Since this is simply \( N \) linear equations in the \( N \) unknowns, \( V(y_{j,N}) \), it has a trivial solution.

The discrete state-space approximation interpretation of (34) is straightforward. Suppose that \( x_t \), rather than being normally distributed, had, in fact, an i.i.d. discrete distribution such that \( x_t = y_{j,N} \) with probability \( w_{j,N} \). Then (34) would provide the exact solution for \( V_t \) given that \( x_t = y_{j,N} \).

In this example, since nothing inside the sum depends on \( j \), it is clear that \( V(y_{j,N}) = V \), a constant, for all \( j \). This implies that the approximate solution for \( V_t \) is the constant

\[
V = \frac{\sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) w_{i,N}}{1 - \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) w_{i,N}}.
\]

The reader may verify numerically that even though both the true solution and the approximation to it are constants, they will not be equal, in general, for all \( N \).

### 3.3 The Case of Multiple I.I.D. Shocks

Now suppose that there are \( K > 1 \) assets. We can imagine that the law of motion of the endowments from the \( K \) assets is such that \( x_{1t} = \ln(D_{1t}) - \ln(D_{1t-1}) \) and \( x_{2t} = \ln(C_t) - \ln(C_{t-1}) \).

\(^{10}\)Suppose that the points and weights for an \( N \)-point rule for the standard normal are denoted \( \bar{y}_{i,N} \) and \( \bar{w}_{i,N} \) respectively. It turns out that when we generate an \( N \)-point rule for a normal random variable with arbitrary mean \( \mu \) and variance \( \sigma^2 \), the quadrature points are given by \( y_{i,N} = \mu + \sigma \bar{y}_{i,N} \) and the weights are \( w_{i,N} = \bar{w}_{i,N} \).
\[
\ln (C_{t-1}), \text{ where } C_t = \sum_{k=1}^{K} D_{kt}, \text{ is jointly normal. So, if we were interested in solving for the price-dividend ratio of the first asset, we might assume that } \begin{pmatrix} x_{1t} & x_{2t} \end{pmatrix}' \text{ is distributed as an i.i.d. normal random vector with mean } \mu \text{ and variance-covariance matrix } \Sigma. \text{ Defining } V_{1t} = P_{1t}/D_{1t}, \text{ and assuming isoelastic utility, we have}
\]

\[
V_{1t} = \beta E_t \exp (\alpha' x_{t+1}) (V_{1t+1} + 1), \tag{36}
\]

where \( \alpha = \begin{pmatrix} 1 & -\gamma \end{pmatrix}' \).

It is, again, possible to show that the price-dividend ratio is a constant:

\[
V_{1t} = \frac{\beta \exp (\alpha' \mu + \frac{1}{2} \alpha' \Sigma \alpha)}{1 - \beta \exp (\alpha' \mu + \frac{1}{2} \alpha' \Sigma \alpha)} \tag{37}
\]

as long as

\[
\beta \exp \left( \alpha' \mu + \frac{1}{2} \alpha' \Sigma \alpha \right) < 1. \tag{38}
\]

If we were unaware of this solution and tried to determine an approximate solution for \( V_{1t} \), as a function of \( x_t \), using a discrete state-space method, we would try to find some function \( V_1 \) such that

\[
V_1 (x_t) = \beta E_t \exp (\alpha' x_{t+1}) [V_1 (x_{t+1}) + 1]
\]

\[
= \int \beta \exp (\alpha' x_{t+1}) [V_1 (x_{t+1}) + 1] f (x_{t+1}) dx_{t+1} \tag{39}
\]

where \( f \) is the density function for the vector \( x_{t+1} \).

