Competitive Equilibrium and the Welfare Theorems

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Rather than having a social planner, set up a market structure with firms (who maximize profits) and households (who maximize utility).

- time-0 market structure
- sequential market structure

**The Two Welfare Theorems**

- 1st Welfare Theorem: circumstances under which a competitive equilibrium is Pareto optimal (i.e. it corresponds to the solution to a social planning problem).
- 2nd Welfare Theorem: circumstances under which a Pareto optimum (the solution to a social planning problem) can be supported as a competitive equilibrium.

**Stochastic Models**
Many market structures are possible: we will look at two examples.

Households own all the factors of production and shares in the firms.

- Endowments of factors and assets are distributed equally across households—allows us to abstract from trade in the assets

Households sell factor services (labor and capital) to firms

Households use their income to either consume or accumulate more capital.

Households wish to maximize lifetime utility
Firms own nothing, hire factors of production to produce output which they sell to households.

Profits are distributed to owners.

Since the firm’s problem is not dynamic the firm’s goal is to maximize profits.
We will look first at the time 0 market structure—come back to the sequential market structure at the end.

Trading and pricing of contracts all takes place at time 0, determining the future sequences of prices and quantities.

After time 0 there is no more trade, simply the delivery of the services and goods promised under the contracts drawn up at time 0.

\( p_t \): time-0 price of a unit of output delivered at time \( t \) in an arbitrary unit of account

\( w_t \): price of a unit of labor delivered in period \( t \) expressed in units of goods delivered in period \( t \) (real wage)

\( r_{kt} \): rental rate of capital in units of goods delivered in period \( t \).
The Firm’s Problem

- The firm chooses \( \{y_t, k_t^d, L_t^d\}_{t=0}^\infty \), to maximize

\[
\Pi = \sum_{t=0}^{\infty} p_t (y_t - r_k t k_t^d - w_t L_t^d)
\]

subject to \( y_t \leq F(k_t^d, L_t^d) \), \( t \geq 0 \), and taking the sequences \( \{p_t, w_t, r_k t\}_{t=0}^\infty \) as given.
- Equivalent to a sequence of static problems where the firm maximizes \( y_t - r_k t k_t^d - w_t L_t^d \).
The Household’s Problem

- Taking the price sequences \( \{p_t, w_t, r_{kt}\}_{t=0}^{\infty} \) as given and the firm’s profits, \( \Pi \), as given the household maximizes

\[
\sum_{t=0}^{\infty} \beta^t u(c_t)
\]

subject to

\[
\sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_{kt}k_t^s + w_tL_t^s) + \Pi
\]

\[
k_{t+1} = (1 - \delta)k_t + i_t, \ t \geq 0
\]

\[
0 \leq L_t^s \leq 1, \ 0 \leq k_t^s \leq k_t, \ t \geq 0
\]

\[
c_t \geq 0, \ k_{t+1} \geq 0, \ t \geq 0.
\]
A competitive equilibrium is a set of prices \( \{ p_t, r_{kt}, w_t \}^T_{t=0} \), and allocations \( \{ k^d_t, L^d_t, y_t \}_{t=0}^{\infty} \) and \( \{ c_t, l_t, k_{t+1}^s, k^s_t, L^s_t \}_{t=0}^{\infty} \) for firms and households, respectively, such that

- \( \{ k^d_t, L^d_t, y_t \}_{t=0}^{\infty} \) solves the firm’s problem given \( \{ p_t, r_{kt}, w_t \}_{t=0}^{\infty} \),
- \( \{ c_t, l_t, k_{t+1}^s, k^s_t, L^s_t \}_{t=0}^{\infty} \) solves the household’s problem given \( \{ p_t, r_{kt}, w_t \}_{t=0}^{\infty} \), and \( \Pi \),
- all markets clear: \( k^d_t = k^s_t, L^d_t = L^s_t, c_t + i_t = y_t, t \geq 0 \).
Conjecture that $p_t$, $w_t$, $r_{kt}$ are strictly positive for all $t$.

The firm, essentially, has a sequence of static problems.

- For each $t$, given $p_t > 0$ it picks $k_t^d$ and $L_t^d$ to maximize $F(k_t^d, L_t^d) - r_{kt} k_t^d - w_t L_t^d$. Hence

$$r_{kt} = F_k(k_t^d, L_t^d)$$
$$w_t = F_n(k_t^d, L_t^d).$$

Since $F$ is CRTS it follows that $F(k_t^d, L_t^d) - r_{kt} k_t^d - w_t L_t^d = 0$, $\forall t$, and therefore that $\Pi = 0$. 
Optimal for the household to set \( L_t^s = 1 \) and \( k_t^s = k_t \).

Budget constraint will always hold with equality, given the properties of \( u \).

