Prospective Deficits and the Asian Currency Crisis

Technical Appendix on Numerical Computation of the Equilibrium

Craig Burnside*, Martin Eichenbaum†, and Sergio Rebelo‡

April 2001

The algorithm we use in the paper works as follows. We compute the equilibrium of the model numerically. We do this by iterating over the values of 3 variables in a system of nested do-loops. The other variables are determined analytically or using numerical integration given the values of the variables we are iterating over.

The outer loop of our algorithm is over the value of μ, the money growth rate for $t \geq T$. The middle do loop in the algorithm is over the value of $\bar{c}$, the level of consumption for $0 < t < t^*$. The inner do loop is over the value of t^*, the time of the speculative attack. I will describe each of these do loops in reverse order.

We treat some variables as fixed parameters. I will list these as follows with their values in our benchmark run stated as well: $y = 1$ (the level of output), $S = 1$ (the fixed exchange rate), $\Psi = -0.034$ (the parameter of the threshold rule), $f_0 = 0.091$ (government's net foreign assets), $b_0 = 0$ (government's net dollar denominated domestic debt), $d_0 = -0.449$ (household's net foreign assets), $r = 0.037$ (the real interest rate), $B = 0.052$ (the face value of outstanding consols at time 0), g (real government purchases, endogenously determined below), $v = 0$, (steady state real transfers), $\tau = 0.438$ (real taxes–this was set so that equilibrium c/y would look something like Thailand-it doesn't match τ/y for Thailand), to economize on notation I will often use the notation $\delta = g + v - \tau$ (endogenous–see below), M (the initial money supply, endogenous see below), T (the date at which new monetary policy is implemented), in the programs we parameterize M_T as Up$\times M$, where we set Up$= 1.211$ (the jump in the money supply at date T relative to steady state M).

*The World Bank
†Northwestern University, NBER and Federal Reserve of Chicago.
‡Northwestern University and NBER.

1 Initial Calculations

Calculate steady state consumption

$$c = r(b_0 + d_0 + B/S) + y + v - \tau$$

Given a choice for initial S this implies that initial M is $M = cS$. From this you can compute $M_T = \text{Up} \times M$. The steady state government budget constraint requires that

$$\delta = -r(b_0 + \frac{B}{S} - f_0).$$

Given that we've chosen v, and τ, a solution for δ implies a solution for $g = \delta - v + \tau$, though it plays no role in later calculations. I now turn to a description of the innermost of the 3 loops.

2 Solving for t^* conditional on μ and $\bar{c}$

For given values of μ and $\bar{c}$, we have (from the cash-in-advance constraint)

$$\bar{M} = S\bar{c}, \tag{1}$$

where $\bar{M}$ is the nominal money supply for $0 < t < t^*$. Also we have

$$\underline{c} = \bar{c}\left(\frac{1+r}{1+r+\mu}\right)^{1/\sigma} \tag{2}$$

where $\underline{c}$ is the level of consumption for $t \geq T$. The cash-in-advance constraint implies that

$$P_T = M_T/\underline{c}. \tag{3}$$

1. The loop over t^* initializes with a guess, say $t^* = T$.

2. Notice that $b_{t^*} + B/S - f_{t^*} = \Psi$, given the definitions of t^* and Ψ. The expression for $b_{t^*} + B/S - f_{t^*}$ can be derived by integrating the budget constraint from 0 to t^*: it states that

$$\begin{aligned} b_{t^*} + B/S - f_{t^*} &= e^{rt^*}(b_0 + \frac{B}{S} - f_0) + \int_0^{t^*} \delta e^{-r(t-t^*)}dt + \\ &\quad \frac{M - \bar{M}}{S}e^{rt^*} + \frac{\bar{M} - M^*}{S} \end{aligned} \tag{4}$$

Using the fact that Ψ is equal to the left hand side you can write this as

$$\Psi = e^{rt^*}(b_0 + \frac{B}{S} - f_0) + \int_0^{t^*} \delta e^{-r(t-t^*)}dt + \frac{M - \bar{M}}{S}e^{rt^*} + \frac{\bar{M} - M^*}{S} \tag{5}$$

There is a typo in the paper because the paper writes the integral $\int_0^{t^*} \delta e^{-r(t-t^*)} dt$ as $\int_0^{t^*} \delta e^{rt} dt$. Because δ is constant both integrals equal $\delta \left(e^{rt^*} - 1\right)/r$, but strictly speaking the one in the paper needs to be rewritten.

Now solve (5)for M^*, the nominal money supply for $t^* \leq t < T$:

$$M^* = -S\Psi + Se^{rt^*}(b_0 + \frac{B}{S} - f_0) + S\frac{g+v-\tau}{r}(e^{rt^*} - 1) + (M - \bar{M})e^{rt^*} + \bar{M}. \qquad (6)$$

This is possible, because everything on the right hand side is either a parameter or we've already solved for it.