The question, in this case, is how to set up the quadrature rule when the state vector \( x_t \) is multidimensional. If the elements of \( x_t \) were uncorrelated, we might guess that a natural way to set up the quadrature rule would be to use separate grids of points for each dimension and simply use all possible combinations of these points in computing the relevant sum. The weights would be the cross products of the weights for the individual grids. Let the univariate \( N \)-point quadrature rule for the standard normal be given by the points \( \{\bar{y}_{i,N}\}_{i=1}^{N} \) and the weights \( \{w_{i,N}\}_{i=1}^{N} \). Suppose we set up a quadrature rule for an \( M \times 1 \) vector \( x_t \), distributed normally with mean 0 and variance-covariance matrix, \( I_M \). This rule might use \( N_m \) points for the \( m \)th element of \( x_t, x_{mt} \). The most likely choice would be to have \( N_m = N \) for all \( m \). In any case, it would be natural to set up a rule such that

\[
y_{i,N} = \begin{pmatrix} \bar{y}_{j_1,N_1} & \bar{y}_{j_2,N_2} & \cdots & \bar{y}_{j_M,N_M} \end{pmatrix}' \tag{40}
\]

and

\[
w_{i,N} = \bar{w}_{j_1,N_1} \bar{w}_{j_2,N_2} \cdots \bar{w}_{j_M,N_M}, \tag{41}
\]
where
\[ j_m = 1, \ldots, N_m, \] (42)
\[ i = (j_1 - 1) N_2 \cdots N_M + (j_2 - 1) N_3 \cdots N_M + \cdots + (j_{M-1} - 1) N_M + j_M, \] (43)
\[ N = N_1 N_2 \cdots N_M. \] (44)

The rule for \( i \) is just a way of indexing the possible states using a single integer.

Now suppose the mean of \( x_t \) is \( \mu \) while its covariance matrix is given by \( \Sigma \). Notice that \( \xi_t = C'^{-1} (x_t - \mu) \) is distributed \( N (0, I_M) \), when \( C \) is the Cholesky decomposition of \( \Sigma \). That is \( C' C = \Sigma \). Since this means we can write \( x_t = \mu + C' \xi_t \), it is natural to consider as a quadrature rule
\[ y_{i,N} = \mu + C' \left( \begin{array}{c} \bar{y}_{j_1,N_1} \\ \bar{y}_{j_2,N_2} \\ \vdots \\ \bar{y}_{j_M,N_M} \end{array} \right)' \] (45)
and
\[ w_{i,N} = \bar{w}_{j_1,N_1} \bar{w}_{j_2,N_2} \cdots \bar{w}_{j_M,N_M}. \] (46)

This means that our approximate solution to (39) based on quadrature will be
\[ V_1 (y_{j,N}) = \sum_{i=1}^{N} \beta \exp (\alpha' y_{i,N}) [V_1 (y_{i,N}) + 1] w_{i,N}, \quad j = 1, \ldots, N, \] (47)
which is a system of \( N \) linear equations in \( N \) unknowns. Again, because nothing on the right-hand side is dependent on \( j \) we get a constant approximate solution for \( V_{1t} \) given by
\[ V_1 = \frac{\sum_{i=1}^{N} \beta \exp (\alpha' y_{i,N}) w_{i,N}}{1 - \sum_{i=1}^{N} \beta \exp (\alpha' y_{i,N}) w_{i,N}}. \] (48)

What is interesting about the example with correlated shocks is that if we give it the discrete state-space interpretation, the state-space for the variable \( x_{jt} \) will be different depending on the values of \( x_{1t} \) through \( x_{jt-1} \). This is best illustrated for the case of two shocks which are positively correlated, as in Figure 1. The matrix \( C \) applies a trapezoidal transformation to the rectangular grid implied if the elements of \( x_t \) are uncorrelated.

### 3.4 The Case of Serially Correlated Shocks

More complexity is introduced if the growth rates of the endowments are serially correlated. To avoid burdensome notation I will return to the univariate example, and suppose that there is a single asset whose endowment behaves according to
\[ x_t = \mu (1 - \rho) + \rho x_{t-1} + \epsilon_t, \] (49)
where $|\rho| < 1$ and $\epsilon_t$ is an i.i.d. $N(0, \sigma^2)$ process.

