Power Series and Uniform Convergence

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1 Last Time

Sequences and series of functions. Defined uniform convergence on a set $S \subseteq \mathbb{R}$. Important theorem was that the uniform limit of continuous functions is continuous.

2 Uniformly Cauchy

Recall: A sequence $(s_n)$ is Cauchy if $\forall \epsilon > 0, \exists N$ such that $m, n > N \rightarrow |s_n - s_m| < \epsilon$. Cauchy is equivalent to convergence. Sometimes its easier to check Cauchy, since it doesn’t refer to any limits. Want a similar idea for sequences of functions, in particular series.

Definition: a sequence of functions on $(f_n)$ on $S \subseteq \mathbb{R}$ is uniformly Cauchy on $S$ if $\forall \epsilon > 0, \exists N$ such that $n, m > N \rightarrow |f_n(x) - f_m(x)|, \forall x \in S$.

Notes: Uniform part of the name refers to the fact that it holds simultaneously for all $x \in S$.
1) Always need the underlying set $S$ as always. $S$ can be any set $\subseteq \mathbb{R}$, not just intervals.
2) If $(f_n)$ is uniformly Cauchy in $S$ and some $x_0 \in S$, then $(f(x_0))$ is a Cauchy sequence in $\mathbb{R}$.

Proof: show that the Cauchy statement for all $x \in S$ is a stronger statement that implies the Cauchy statement at any particular point, including $x_0$.
3) If $(f_n)$ is a sequence of functions that is uniformly convergent on $S \subseteq \mathbb{R}$, then its also uniformly Cauchy (just like any convergent sequence is automatically Cauchy).
Proof: let $\epsilon > 0$ be given. Assume $(f_n)$ is the uniformly convergent set of functions on $S$ and let $f$ be the limit function. \(\exists N\) such that $n > N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}, \forall x \in S$. If $m, n > N$, then $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $(f_n)$ is uniformly Cauchy on $S$.

Recall that the same theorem holds for sequences in $\mathbb{R}$. The converse is also true. This is the more interesting half of this theorem. The same thing will be true in the case of functions, but is easier to prove because we will use this property of real numbers in the proof.

Theorem: if $(f_n)$ is a sequence of functions on some set $S \subseteq \mathbb{R}$, and $(f_n)$ is uniformly Cauchy on $S$, then (we want to say that $(f_n)$ is uniformly continuous) \(\exists f\) on $S$ such that $(f_n)$ converges to $f$ uniformly.

Proof: \(\forall x_0 \in S\), note that $(f_n(x_0))$ is a Cauchy sequence in $\mathbb{R} \rightarrow (f_n(x_0))$ is convergent. Let $f(x_0) = \lim f_n(x_0), \forall x_0 \in S$. Immediately, we have that $(f_n)$ converges to $f$ pointwise.

Let $\epsilon > 0$. \(\exists N\) such that $|f_n(x) - f(m)(x)| < \frac{\epsilon}{2}$ by the Cauchy criterion. Think of $m$ as being fixed with $m > N$ and let $n \rightarrow \infty$. $f_n(x) \in [f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}] \rightarrow \lim f_n(x) \in [f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}]$ by exercise $(8.9c)$, $\Rightarrow |f(x) - f_n(x)| \leq \frac{\epsilon}{2}, \forall x \in S \rightarrow f_m$ converges to $f$ uniformly as $m$ goes to $\infty$.

All that goes in is that we have pointwise convergence, fix our $m$, and then using our uniform Cauchy estimate.

## 3 Weierstrass M Test

Note: if $(g_k)$ is a sequence if continuous functions and $s_n = \sum_{k=1}^{n} g_k$, then $s_n$ is continuous $\forall n$, so if $s_n$ converges to $s$ uniformly, then $s$ is continuous on $S$. So $s(x) = \sum_{k=1}^{\infty} g_k(x)$ is continuous. We usually say the infinite sum converges uniformly on $S$. How do we check for this uniform convergence? The Cauchy criterion. It is enough by the theorem to check that $(s_n)$ is a uniformly Cauchy sequence on $S$. Want $\forall \epsilon > 0$, \(\exists N\) such that $n > m > N \Rightarrow |s_n(x) - s_m(x)| < \epsilon, \forall x \in S$. $|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^{n} g_k(x) \right| \leq \sum_{k=m+1}^{n} |g_k(x)| \leq \sum_{k=m+1}^{n} \sup \{ |g_k(x)| : x \in S \}, \forall x \in S$. Each one of these sup is just a fixed number, so we can ask if this series converges. If $M_k = \sup \{ |g_k(x)| : x \in S \}$, then if $\sum_{k=1}^{\infty} M_k$ converges, then $s_n$ is uniformly Cauchy on $S$ and $\sum_{k=1}^{\infty} g_k$ converges uniformly on $S$. So the $s(x) = \sum_{k=1}^{\infty} g_k(x)$ is Continuous on $S$.

Its easier to check whether a series of numbers converges than a sequence
of functions.

Theorem: if \((M_k)\) is a sequence \(\in \mathbb{R}\) such that \(\sum_{k=1}^{\infty} M_k\) converges, and \(|g_k(x)| \leq M_k, \forall x \in S\), then \(\sum_{k=1}^{\infty} g_k\) converges uniformly on \(S\).

Note you don’t even have to have the best estimate for \(M_k\). Also, \(M_k \geq 0, \forall k\).

4 Examples

1) \(g_k = \frac{1}{k^2}x^k, S = [-1, 1]\). On \(S, |x^k| \leq 1 \rightarrow |g_k(x)| \leq \frac{1}{k^2}, \forall x \in S\). Let \(M_k = \frac{1}{k^2}\). Know by the P-series test that \(\sum_{k=1}^{\infty} \frac{1}{k^2}\) converges \(\rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}x^k\) converges uniformly on \([-1, 1]\). This is a power series. Note that we couldn’t do any better than \(S = [-1, 1]\). This is the whole interval of convergence.

2) \(g_k = \frac{1}{k!}x^k, S = [-a, a], a \in \mathbb{R}, a > 0\). \(|x^k| \leq a^k\) on \(S\), \(|g_k(x)| \leq \frac{1}{k!}a^k, \forall x \in S\). Let \(M_k = \frac{1}{k!}a^k\).

\[
\lim_{k \to \infty} \frac{M_{k+1}}{M_k} = \lim_{k \to \infty} \frac{a}{k+1} = 0
\]

So \(\sum_{k=1}^{\infty} M_k\) converges by the ratio test \(\rightarrow \sum_{k=1}^{\infty} \frac{1}{k!}x^k\) converges uniformly on \([-a, a]\) \(\rightarrow f(x) = \sum_{k=1}^{\infty} \frac{1}{k!}x^k\) is continuous on \([-a, a], \forall a \in \mathbb{R}\) \(\rightarrow f\) is continuous on \(\mathbb{R}\).

Note \(\sum_{k=1}^{\infty} \frac{1}{k!}x^k\) does not converge uniformly on \(\mathbb{R}\)! Look at the partial sums, which are not uniformly Cauchy. Let \(n = m + 1, |s_n - s_m| = |g_n(x)|\), which is not \(< \epsilon \forall x\) because \(g_n(x)\) is unbounded \(\forall n\).