3. Recall from the paper that P_t is governed by a differential equation $\dot{P}_t = aP_t^{1+\sigma} - bP_t$, for $t^* < t < T$, where

$$\begin{aligned} a &= (M^*)^{-\sigma}(1+r)\bar{c}^{\sigma} \\ b &= 1 + r. \end{aligned}$$

Notice that we can now evaluate a and b.

4. The paper argues that by continuity, $P_{t^*} = S$. Since the solution of the differential equation is

$$P_t = \left[\frac{b}{a - e^{(t-\gamma)\sigma b}}\right]^{1/\sigma}$$

this condition implies

$$\gamma = t^* - \frac{1}{\sigma b}\ln(a - bS^{-\sigma}).$$

Hence

$$P_t = \left[\frac{b}{a - e^{(t-t^*)\sigma b}(a - bS^{-\sigma})}\right]^{1/\sigma}. \qquad (7)$$

But continuity of the price level also requires that

$$P_T = \left[\frac{b}{a - e^{(T-t^*)\sigma b}(a - bS^{-\sigma})}\right]^{1/\sigma}.$$

Given that we already have a solution for P_T this equation can be rearranged and solved for t^*:

$$t^* = T - \frac{1}{\sigma b}\ln\left(\frac{a - bP_T^{-\sigma}}{a - bS^{-\sigma}}\right). \qquad (8)$$

This is our new guess for t^* and we go back to step 2. When our guess stops changing, we stop iterating over t^*.

3 Solving for $\bar{c}$ conditional on μ

Above we showed how to solve for $\bar{M}$, t^*, $\underline{c}$, P_T, t^*, M^*, a and b, given μ and $\bar{c}$. In this section I show how the middle loop over $\bar{c}$ works. We use bisection in this loop. We (hopefully) bracket the solution between two values $\bar{c}_l$ and $\bar{c}_h$, then make an initialize the loop by guessing that

$$\bar{c} = \frac{1}{2}\bar{c}_l + \frac{1}{2}\bar{c}_h.$$

Conditional on this guess we solve, as above, for or $\bar{M}$, t^*, $\underline{c}$, P_T, t^*, M^*, a and b. Then we define a variable $\tilde{c}_T = \lim_{t\uparrow T} c_t$. (This is my notation for the limit from the left.) Notice that since $M_t = M^*$ for $t^* \le t < T$ and because the price level is a continuous function of time at T, we have

$$\tilde{c}_T = M^*/P_T, \tag{9}$$

from the cash-in-advance constraint. Of course, it's also true that $\tilde{c}_T = \lim_{t\uparrow T} m_t$.

What we want to do next is evaluate the household's lifetime budget constraint given our guess, $\bar{c}$. Directly from the paper, the household's lifetime budget constraint is given by

$$b_0 + d_0 + \int_0^\infty e^{-rt}(y + v_t + rB/S_t)dt = \int_0^\infty e^{-rt}(c_t + \tau + \pi_t m_t + \dot{m}_t)dt + \sum_{i\in I} e^{-ri}\Delta m_i. \tag{10}$$

We rearrange this so that it is

$$b_0 + d_0 + \int_0^\infty e^{-rt}(y + v_t - \tau)dt = \int_0^\infty e^{-rt}(c_t - rB/S_t + \pi_t m_t + \dot{m}_t)dt + \sum_{i\in I} e^{-ri}\Delta m_i. \tag{11}$$

Given what we know, so far, about the solution, we can simplify this considerably. We have $\int_0^\infty e^{-rt} y dt = y/r$, $\int_0^\infty e^{-rt} v_t dt = v/r + \phi$, and $\int_0^\infty e^{-rt}\tau dt = \tau/r$. Hence the lefthand side of (11) is given by

$$LHS = b_0 + d_0 + \frac{y + v - \tau}{r} + \phi \tag{12}$$

We break the right hand side into 4 pieces. Consumption spending:

$$\begin{aligned} RHS_1 &= \int_0^\infty e^{-rt} c_t dt \\ &= \int_0^{t^*} \bar{c}e^{-rt}dt + \int_{t^*}^T c_t e^{-rt}dt + \int_T^\infty \underline{c}e^{-rt}dt \\ &= \bar{c}\frac{1}{r}\left(1 - e^{-rt^*}\right) + M^*\int_{t^*}^T P_t^{-1}e^{-rt}dt + \underline{c}\frac{1}{r}e^{-rT} \end{aligned} \tag{13}$$