The Euler equation in this case is

$$V_t = \beta E_t \exp(\alpha x_{t+1}) (V_{t+1} + 1), \quad (50)$$

where $\alpha = 1 - \gamma$. It has a solution derived in Burnside (1997) given by

$$V_t = \sum_{i=1}^{\infty} \beta^i \exp \left\{ \alpha \left[ \mu \left[ i - \frac{\rho(1-\rho^i)}{1-\rho} \right] + \frac{\rho(1-\rho^i)}{1-\rho} x_t \right] + \frac{1}{2} \frac{\alpha^2}{(1-\rho)^2} \left[ i - 2 \rho(1-\rho^i) + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2} \right] \sigma^2 \right\}, \quad (51)$$

as long as

$$\beta \exp \left( \frac{\alpha \mu + \frac{1}{2} \frac{\alpha^2}{(1-\rho)^2} \sigma^2}{1-\rho} \right) < 1. \quad (52)$$

While this is a solution, it is not a particularly useful one for at least two reasons. First, it is an infinite series of expressions rather than a single expression, so it must be recursively calculated for any specific value of $x_t$. Secondly, the series does not converge rapidly for some parameter values. This is easily verified for a trivial special case of log utility ($\gamma = 1$) in which case $\alpha = 0$. This yields $V_t = \sum_{i=1}^{\infty} \beta^i$. For a value of $\beta$ close to 1, say $\beta = 0.99$, it takes many terms in the series before $v_t$ approaches its true value of $\beta/(1-\beta) = 99$. For example, well over 600 terms must be added in order for the sum to be within 0.1 of 99, and over 900 terms to be within 0.01.

So, although the model has an exact solution which can be written in an analytical form, this form may not be useful in all circumstances, as it is an infinite series which will converge slowly for typical parameterizations. Furthermore, if the $x_t$ process is something more general than a Gaussian process, there may not even be an analytical form for each term in the series.

We have see, from equation (51), that the solution for $V_t$ is of the form $V(x_t)$. If we tried to obtain an approximation to this using a discrete state-space method, we would start from the Euler equation written as an integral equation the

$$V(x_t) = \int \beta \exp(\alpha x_{t+1}) [V(x_{t+1}) + 1] f(x_{t+1}|x_t) dx_{t+1}. \quad (53)$$

This equation differs significantly from the equation we had in the i.i.d. case because the density for $x_{t+1}$ is conditional on the value for $x_t$. This implies that if we used quadrature to approximate the integral in (53), the quadrature rule would need to be different for each $x_t$. 
To get around this problem Tauchen and Hussey (1991) suggest the following transformation. Notice that (53) is equivalent to

$$ V(x_t) = \int \beta \exp(\alpha x_{t+1}) [V(x_{t+1}) + 1] \frac{f(x_{t+1}|x_t)}{f(x_{t+1}|\mu)} f(x_{t+1}|\mu) \, dx_{t+1}. $$

(54)

Now, using an $N$-point rule based on the density function $f(x_{t+1}|\mu)$ we could approximate (54) by

$$ V(x_t) \approx \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] \frac{f(y_{i,N}|x_t)}{f(y_{i,N}|\mu)} w_{i,N}, $$

(55)

where the quadrature points are $y_{i,N} = \mu + \sigma \tilde{y}_{i,N}$, and the weights are $w_{i,N} = \tilde{w}_{i,N}$ with $\tilde{y}_{i,N}$ and $\tilde{w}_{i,N}$ defined as before.

If we imposed equality in (55) for $x_t \in \{y_{i,N}\}_{i=1}^{N}$ we would have $N$ linear equations in $N$ unknowns given by

$$ V(y_{j,N}) = \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] \frac{f(y_{i,N}|y_{j,N})}{f(y_{i,N}|\mu)} w_{i,N}. $$

(56)

In this case the solution is not a constant, because the term in the sum do depend on $j$.