Rewrite the household’s problem as

\[
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ subject to }
\sum_{t=0}^{\infty} p_t [c_t + k_{t+1} - (1 - \delta)k_t] = \sum_{t=0}^{\infty} p_t (r_{kt}k_t + w_t) + \Pi
\]  

(3)

\[c_t \geq 0, \quad k_{t+1} \geq 0, \quad t \geq 0.\]
Nonnegativity constraint on $c_t$ never holds with equality, so:

$$\beta^t u'(c_t) - \theta p_t = 0, \quad t \geq 0$$  (4)

$$\theta[(r_{kt+1} + 1 - \delta)p_{t+1} - p_t] \leq 0, \quad t \geq 0,$$  (5)

where $\theta$ is the Lagrange multiplier on the budget constraint.

The inequality is an equality for any $t$ such that $k_{t+1} > 0$ (assume that $k_{t+1} > 0$ for all $t$).

Imposing the equilibrium conditions $k^d_t = k^s_t = k_t$, $L^d_t = L^s_t = 1$ and $c_t + i_t = y_t$, and using (1) we can rewrite (4) and (5) as

$$\beta^t u'(c_t) = \theta p_t, \quad t \geq 0$$

$$[f'(k_{t+1}) + 1 - \delta]p_{t+1} = p_t, \quad t \geq 0$$

and we also have

$$c_t + k_{t+1} - (1 - \delta)k_t = f(k_t), \quad t \geq 0.$$
The Two Welfare Theorems

- Notice that if we substitute $p_t$ out of our equilibrium conditions we have

$$\beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta] = u'(c_t), \quad t \geq 0$$

$$c_t + k_{t+1} - (1 - \delta)k_t = f(k_t), \quad t \geq 0.$$ 

- These are the same as the optimality conditions from the social planner’s problem.

- Although this is not a formal proof of the two welfare theorems, we have constructed a competitive equilibrium which is characterized by the same conditions as the social planner’s problem. Thus we have shown that
  - the competitive equilibrium is pareto optimal
  - that we can support the social planner’s solution with this competitive equilibrium
A Sequential Market Structure
Recursive Representation

- We can consider an alternative market structure in which agents trade contracts in each period.
- Write prices and single-period profits as functions of the state variables, so that they can be represented in a dateless formulation of the household’s problem
  \[
  r_{kt} = r_k(k_t), \ w_t = w(k_t), \ \pi_t = \pi(k_t)
  \]
- Continue to assume household supplies labor inelastically
- Continue to abstract from trade in shares of the firms.
- Could add trade in single period securities that pay a unit of consumption in the next period to show comparability to time 0 market structure
- The firm’s problem remains the same because it is static.
Let $K$ and $C$ be the household’s own capital and consumption, $k$ the aggregate capital stock, which is the state variable.

Household solves

$$V(K, k) = \max_{C, K'} \left\{ u(C) + \beta V[K', h(k)] \right\}$$

subject to

$$C + K' - (1 - \delta)K \leq Kr_k(k) + w(k) + \pi(k)$$
A recursive competitive equilibrium is a value function, \( V \), a policy function for the household, \( H \), a law of motion for the aggregate capital stock, \( h \), and functions \( r \), \( w \) and \( \pi \), such that

- \( V \) satisfies (6),
- \( H \) is the optimal policy function for (6),
- \( H(k, k) = h(k) \) for all \( k \),
- \( r_k(k) \) and \( w(k) \) satisfy the firm’s first order conditions; i.e.

\[
    r_k(k) = F_k(k, 1) \quad \text{and} \quad w(k) = F_n(k, 1)
\]

- \( \pi(k) = F(k, 1) - r_k(k)k - w(k) \).
The first order conditions for the firm’s problem are the same as before

\[ r_k(k) = F_k(k^d, L^d) \quad \text{and} \quad w(k) = F_n(k^d, L^d) \]

In equilibrium we must have \( k^d = k \) and \( L^d = 1 \) so that

\[ r_k(k) = F_k(k, 1) \quad \text{and} \quad w(k) = F_n(k, 1) \]

The firm’s profits single period profits are

\[ \pi = F(k^d, L^d) - r_k(k)k^d - w(k)L^d \]

In equilibrium profits are zero from CRTS and the fact that \( k^d = k \) and \( L^d = 1 \).