Flow seignorage:

$$\begin{aligned} RHS_2 &= \int_0^\infty e^{-rt}(\pi_t m_t + \dot{m}_t)dt = \int_0^\infty e^{-rt}(\dot{M}_t/S_t)dt \\ &= \int_0^{t^*} e^{-rt}(\dot{M}_t/S_t)dt + \int_{t^*}^T e^{-rt}(\dot{M}_t/S_t)dt + \int_T^\infty e^{-rt}(\dot{M}_t/S_t)dt \end{aligned}$$

$$\begin{aligned} &= \int_T^\infty e^{-rt}\mu\underline{c}dt \\ &= \mu\underline{c}\frac{1}{r}e^{-rT}. \end{aligned} \tag{14}$$

Jump seignorage:

$$\begin{aligned} RHS_3 &= \sum_{i\in I} e^{-ri}\Delta m_i = \Delta m_0 + e^{-rt^*}\Delta m_{t^*} + e^{-rT}\Delta m_T \\ &= \frac{\bar{M}-M}{S} + e^{-rt^*}\frac{M^*-\bar{M}}{S} + e^{-rT}\left(\frac{M_T}{P_T} - \lim_{t\uparrow T} m_t\right) \\ &= \bar{c} - c + e^{-rt^*}\left(\frac{M^*}{S} - \bar{c}\right) + e^{-rT}(\underline{c} - \tilde{c}_T) \end{aligned} \tag{15}$$

and the new present-value of the consol payments:

$$\begin{aligned} RHS_4 &= \int_0^\infty e^{-rt} rB/S_t dt = rB\int_0^\infty S_t^{-1}e^{-rt}dt \\ &= r\frac{B}{S}\int_0^{t^*} e^{-rt}dt + rB\int_{t^*}^T P_t^{-1}e^{-rt}dt + rB\int_T^\infty \underline{c}M_T^{-1}e^{-\mu(t-T)}e^{-rt}dt \\ &= \frac{B}{S}\left(1-e^{-rt^*}\right) + rB\int_{t^*}^T P_t^{-1}e^{-rt}dt + r\frac{B}{P_T}e^{\mu T}\int_T^\infty e^{-(r+\mu)t}dt \\ &= \frac{B}{S}\left(1-e^{-rt^*}\right) + rB\int_{t^*}^T P_t^{-1}e^{-rt}dt + r\frac{B}{P_T}\frac{1}{r+\mu}e^{-rT} \end{aligned} \tag{16}$$

The lifetime budget constraint becomes

$$LHS = RHS_1 + RHS_2 + RHS_3 - RHS_4 \tag{17}$$

We evaluate the two sides of (17). We use numerical integration to evaluate the integral that appears both in (13) and (16) knowing that P_t has the functional form given in (7) for $t^* < t < T$.

The left-hand side is resources, the right hand-side is spending, so to speak. So if $LHS < RHS$ our guess of $\bar{c}$ is too large, and we set

$$\begin{aligned} \bar{c}^{new} &= \frac{1}{2}\bar{c}_l + \frac{1}{2}\bar{c} \\ \bar{c}_h^{new} &= \bar{c} \end{aligned}$$

and we iterate. On the other hand if $LHS > RHS$ our guess of $\bar{c}$ is too low and we set

$$\begin{aligned} \bar{c}^{new} &= \frac{1}{2}\bar{c} + \frac{1}{2}\bar{c}_h \\ \bar{c}_l^{new} &= \bar{c}. \end{aligned}$$

Eventually (hopefully) the iterations continue to a point where $LHS \approx RHS$, and we stop.

4 Solving for μ

Now we can describe the outer loop over μ. Again, this loop works by bisection. We (hopefully) bracket the solution between two values μ_l and μ_h, then make an initialize the loop by guessing that

$$\mu = \frac{1}{2}\mu_l + \frac{1}{2}\mu_h.$$

Given this guess we then run the $\bar{c}$-loop, which nests the t^*-loop. So now we have solutions for $\bar{M}$, t^*, $\underline{c}$, P_T, t^*, M^*, a and b, and $\bar{c}$ given μ. Next we evaluate the lifetime government budget constraint which implies that

$$\phi = \int_0^\infty (\dot{m}_t + \pi_t m_t)e^{-rt}dt + \sum_{i\in I} e^{-ri}\Delta m_i + \frac{B}{S} - \int_0^\infty \frac{rB}{S_t}e^{-rt}dt$$

Notice that we can use the results (14), (15) and (16) from above to write this as

$$\phi = RHS_2 + RHS_3 - RHS_4 + B/S.$$

The left-hand side of this equation is the fiscal obligation due to the increase in transfers, the right hand-side is extra revenue. So if $\phi < RHS$ our guess of μ is too large, and we set

$$\begin{aligned} \mu^{new} &= \frac{1}{2}\mu_l + \frac{1}{2}\mu \\ \mu_h^{new} &= \mu \end{aligned}$$

and we iterate. On the other hand if $\phi > RHS$ our guess of μ is too low and we set

$$\begin{aligned} \mu^{new} &= \frac{1}{2}\mu + \frac{1}{2}\mu_h \\ \mu_l^{new} &= \mu. \end{aligned}$$

Eventually (hopefully) the iterations continue to a point where $\phi \approx RHS$, and we stop.