Burnside (1993) derives a number of theoretical properties of the solution generated by (56). Primary among these is that, subject to a minor modification of the quadrature rule, if the solution in (56) is extended to the real line by using step functions over the intervals $(z_{i-1,N}, z_{i,N}) = (\mu + \sigma \tilde{z}_{i-1,N}, \mu + \sigma \tilde{z}_{i,N})$ then it can be shown to converge, as $N \to \infty$ in $L^2$ to the true solution to (53).

Unfortunately, the approximation in (56) does not have a direct interpretation as the exact representation of the Euler equation when $x_t$ is a discrete state-space process. To see this, suppose that $x_t$ was a simple first-order Markov process, with a discrete state-space $\{y_{i,N}\}_{i=1}^{N}$. Suppose that the probability that $x_{t+1} = y_{i,N}$ given that $x_t = y_{j,N}$ was given by $\pi_{ji}$. The exact representation of the Euler equation for this model would be

$$ V(y_{j,N}) = \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] \pi_{ji}. $$

(57)

This closely resembles (56) if we think of the terms $[f(y_{i,N}|y_{j,N})/f(y_{i,N}|\mu)] w_{i,N}$ as equivalent to $\pi_{ji}$. Since the $\pi_{ji}$ terms are conditional probabilities, $\sum_i \pi_{ji} = 1$. However, in general, $\sum_i [f(y_{i,N}|y_{j,N})/f(y_{i,N}|\mu)] w_{i,N} \neq 1$. For this reason, Tauchen and Hussey (1991) suggest solving

$$ V(y_{j,N}) = \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] \frac{f(y_{i,N}|y_{j,N}) w_{i,N}}{f(y_{i,N}|\mu)} \frac{1}{s_j}. $$

(58)
where
\[ s_j = \sum_{i=1}^{N} \frac{f(y_i,N|y_j,N)}{f(y_i,N|\mu)} w_{i,N}. \] (59)

The approximate solution generated by (56) is the same as the exact solution for the price-dividend ratio in a model where \( x_t \) is a first-order Markov process with a discrete state-space \( \{y_i,N\}_{i=1}^{N} \) and transition probabilities given by
\[ \pi_{ji} = \frac{f(y_i,N|y_j,N) w_{i,N}}{s_j}. \] (60)

It can be shown that \( \lim_{N \to \infty} s_j = 1 \) for all \( j \) so that the solutions to (56) and (58) converge to the same limit.

In the final section of the paper I discuss extensions and limitations to the use of quadrature as the basis of a numerical solution method.

4 Solving Models with Endogenous State Variables

4.1 A One-Sector Neoclassical Growth Model

To illustrate how discrete state-space methods may be applied to problems with endogenous state variables, I will take as an example a standard one-sector neoclassical growth model. Suppose that there is an economy populated by \( N \) identical agents who supply one unit of labor inelastically and have logarithmic preferences over the single consumption good. Output at date \( t \), \( Y_t \) is given by \( Y_t = A_t K_t^\theta \) where \( A_t \) is a stationary stochastic shock to the level of technology and \( K_t \) is the capital stock at the beginning of date \( t \). The resource constraint for the economy is
\[ C_t + K_{t+1} - (1 - \delta)K_t \leq Y_t \] (61)

where \( \delta \) is the depreciation rate of capital. A competitive equilibrium for a decentralized version of this economy is equivalent to the social planning problem in which the planner maximizes
\[ E_0 \sum_{t=0}^{\infty} \beta^t \ln(C_t) \] (62)

by choosing contingency plans for \( C_t \) and \( K_{t+1} \) subject to the resource constraint and \( K_0 \).