Hence \( \pi(k) = 0 \) for all \( k \).
Solving for the Recursive Equilibrium
The Household’s Problem

- After substituting out $C$, the first-order and envelope conditions for the household are
  \[ u'(C) = \beta V_1 [K', h(k)] \]
  \[ V_1(K, k) = u'(C) [r_k(k) + 1 - \delta] \]

- Combining these we have the usual Euler equation and the budget constraint
  \[ u'(C) = \beta u'(C') [r_k(k') + 1 - \delta] \]
  \[ C + K' - (1 - \delta)K \leq K r_k(k) + w(k) + \pi(k) \]

- Imposing $C = c$ and $K = k$, and given the results from the firm’s problem which determined $r_k(k)$, $w(k)$ and $\pi(k)$ we have
  \[ u'(c) = \beta u'(c') [f'(k') + 1 - \delta] \]
  \[ c + k' - (1 - \delta)k \leq f(k) \]

- This is equivalent to what we got from the time 0 structure.
If we had allowed households to trade single period bonds we would have had to modify the budget constraint to be:

\[ C + K' - (1 - \delta)K + q(k, b)B' \leq Kr_k(k, b) + w(k, b) + \pi(k, b) + B. \]

Since the aggregate quantity of bonds must be \( b = 0 \) in equilibrium, the first-order and envelope conditions for \( B' \) would have been

\[ q(k, 0) u'(C) = \beta V_2 [K', B', h(k, b), 0] \]

\[ V_2 [K, B, k, 0] = u'(C). \]

Hence the price of a one period bond is

\[ q(k, 0) = \beta u'(C') / u'(C) \]

At date \( t \), \( q_t \) is the same as \( p_{t+1} / p_t \) from the time 0 problem.
Alternative Market Arrangements

- There are many possible market arrangements that we have not explored that would lead to equivalent outcomes.
- An important case is when the households do not own the capital stock, and instead it is owned by firms who also make the investment decisions.
- With this setup the firms and the households both have dynamic problems, and it is critical to allow the households to trade the one period bonds.
- Firms have to discount their profit flow, and do so using the prices of the bonds.
  - This ensures that the firms choose the same investment the household would have.
We described a model in which output per capita is $z_t f(k_t)$.

To set up a market structure we need to be formal and write

$$z_t = z_t(s^t)$$

where $s^t$ is the *history* of a stochastic event $s_t$ up to date $t$. I.e.

$$s^t = (s_t, s_{t-1}, \ldots, s_0).$$

Unconditional probability of observing a particular history is

$$\pi_t(s^t)$$

Also have conditional probabilities

$$\pi_\tau(s^\tau | s^t)$$

Assume that $s_0$ is known.
The Social Planner’s Problem in the Stochastic Model

Basic Setup

- Recall that the social planner maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. \( c_t = z_t f(k_t) + (1 - \delta)k_t - k_{t+1} \), for \( t \geq 0 \), and \( k_0 \) given.

- The planner has to choose contingency plans—choices of the future \( k_t \)'s that are contingent on realizations of the state.

  - The planner chooses \( c_t(s^t) \), \( k_{t+1}(s^t) \) for each \( t \) and each possible \( s^t \).

- Assuming a discrete distribution for the shocks, this can be rewritten as

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u(c_t(s^t))$$

s.t. \( c_t(s^t) = z_t(s^t)f[k_t(s^{t-1})] + (1 - \delta)k_t(s^{t-1}) - k_{t+1}(s^t) \) for each \( t \) and \( s^t \).
Abstracting from issues arising from infinite numbers of choice variables form the Lagrangian

\[ L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \left( u \left[ c_t(s^t) \right] + \mu_t(s^t) \left\{ z_t(s^t) f[k_t(s^{t-1})] + \cdots + (1 - \delta) k_t(s^{t-1}) - k_{t+1}(s^t) - c_t(s^t) \right\} \right) \]

The first order conditions are

\[ u'[c_t(s^t)] = \mu_t(s^t) \]
\[ \beta^t \pi_t(s^t) \mu_t(s^t) = \sum_{s^{t+1}|s^t} \beta^{t+1} \pi_{t+1}(s^{t+1}) \mu_{t+1}(s^{t+1}) \times \left\{ z_{t+1}(s^{t+1}) f'[k_{t+1}(s^t)] + (1 - \delta) \right\} \]
Rewritten these become the familiar Euler equation

\[ u'[c_t(s^t)] = \sum_{s^{t+1}|s^t} \beta \pi_{t+1}(s^{t+1}|s^t) u'[c_{t+1}(s^{t+1})] \times \]
\[ \{ z_{t+1}(s^{t+1}) f'(k_{t+1}(s^t)) + (1 - \delta) \} \]

or

\[ u'(c_t) = E_t \beta u'(c_{t+1})[z_{t+1}f'(k_{t+1}) + (1 - \delta)]. \]

This is the same as the Euler equation we got in the notes on dynamic programming.

