Double Pendulum and Lagrange Multipliers

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1 Double pendulum

1.1 Motivation

This problem is a straightforward application of the method we have developed thus far. As with all mechanics problems, the art is choosing the correct variables to parameterize the problem. It will also illustrate how to take appropriate limits to simplify the equations of motion.

1.2 Setup

First mass hangs from a rod of length $b$ at angle $\theta_1$ from the vertical. The second mass hangs from a rod of length $a$ from the first mass at angle $\theta_2$ from the vertical. Call the vector from the origin to the first mass $\vec{r}_1$, and the vector from the origin to the second mass $\vec{r}_2$.

$$
\vec{r}_1 = b \sin \theta_1 \hat{x} - b \cos \theta_1 \hat{y}
$$

$$
\vec{r}_2 = \vec{r}_1 + (a \sin \theta_2 \hat{x} - a \sin \theta_2 \hat{y})
$$

$$
= (b \sin \theta_1 + a \sin \theta_2) \hat{x} - (b \cos \theta_1 + a \cos \theta_2) \hat{y}
$$
1.3 Compute the Lagrangian

1.3.1 Kinetic term

We need the squares of the time derivatives.

\[ \dot{\vec{r}}_1 = b\dot{\theta}_1 \cos \theta_1 \hat{x} + b\dot{\theta}_1 \sin \theta_1 \hat{y} \]
\[ \dot{\vec{r}}_2 = \dot{\vec{r}}_1 + (a\dot{\theta}_2 \cos \theta_2 \hat{x} + a\dot{\theta}_2 \sin \theta_2 \hat{y}) \]
\[ \dot{\vec{r}}_2 = b\dot{\theta}_1 \cos \theta_1 \hat{x} + (b\dot{\theta}_1 \sin \theta_1 + a\dot{\theta}_2 \sin \theta_2 \hat{y}) \]
\[ \dot{\vec{r}}_2 = b^2 \dot{\theta}_1^2 \cos^2 \theta_1 + b^2 \dot{\theta}_2^2 \sin^2 \theta_1 \]
\[ \dot{\vec{r}}_2 = b^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a^2 \dot{\theta}_2^2 + 2m_2 ab \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \]

The last term of \( \dot{r}_2 \) is the cross term that mixes the motion of the two masses and introduces nonlinear behavior.

\[ T = \frac{1}{2} (m_1 + m_2) b^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a^2 \dot{\theta}_2^2 + m_2 ab \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \]

1.3.2 Potential energy

Gravity is the only relevant potential.

\[ U = m_1 gb(1 - \cos \theta_1) + m_2 g (a(1 - \cos \theta_2) + b(1 - \cos \theta_1)) \]

1.4 Euler-Lagrange equations

1.4.1 \( \theta_1 \)

\[ \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) b\dot{\theta}_1 + m_2 ab \dot{\theta}_2 \cos(\theta_1 - \theta_2) \]
\[ \frac{\partial L}{\partial \theta_1} = -m_2 ab \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 gb \sin \theta_1 - m_2 gb \sin \theta_1 \]

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Normally we would take time derivatives of the first equation, but we hold off because we first want to linearize the equations. We will apply this approximation below.

### 1.4.2 \( \theta_2 \)

\[
\frac{\partial L}{\partial \theta_2} = ma^2 \dot{\theta}_2 + m_2 ab \dot{\theta}_1 \cos(\theta_1 - \theta_2)
\]

\[
\frac{\partial L}{\partial \theta_2} = m_2 ab \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 ga \sin \theta_2
\]

### 1.5 Linearization

**Definition.** *Linearization* is taking the approximation that perturbations in the problem are very small, and that terms of second order and higher are negligible.

In this limit,

\[
\cos \theta \approx 1 \\
\sin \theta \approx \theta
\]

We have not defined how small these angles have to be in practice to apply the linearization assumptions. To decide, we should keep higher order terms and examine their effect on the equations of motion. In this case, it turns out that small \( \theta \) means \( \theta < 5^\circ \).

### 1.6 Equations of motion

We apply our small angle approximations to linearize before taking time derivatives.