As in the asset pricing example, it is convenient to rewrite the problem by substituting the resource constraint into the planner’s objective to obtain
\[ E_0 \sum_{t=0}^{\infty} \beta^t \ln \left[ A_t K_t^\theta + (1 - \delta)K_t - K_{t+1} \right]. \] (63)
The Euler equation for this problem is

\[
\frac{1}{A_t K_t^\theta + (1 - \delta) K_t - K_{t+1}} = \beta E_t \frac{\theta A_{t+1} K_{t+1}^{\theta - 1} + (1 - \delta)}{A_{t+1} K_{t+1}^\theta + (1 - \delta) K_{t+1} - K_{t+2}}.
\] (64)

Unlike the asset pricing example, this equation is not linear in the variable we wish to solve for, in this case \(K_{t+1}\), as a function of the state variables at time \(t\), \(A_t\) and \(K_t\). Furthermore, one of the state variables, \(K_t\) is endogenously determined over time. As a result of these two features of the model, for many assumptions about the distribution of the shocks to technology, \(A_t\), there will be no analytic form for the optimal level of \(K_{t+1}\) in terms of the relevant state variables. One special case for which a closed form is well known, is the case where \(\delta = 1\), so that capital depreciates completely from period to period. In this case the solution is \(K_{t+1} = \beta \theta A_t K_t^\theta\).\(^{11}\)

In practice, we will assume that \(a_t = \ln(A_t)\) is a Gaussian AR(1) process with the law of motion

\[
a_t = \rho a_{t-1} + \epsilon_t,
\] (65)

where \(\epsilon_t\) is i.i.d. \(N(0, \sigma^2)\). The zero mean assumption for \(a_t\) is for convenience and will play no important role in the solution of the model.

### 4.2 Discrete State-Spaces and the Growth Model

Tauchen’s (1990) method for solving the growth model is perhaps the most straightforward given our discussion in the section on exogenous state variables. Rather than working with the Euler equation for the growth model, Tauchen works with the Bellman equation for the dynamic programming problem associated with it. This is given by

\[
V(K, a) = \max_{K' \in \Gamma(K, a) \subset \mathcal{K}_C} r(K, K'; a) + \beta \int V(K', a') f(a'|a) \, da',
\]

where \(r(K, K'; a) = \ln \left[ e^{a} K^\theta + (1 - \delta) K - K' \right], \Gamma(K, a) = \{ K' | 0 \leq K' \leq e^{a} K^\theta + (1 - \delta) K \}\), and the set \(\mathcal{K}_C\) is the continuous state space for \(K\). Also

\[
f(a'|a) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[ -\frac{1}{2\sigma^2} (a_t - \rho a_{t-1})^2 \right],
\]
given the assumption of normality, made above.

The assumption of normality is somewhat problematic, as it means there is no upper bound on \(a\), and consequently no upper bound on the capital stock. If, on the other hand,

\(^{11}\text{See Sargent (1987b).}\)
there were an upper bound on $a$, denoted $\bar{a}$, then we could derive an upper bound on the capital stock:

$$\bar{K} = (e^{\bar{a}/\delta})^{1/(1-\theta)},$$

(66)

The capital stock could never exceed this number because this is the capital stock that would result, in the limit, from zero consumption in every period, and the good luck of realizing $a_t = \bar{a}$ for all $t$. With this result we would be able to define $\mathcal{K}_C = [0, \bar{K}]$. But with the assumption of normality, $\bar{K}$ is, effectively, infinite.

To convert the dynamic programming problem in (??) to one involving discrete state-spaces, Tauchen proposes the following procedure. First, approximate the law of motion of $a$ using a discrete state-space process defined in exactly the same way as in section 3.4. That is, redefine $a$ to be a process which lies in a set $\mathcal{A} = \{a_i\}_{i=1}^N$ where $a_i = \sigma \bar{y}_{i,N}$ and $\{\bar{y}_{i,N}\}_{i=1}^N$, as before, is the set of quadrature points corresponding to an $N$-point rule for a standard normal. Let the probability that $a' = a_i$ given that $a = a_j$ be given by

$$\pi_{ji} = \frac{f(a_i|a_j) \bar{w}_{i,N}}{f(a_i|0) s_j}$$

(67)

where

$$s_j = \sum_{i=1}^{N} \frac{f(a_i|a_j)}{f(a_i|0)} \bar{w}_{i,N}$$

(68)

and $\{\bar{w}_{i,N}\}_{i=1}^N$ are the quadrature weights for an $N$-point rule for the standard normal. After this first step in the approximation we might imagine a variation on the Bellman equation given by

$$\tilde{V}(K, a_j) = \max_{K' \in \Gamma(K, a_j)} r(K, K'; a_j) + \beta \sum_{i=1}^{N} \tilde{V}(K', a_i) \pi_{ji},$$

(69)

for $j = 1, 2, \ldots, N$.