5 Plots

In plotting the solutions for c_t (which is also m_t), π_t, $b_t - f_t$, M_t and S_t, we note the following results from above, and some more derived here. We always use $M_t = c_t P_t = c_t S_t$ to derive the path of the money supply.

$$t < 0$$

Here $c_t = c$, $\pi_t = 0$, $b_t - f_t = b_0 - f_0$ and $S_t = S$.

$$0 \le t < t^*$$

Here we note that $c = \bar{c}$, $\pi_t = 0$ and $S_t = S$. We need an expression for $b_t - f_0$. The government's flow budget constraint implies that $\Delta f_0 - \Delta b_0 = \Delta m_0$ and that for $0 < t < t^*$

$$\dot{b}_t - \dot{f}_t = r(b_t - f_t) + \delta + rB/S.$$

This implies that the government budget constraint rolled forward from 0 to $t < t^*$ is

$$\begin{aligned} b_t - f_t &= e^{rt}(b_0 - f_0) + \int_0^t \left(\delta + r\frac{B}{S}\right) e^{-r(s-t)} ds + \frac{M - \bar{M}}{S} e^{rt} \\ &= e^{rt}(b_0 - f_0) + (e^{rt} - 1)\left(\frac{\delta}{r} + \frac{B}{S}\right) + (c - \bar{c})e^{rt}. \end{aligned}$$

$t^* \leq t < T$

Here we note that $S_t = P_t$, where P_t is given by (7), above. Also, recall, from above, that (7) was derived from the differential equation $\dot{P}_t = aP_t^{1+\sigma} - bP_t$. Since $\pi_t = \dot{P}_t/P_t$ we note that this means

$$\begin{aligned} \pi_t &= aP_t^{\sigma} - b \\ &= a\frac{b}{a - e^{(t-t^*)\sigma b}(a - bS^{-\sigma})} - b. \end{aligned}$$

Since $M_t = M^*$ we have $c_t = M^*/P_t$. Since $M_t = M^*$ for $t^* < t < T$, the flow budget constraint for $t^* < t < T$ is

$$\dot{b}_t - \dot{f}_t = r(b_t - f_t) + \delta + rB/S_t$$

This means that the budget constraint rolled forward from t^* to $t < T$ is

$$\begin{aligned} b_t - f_t &= e^{r(t-t^*)}(b_{t^*} - f_{t^*}) + \int_{t^*}^t (\delta + r\frac{B}{S_t})e^{-r(s-t)} ds \\ &= e^{r(t-t^*)}(\Psi - \frac{B}{S}) + [e^{r(t-t^*)} - 1]\frac{\delta}{r} + rBe^{rt} \int_{t^*}^t P_t^{-1} e^{-rs} ds. \end{aligned}$$

The integral can be computed numerically just as we computed $\int_{t^*}^T P_t^{-1} e^{-rs} ds$, above.

$t \geq T$

Here we note that $c_t = \underline{c}$, $\pi_t = \mu$ and $S_t = P_t = P_T e^{\mu(t-T)}$. From the previous subsection notice that

$$Q \equiv \lim_{t \to T} b_t - f_t = e^{r(T-t^*)}(\Psi - \frac{B}{S}) + [e^{r(T-t^*)} - 1]\frac{\delta}{r} + rBe^{rT} \int_{t^*}^T P_t^{-1} e^{-rs} ds.$$

Then at date T the budget constraint has a jump, so that

$$b_T - f_T = Q - \Delta m_T = Q + \frac{M^*}{P_T} - \frac{M_T}{P_T} = Q + \tilde{c}_T - \underline{c}.$$

For $T < t < T'$ (and we always ignore T') the flow budget constraint is

$$\begin{aligned}\dot{b}_t - \dot{f}_t &= r(b_t - f_t) + \delta + rB/S_t - \dot{m}_t - \pi_t m_t \\ &= r(b_t - f_t) + \delta + \frac{rB}{P_T} e^{-\mu(t-T)} - \mu \underline{c}.\end{aligned}$$

Hence the budget constraint rolled forward from T to $t < T'$ is

$$\begin{aligned}b_t - f_t &= e^{r(t-T)}(b_T - f_T) + \int_T^t \left[\delta - \mu\underline{c} + r\frac{B}{P_T} e^{-\mu(s-T)}\right] e^{-r(s-t)} ds \\ &= e^{r(t-T)}(Q + \tilde{c}_T - \underline{c}) + [e^{r(t-T)} - 1]\frac{\delta - \mu\underline{c}}{r} + \frac{rB}{P_T}\frac{1}{\mu + r}[e^{r(t-T)} - e^{-\mu(t-T)}].\end{aligned}$$