Now we want to show equivalence of the social planning problem to a competitive equilibrium.
The Decentralized Model

The Firm’s Problem

The firm maximizes

$$\Pi = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) \left\{ z_t(s^t) F \left[ k_t^d(s^t), L_t^d(s^t) \right] - r_{kt}(s^t) k_t^d(s^t) - w_t(s^t) L_t^d(s^t) \right\}$$

Firm’s problem is fundamentally static:

$$r_{kt}(s^t) = z_t(s^t) F_k \left[ k_t^d(s^t), L_t^d(s^t) \right]$$

$$w_t(s^t) = z_t(s^t) F_n \left[ k_t^d(s^t), L_t^d(s^t) \right]$$

CRTS technology implies zero profits.
The Decentralized Model

The Household’s Problem

The household maximizes

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) u[c_t(s^t)]$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) [c_t(s^t) + k_{t+1}(s^t) - (1 - \delta) k_t(s^{t-1})] \leq$$

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) [r_{kt}(s^t) k_t^s(s^t) + w_t(s^t) L_t^s(s^t)] + \Pi$$
The Decentralized Model
The Household’s First Order Conditions

- The household will set $L^s_t(s^t) = 1$ for all $t, s^t$ and $k^s_t(s^t) = k_t(s^{t-1})$ for all $t, s^t$.
- The household’s first order conditions for $c_t(s^t)$ and $k_{t+1}(s^t)$ are
  \[
  \beta^t \pi^t_t(s^t) u'[c_t(s^t)] = p_t(s^t)
  \]
  \[
  p_t(s^t) = \sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1}) \left[ r_{kt+1}(s^{t+1}) + (1 - \delta) \right]
  \]
Substituting out $p_t(s^t)$ and using
\[ r_{kt}(s^t) = F_k \left[ k_t(s^{t-1}), 1 \right] = f'\left[ k_t(s^{t-1}) \right] \]
we have
\[
\begin{aligned}
u'[c_t(s^t)] &= \sum_{s^{t+1}|s^t} \beta \pi_{t+1}(s^{t+1}|s^t) u'[c_{t+1}(s^{t+1})] \times \\
&\quad \left\{ z_{t+1}(s^{t+1}) f'[k_{t+1}(s^t)] + (1 - \delta) \right\}
\end{aligned}
\]

This is just the Euler equation again!

We also impose market clearing in the goods market,
\[
c_t(s^t) + k_{t+1}(s^t) - (1 - \delta)k_t(s^{t-1}) = z_t(s^t) f \left[ k_t(s^{t-1}) \right]
\]
which guarantees that we replicate the social planner problem.
As you will see if you try to read Ljunqvist-Sargent, formulating the sequential markets representation of the decentralized economy is hideous unless you assume that $s_t$ is a Markov process.

Since we did this when thinking about the social planning problem in the previous set of slides, we will immediately go the Markov case here.

We will use the big $K$-little $k$ trick we used earlier in this chapter to represent household/firm choices versus aggregate variables.
The Sequential Markets Decentralized Model

The Firm’s Problem

- The representative firm’s problem remains fundamentally static. It maximizes

\[ \pi(k, s) = \max_{K^d, L^d} z(s) F \left( K^d, L^d \right) - r_k(k, s) K^d - w(k, s) L^d \]

- First order conditions:

\[
\begin{align*}
  r_k(k, s) & = z(s) F_k \left( K^d, L^d \right) \\
  w(k, s) & = z(s) F_n \left( K^d, L^d \right)
\end{align*}
\]

- CRTS technology implies \( \pi(k, s) = 0 \) for all \( k, s \).

- The firm’s problem determines \( K^d \) and \( L^d \) as functions of the current aggregate states, \( k \) and \( s \).
The household’s problem, which is recursive, can be represented by the following Bellman equation

\[
V(K, k, s) = \max_{C, K'} u(C) + \beta \sum_{s'} V[K', h(k, s), s'] \pi(s'|s)
\]

subject to

\[
C + K' - (1 - \delta)K \leq Kr_k(k, s) + w(k, s) + \pi(k, s)
\]
The Sequential Markets Decentralized Model
The Household’s First Order Conditions

- If we substitute in the constraint and differentiate with respect to $K'$ we get
  \[ u'(C) = \beta \sum_{s'} V_1 [K', h(k, s), s'] \pi(s'|s) \]

- The envelope condition is
  \[ V_1(K, k, s) = u'(C) [r_k(k, s) + (1 - \delta)] \]

- Combining these we have
  \[ u'(C) = \beta \sum_{s'} u'(C') \{ r [h(k, s), s'] + (1 - \delta) \} \pi(s'|s) \]

  which is the same old Euler equation.
The Sequential Markets Decentralized Model
Recursive Competitive Equilibrium

- Imposing market clearing we have

\[ r_k(k, s) = z(s)F_k(k, 1) \]
\[ w(k, s) = z(s)F_n(k, 1) \]

and

\[ c(k, s) + h(k, s) - (1 - \delta)k = z(s)f(k) \]

with the Euler equation becoming

\[ u'[c(k, s)] = \beta \sum_{s'} u'[c[h(k, s), s]] \{ z(s')f'[h(k, s)] + (1 - \delta) \} \pi(s'|s) \]

- Once again, the decentralized economy replicates the social planning solution.