In the next step, which I will describe below, the set $\mathcal{K}_C$ will be replaced by a discrete set $\mathcal{K}_D \subset \mathcal{K}_C$. That is, we will define a set $\mathcal{K}_D = \{K_m\}_{m=1}^M$ from which the social planner must choose the optimal level of capital in the next period, $K'$. In other words we will approximate the original dynamic programming problem with the Bellman equation

$$\hat{V}(K, a_j) = \max_{K' \in \Gamma_D(K, a_j)} r(K, K'; a_j) + \beta \sum_{i=1}^{N} \hat{V}(K', a_i) \pi_{ji},$$

(70)

for $j = 1, 2, \ldots, N$, and with

$$\Gamma_D(K, a) = \mathcal{K}_D \cap \Gamma(K, a).$$

(71)
Ideally, we would have a rule for constructing the discrete set $K_D$ which would resemble the quadrature rule for constructing the set $A$. Unfortunately, without knowledge of the solution to the model, our knowledge of the solution for $K$ will be limited. We will, therefore, not have the information necessary to construct a Gaussian quadrature-based discrete state-space for $K$.

A number of alternative rules for constructing $K_D$ are available. One is to locate a grid of evenly spaced points, $K_m$ in the set $K_C$. Another rule, proposed by Tauchen (1990), is to assume linear utility and compute the mean and standard deviation of the capital stock implied by the solution to that example. Then, he suggests using an equal spaced grid in a $\pm 4$ standard deviation band around the mean. Another possibility is to set up a grid in the logarithm of the capital stock based on the mean and standard deviation of the capital stock implied by the log-linear approximation to the model.

Results in Bertsekas (1976) imply that there will be unique bounded solutions to (69) and (70). In fact, there will be a well-defined sense in which the solution to (70) can be made arbitrarily close to the solution to (69), as long as in the limit, as $M \to \infty$, with $K_1 = 0$, $K_M = \bar{K}$, $K_m > K_{m-1}$, $\lim_{M \to \infty} \sup_m (K_m - K_{m-1}) = 0$.

In order to implement (70) on a computer, it is helpful to rely on standard results from dynamic programming. Suppose we guessed that the value function $\hat{V}$ which solves (70) takes some particular form $\hat{V}_0$. If we used this on the right hand side of the Bellman equation and performed the necessary maximization over $K'$ we would, unless our guess was correct, obtain a new function $\hat{V}_1$, on the left-hand side of the equation. This process could then proceed iteratively as

$$\hat{V}_S(K_m, a_j) = \max_{K' \in \Gamma_D(K_m, a_j)} r(K_m, K'; a_j) + \beta \sum_{i=1}^{N} \hat{V}_{S-1}(K', a_i) \pi_{ji},$$

for $m = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. Given a choice for $\hat{V}_0$ the process could continue until $S$ became sufficiently large that $\sup_{m,j} |\hat{V}_S(K_m, a_j) - \hat{V}_{S-1}(K_m, a_j)|$ was less than some tolerance. In fact, Tauchen suggests scaling the tolerance to the minimum value of the value function $\inf_{m,j} |\hat{V}_S(K_m, a_j)|$.

That convergence will be obtained is a standard result in dynamic programming, discussed further in this volume by Diaz-Jiminez. Furthermore, as a by-product of iterating over the value function the optimal decision rule for capital is obtained. That is, we obtain a rule for setting the capital stock which defines $K' = \hat{h}(K, a)$. 