#### 1.6.1 \( \theta_1 \)

\[
\frac{d}{dt} \frac{\partial L}{\partial \theta_1} = (m_1 + m_2)b^2 \ddot{\theta}_1 + m_2 ab \ddot{\theta}_2
\]

\[
\frac{d}{dt} \frac{\partial L}{\partial \theta_1} = \frac{\partial L}{\partial \theta} \\
\Rightarrow (m_1 + m_2)b^2 \ddot{\theta}_1 + m_2 ab \ddot{\theta}_2 \approx -(m_1 + m_2)gb \theta_1
\]
1.6.2 $\theta_2$

$$m_2a^2\ddot{\theta}_2 + mab\ddot{\theta}_1 \approx -mg\theta_2$$

1.6.3 Combined equation

These are a set of coupled second order differential equations with constant coefficients. Divide the first by $\frac{b}{m_1+m_2}$ to get

$$b\ddot{\theta}_1 + \left(\frac{m_2}{m_1+m_2}\right)a\ddot{\theta}_2 \approx -g\theta_1$$

Divide the second by $\frac{a}{m_2}$ to get

$$a\ddot{\theta}_2 + b\ddot{\theta}_2 \approx -g\theta_2$$

Using the reduced mass,

$$\mu = \frac{m_2}{m_1+m_2}$$

we can write these equations in a matrix,

$$
\begin{pmatrix}
 b & \mu a \\
 b & a
\end{pmatrix}
\begin{pmatrix}
 \ddot{\theta}_1 \\
 \ddot{\theta}_2
\end{pmatrix}
= -g
\begin{pmatrix}
 \theta_1 \\
 \theta_2
\end{pmatrix}
$$

1.7 Comparison to regular pendulum

Consider a regular pendulum moving close to equilibrium. The equation of motion is

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

This equation is nonlinear in its present form. We linearize with the small angle approximation to get

$$\ddot{\theta} + \omega_0^2 \theta \approx 0$$

where

$$\frac{g}{l} = \omega_0^2$$
This type of differential equation should be familiar; the solutions are sinusoids.

\[ \theta(t) = \eta e^{pt} \]
\[ \theta(t) = A \cos \omega_0 t + B \sin \omega_0 t = F \cos(\omega_0 + \phi_0) = De^{i\omega_0 t} + E e^{-i\omega_0 t} = G e^{i(\omega_0 t + \theta_0)} \]

This motivates us to choose exponential solutions in the current case

\[ \begin{align*}
\theta_1 &= \eta_1 e^{i\omega t} \\
\dot{\theta}_1 &= i\omega \eta_1 e^{i\omega t} = i\omega \theta_1 \\
\ddot{\theta}_1 &= -\omega^2 \theta_1 \\
\theta_2 &= \eta_2 e^{i\omega t} \\
\dot{\theta}_2 &= i\omega \eta_2 e^{i\omega t} = i\omega \theta_2 \\
\ddot{\theta}_2 &= -\omega^2 \theta_2 
\end{align*} \]

1.8 Solving the differential equation

We go back to the matrix equation and replace the double time derivatives

\[ \begin{pmatrix} -b\omega^2 & -\mu a\omega^2 \\ -b\omega^2 & -a\omega^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = -g \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \]
\[ \begin{pmatrix} g - b\omega^2 & -\mu a\omega^2 \\ b\omega^2 & g - a\omega^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0 \]

Solving the differential equation has now become a problem of finding the eigenvalues of the above matrix. As with any eigenvalue problem, we take the determinant and set it equal to 0.

\[ (g - b\omega^2)(g - a\omega^2) - ab\mu\omega^4 = 0 \]
\[ g^2 - a\omega^2 g - b\omega^2 g + ab\omega^4 - ab\mu\omega^4 = 0 \]
\[ ab(1 - \mu)\omega^4 - (a + b)g\omega^2 + g^2 = 0 \]

Solve for \( \omega^2 \) using the quadratic formula. There are two solutions representing two modes of behavior for the pendulum system, each with a different frequency.

\[ \omega^2 > \omega^2_\pm \]
In the high frequency case of $\omega_+\theta_1$ and $\theta_2$ have opposite sign and are out of phase. In the low frequency case of $\omega_-\theta_1$ and $\theta_2$ have the same sign and are in phase.

2 Lagrange multipliers

2.1 Example: Spherical pendulum
Recall that we had a pendulum that was at a fixed distance from the axis. We had only two degrees of freedom, $\theta$ and $\phi$. We considered the constraint on the system separately from the Hamiltonian.

$$r = R = \sqrt{x^2 + y^2 + z^2}$$

We solved for $R$ as a function and $\phi$ and $\theta$ and substituted this expression directly into the Lagrangian.