Baxter, Crucini and Rouwenhorst (1990) use a discrete state-space method which directly approximates the decision rules as opposed to the value function. In this respect, their method is symmetric to the methods we described for solving the Euler equations in asset pricing problems. The basis of their method is the observation that for the true model, the Euler equation for capital is given by

$$\frac{1}{g(K, a) - h(K, a)} = \beta \int \frac{g_K[h(K, a), a']}{g[h(K, a), a'] - h[h(K, a), a']} f(a'|a) da', \quad (73)$$

where $g(K, a) = e^{aK^\theta} + (1 - \delta)K$, $g_K(K, a) = \theta e^{aK^{\theta-1}} + 1 - \delta$ and $h(K, a)$ is the optimal policy rule for $K_0$.

As the first step in their algorithm, they set up discrete state-spaces for the capital stock and the technology shock in the same way we did for the dynamic programming problem. Assuming that one has a candidate decision rule $h_S : K_D \times A \rightarrow K_D$ the next decision rule is obtained by finding the function $h_{S+1} : K_D \times A \rightarrow K_D$ that best fits the Euler equation. I.e. for each $K_m$ and $a_j$ the method chooses $K' \in \Gamma_D(K_m, a_j)$ to minimize the difference between

$$\frac{1}{g(K, a) - K'} \quad (74)$$

and

$$\beta \sum_{i=1}^{N} \frac{g_K(K', a_i)}{g(K', a_i) - h_S(K', a_i)} \pi_{ji}. \quad (75)$$

This process implicitly defines a new decision rule $h_{S+1}(K, a)$.

The algorithm could be stopped if $h_{S+1}(K, a) = h_S(K, a)$, which frequently occurs in practice, or if the maximum change in the decision rule eventually satisfied some tolerance for the difference between (74) and (75).

Unfortunately, there is little known about the convergence properties of this algorithm. Although it is perfectly reasonable on intuitive grounds, it is not justified by the same fixed point arguments that justify the dynamic programming approach suggested by Tauchen (1990).

5 Extensions, Limitations and Concluding Remarks

When we obtain a discrete state-space approximation to a model, we may be satisfied with examining the properties of the approximating model. However, in some cases our main interest is in the original continuous state-space model. In all cases there is a trivial extension
of the discrete state-space solution to the continuous state-space that involves using step functions. This way of extending the solution is perhaps best illustrated using the univariate asset pricing example, where the discrete state-space method gave us a solution \( V(y_{j,N}) \) \( j = 1, \ldots, N \), at \( N \) points on the real line. However, recall that the quadrature rule divided the real line into nonoverlapping segments according to (15), \( (z_{j-1,N}, z_j,N) \) such that \( y_{j,N} \in (z_{j-1,N}, z_j,N) \). In this case we can extend the discrete state-space solution to the real line by letting

\[
V(x) = \sum_{j=1}^{N} V(y_{j,N}) 1_{(z_{j-1,N}, z_j,N)}(x),
\]

where

\[
1_{(z_{j-1,N}, z_j,N)}(x) = \begin{cases} 
1 & \text{if} \ x \in (z_{j-1,N}, z_j,N) \\
0 & \text{otherwise}.
\end{cases}
\]

Similar trivial extensions are available for any of the approximate solutions we have examined in this paper.

To avoid the discontinuities implied by these sorts of extensions, another trivial form of extension involves using the step function in the outer limits of the state-space, while using sloping functions to connecting the inner quadrature points in a continuous way. For example, in the asset pricing example we could define

\[
V(x) = \begin{cases} 
V(y_{1,N}) & \text{if} \ x \leq y_{1,N} \\
V(y_{j-1,N}) + \frac{(x-y_{j-1,N})}{(y_{j,N}-y_{j-1,N})} [V(y_{j,N}) - V(y_{j-1,N})] & \text{if} \ y_{j-1,N} < x \leq y_{j,N} \\
V(y_{N,N}) & \text{if} \ x > y_{N,N}.
\end{cases}
\]

Finally, in the asset pricing context, there is the Nyström extension. This involves substituting the solution to the discrete state-space model, \( \{V(y_{i,N})\}_{i=1}^{N} \), into the approximate Euler equation (55), for any \( x \),

\[
V(x) \approx \sum_{i=1}^{N} \beta \exp(\alpha y_{i,N}) [V(y_{i,N}) + 1] \frac{f(y_{i,N}|x)}{f(y_{i,N}|\mu)} w_{i,N}.
\]

Notice that this provides not only a continuous, but a differentiable, function of \( x \), as an approximate solution.

One of the major limitations of discrete state-space methods is their computational expense. In the section on asset pricing models with multivariate shocks, we illustrated an example in which, had the shocks been serially correlated, we would have had to obtain the solution to the model by solving \( N \) linear equations in \( N \) unknowns. But \( N \) was equal to \( N_1 N_2 \cdots N_M \) where \( M \) was the number of variables in the state space, and \( N_m \) represented
the number of points in the quadrature grid for each variable. Suppose, for illustration, that $N_m = n$ for all $m$, so that $N = n^M$. In solving the equations this means we need to store at least one matrix with dimension $n^{2M}$. Secondly, and more importantly, computation time in standard methods for solving linear equations is approximately proportional to $N^3$, or in our case $n^{3M}$. So as either the number of state variables, $M$, or the number of quadrature points, $n$, increases, the computation involved rapidly becomes more difficult. This has been called the *curse of dimensionality*.

Related to this is the problem of choosing the grid of points for endogenous state variables, such as in our growth model. It turns out, from examples examined in the literature, that very fine grids of points appear to be needed to obtain any satisfactory level of accuracy from the discrete state-space solutions. This adds to the dimensionality problem by exaggerating the need for a large number of grid points. This also stands in contrast to examples based on the asset pricing model, where small numbers of points seem to be adequate for obtaining quite accurate approximations based on the Nyström extension.

In conclusion, the main advantages of discrete state-space methods are threefold. First, they are easy to implement and are well grounded in intuition. Second, there are numerous theoretical results regarding convergence, described in more detail in references given here, which justify their use. And third, they tend to be rather reliable in providing plausible approximations to the solutions of a wide variety of dynamic models. The main disadvantages of these methods are twofold. First, they can be computationally expensive for relatively complicated problems with a large number of state variables. Second, they generally require the state variables to be stationary so that a process defined on a fixed discrete state-space can adequately describe their laws of motion.

6 Software

There are several m-files for use with MATLAB, associated with this chapter. The first of these is apiid.m which solves the asset pricing model for the single shock i.i.d. case.

The second is apsco.m which solves the model for serially correlated single shock case, and plots the exact solution, the step function extension and the Nyström extension.

The third file, grtauch.m, solves a version of the one sector growth model using the method proposed by Tauchen. While the fourth file, grbcr.m, solves the same model using the method proposed by Baxter, Crucini and Rouwenhorst.
A number of other files which contain procedures necessary to the functioning of the above, all of which are documented with extensive comment lines, are included as well. These include condnorm.m, a procedure for evaluating the conditional normal density function, getprice.m, a procedure for obtaining price-dividend ratios in the asset pricing example, gettrans.m, a procedure for obtaining a probability transition matrix that approximates the law of motion of a serially correlated normal random variable, and ghquad.dat, a table of quadrature points, weights and interval endpoints.

The example programs are implemented in what I would call a “naive” way, to make them as clear as possible. This means that they do not take advantage of many “tricks” for speeding up computation. However, they do serve as useful benchmarks for computation time.

7 References


Note: The grey points are the 9 quadrature points that result from an example where $M = 2$, $N_1 = N_2 = 3$, and $x_t \sim N(\mu, I_2)$. The black points are for the case where $x_t \sim N(\mu, \Sigma)$ where $\Sigma_{11} = \Sigma_{22} = 1$, but $\Sigma_{12} = \Sigma_{21} = 0.5$. 