2.2 General method

2.2.1 Motivation
Lagrange multipliers are a more general approach to systems that are operating under constraints. They allow us to incorporate the constraints of the system directly into the Lagrangian. This method is especially useful for time-dependent constraints, where direct substitution would be difficult.

2.2.2 Setup
Suppose we have a function

$$f(x_1, x_2)$$

that we want to extremize subject to the constraint

$$g(x_1, x_2) = k$$

We could solve for $x_2$ as a function of $x_1$ and $k$ and eliminate $x_2$.

$$f(x_1, x_2(x_1, k)) = f(x_1)$$
This is what we have done in all previous cases.

The more general way is to introduce a function

\[ f(x_1, x_2) + \lambda g(x_1, x_2) \]

\( \lambda \) is an undetermined constant, called a \textit{Lagrange multiplier}. We find the extremum of the new function with respect to both \( x_1 \) and \( x_2 \) and then choose \( \lambda \) to satisfy the constraint.

### 2.3 Example

Suppose we want to minimize

\[ f(x_1, x_2) = 3x_1 + 4x_2 \]

subject to the constraint

\[ x_1^2 + x_2^2 = 1 \]

We look for extrema of

\[ 3x_1 + 4x_2 + \lambda (x_1^2 + x_2^2) \]

We take partial derivatives and set them equal to 0.

\[ 3 + 2\lambda x_1 = 0 \Rightarrow x_1 = -\frac{3}{2\lambda} \]
\[ 4 + 2\lambda x_2 = 0 \Rightarrow x_2 = -\frac{2}{\lambda} \]

We plug these solutions for \( x_1 \) and \( x_2 \) into the constraint equation to determine \( \lambda \),

\[ \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{25}{4} \Rightarrow \lambda = \pm \frac{5}{2} \]

This gives the solutions,

\[ x_1 = -\frac{3}{5} \quad x_2 = -\frac{4}{5} \]
\[ x_1 = \frac{3}{5} \quad x_2 = \frac{4}{5} \]

The first is the minimum and the second is the maximum.
2.4 Justification

For small changes of $dx_1$ and $dx_2$, the change in the given function is

$$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2$$

At the critical points, these partials are 0.

$$\frac{\partial f}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0$$

Adding the two,

$$\frac{\partial}{\partial x_1} (f + \lambda g) = 0$$
$$\frac{\partial}{\partial x_2} (f + \lambda g) = 0$$

We assume that $\lambda$ is not a function of $x_1$ and $x_2$

2.5 Example: Asteroid

2.5.1 Situation

Consider an asteroid of mass $m$ and volume $v_0$. The mass density is

$$\rho = \frac{m}{v_0}$$

We want to find the shape of the asteroid that gives the largest possible vertical gravitational field.

The gravitational force experience by a test mass on the surface of a sphere is as if all the mass of the sphere were at the center of the sphere. The force goes as $\frac{1}{r^2}$, so moving closer to the center should increase the force. However, making the sphere flatter lessens the gravitational pull. We want to flatten one edge of the sphere so that it remains symmetric about the azimuth. Let $r = 0$ be the point of contact between the test mass and the sphere. We want to find $r(\theta, \phi)$. 
2.5.2 Minimization

\[ \vec{g} = -G \int_{\text{vol}} \frac{\rho(r)}{r^2} \hat{r} \, dV \]

We add a cos \( \theta \) factor to get the vertical component.

\[ g(R) = G\rho \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sin \theta \int_{0}^{R(\theta,\phi)} r^2 dr \cdot \frac{\cos \theta}{r^2} \]

We find the volume,

\[ V = \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sin \theta \int_{0}^{R(\theta,\phi)} r^2 dr \]

\[ V(R) = \frac{1}{3} \int_{0}^{\pi} d\theta \int_{0}^{2\pi} \sin \theta d\phi R^3(\theta, \phi) \]

We want to minimize the function

\[ g(R) + \lambda(V(R) - V_0) \]

To do so, we compute partial derivatives,

\[ \frac{\partial}{\partial R} (g(R) + \lambda(V(R) - V_0)) = 0 \]

\[ G\rho \sin \theta \cos \theta - \lambda \frac{1}{3} \sin \theta (3R(\theta, \phi))^2 = 0 \]

\[ R(\theta, \phi) = \sqrt{\frac{G\rho}{\lambda} \cos \theta} \]