4-Vector Transformations

Adrian Down

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1 Review: Analog with 3D rotations

1.1 Three dimensions

3D vectors transform according to

\[ a'_i = \Lambda_{ij} a_j \]

The transformation matrix satisfies

\[ \delta_{kl} \Lambda_{ki} \Lambda_{lj} = \delta_{ij} \]

Second rank tensors transform according to

\[ T'_{ij} = \Lambda_{ik} \Lambda_{jl} T_{kl} \]

1.2 Four dimensions

In 4D, vectors transform according to

\[ a'_\mu = \Lambda^\mu_\nu a^\nu \]

The transformation matrix is now related to the metric tensor,

\[ g_{\rho\sigma} = g_{\nu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \]

Tensors transform according to

\[ F'_{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F_{\rho\sigma} \]
2 Lorentz transformation \( \Lambda \)

We would like to develop an argument to determine the elements of the \( \Lambda \) matrix based only on the requirements that we have thus far placed on \( \Lambda \).

2.1 3D analogy

Consider a transformation in 3D describing a rotation by angle \( \phi \) in the \( xy \) plane. We want to solve for the transformation matrix. Since the transformation must leave the \( z \) direction unchanged, the matrix must be a \( 2 \times 2 \) block matrix in the \( \hat{x} \) and \( \hat{y} \) coordinates and the identity in the \( \hat{z} \) coordinate.

Write \( \Lambda \) in a completely general form,

\[
\Lambda = \begin{pmatrix} c(\phi) & s(\phi) \\ -s(\phi) & c(\phi) \end{pmatrix}
\]

Substitute this matrix into \( \delta_{kl}\Lambda_{ki}\Lambda_{lj} = \delta_{ij} \). This equation determines each component of \( \Lambda \), and thus represents 9 equations. After removing some redundant equations,

\[
\begin{align*}
    c\bar{s} &= \bar{c}s \\
    c^2 + s^2 &= 1 \\
    \bar{c}^2 + \bar{s}^2 &= 1
\end{align*}
\]

Rigorously determining all functions that satisfy these equations is a mathematical problem. One possible solution is \( c = \bar{c} \) and \( s = \bar{s} \). The first equation is satisfied, and the third is also satisfied if the second is. Two functions that satisfy the second equation are \( \sin \phi \) and \( \cos \phi \), so take these as a solution.

2.2 4D transformation

2.2.1 Transformation matrix

Now consider a transformation in 4 dimensions of \( x \) and \( t \). As before, we need only consider a \( 2 \times 2 \) block matrix. With some foresight, write a general formulation of the transformation as,

\[
\Lambda = \begin{pmatrix} c(\eta) & \bar{s}(\eta) \\ -s(\eta) & \bar{c}(\eta) \end{pmatrix}
\]
Note. The functions are not yet specified, so the minus signs are chosen to make the final result appear in a familiar form.

Now, substitute into $g_{\rho\sigma} = g_{\nu\Lambda} \Lambda^\mu_\rho \Lambda^\nu_\sigma$. There independent requirements gained from this system of equations is

\[
\begin{align*}
    cs &= \bar{c}s \\
    c^2 - s^2 &= 1 \\
    \bar{c}^2 - \bar{s}^2 &= 1
\end{align*}
\]

As before, we take the simple solution $c = \bar{c}$ and $s = \bar{s}$. The remaining two solutions are satisfied if we take

\[
\begin{align*}
    c(\eta) &= \cosh \eta = \frac{e^\eta + e^{-\eta}}{2} \\
    s(\eta) &= \sinh \eta = \frac{e^\eta - e^{-\eta}}{2}
\end{align*}
\]

Note.

\[
\cosh \eta + \sinh \eta = \frac{e^\eta + e^{-\eta}}{2} + \frac{e^\eta - e^{-\eta}}{2} = e^\eta
\]

With this identity, we can form the Taylor series expansions of these functions easily,

\[
\begin{align*}
    e^\eta &= 1 + \eta + \frac{\eta^2}{2} + \frac{\eta^3}{3!} + \ldots \\
    \cosh \eta &= 1 + \frac{\eta^2}{2} + \ldots \\
    \sinh \eta &= \eta + \frac{\eta^3}{3!} + \ldots
\end{align*}
\]

2.2.2 Rapidity $\eta$

$\eta$ is the fundamental additive parameter of the Lorentz transformation, analogous to the angle is in 3D rotations. $\eta$ is called either the boost or the rapidity. As we will see, it is defined such that

\[
\frac{v}{c} = \tanh \eta
\]
Referring back to the form of the transformation matrix found above,

\[
\Lambda = \begin{pmatrix}
\cosh \eta & -\sinh \eta \\
-\sinh \eta & \cosh \eta
\end{pmatrix} = \cosh \eta \begin{pmatrix}
1 & - \tanh \eta \\
- \tanh \eta & 1
\end{pmatrix}
\]

By the requirement that the Lorentz transformation reduce to the Galilean transformation, we can assume the relation between \( \beta = \frac{v}{c} \) and \( \eta \),

\[
\beta = \tanh \eta \quad \eta = \tanh^{-1} \beta
\]

With this relationship, the Lorentz transformation is a linear function in Minkowski space. The requirement that light travel at the same velocity in all frames forces this relationship.

Using the identity \( \cosh \eta = \left( \sqrt{1 - \tanh^2 \eta} \right)^{-1} \), the transformation can then be written

\[
\Lambda = \frac{1}{\sqrt{1 - \beta^2}} \begin{pmatrix}
1 & -\beta \\
-\beta & 1
\end{pmatrix}
\]

### 2.3 Aside

Taking \(-\eta\) corresponds to an object moving in the negative direction. Thus, taking \(\Lambda(-\eta)\) should give the inverse transformation.

\[
\Lambda^{-1}(\eta) = \Lambda(-\eta)
\]

Substituting and multiplying these matrices gives the identity matrix, as expected.

### 3 Four-vector formulation

#### 3.1 4-Scalars

**Definition (4-Scalars).** A 4-scalar is a quantity that is invariant to Lorentz transformation.

##### 3.1.1 Rest mass \( m \)

In this course, the rest mass \( m \) will *never* change. We will *not* speak about a “relativistic mass.” Thus the equation \( E = mc^2 \) is not valid, since this equation requires an \( m \) that is a function of velocity.
3.1.2 Proper time $\tau$

**Definition (Proper time).** The *proper time* is the time interval between two events that occur at the same place. The unique frame in which these events occur at the same place is called the *proper frame.*

*Note.* For two events separated by a time-like interval, it is always possible to find a frame in which these events occur at the same place separated by a time interval. The existence of this frame ensures that the proper time is a meaningful quantity.

3.2 4-vectors

3.2.1 Time-position

The basic position four-vector is

$$x^\mu = (ct, x)$$

*Note.* The $c$ is included to ensure that the 0th component has units of length.

3.2.2 4-momentum

Taking the time derivative of the position vector,

$$m \frac{dx^\mu}{d\tau} = p^\mu = \left( \frac{E}{c}, p \right)$$

*Note. Caution:*

$$p \neq mv \quad (= \gamma mv)$$

$$E \neq \frac{1}{2}mv^2 \quad (= \gamma mc^2)$$

3.2.3 4-Wave vector

$$k^\mu = \left( \frac{\omega}{c}, k \right)$$

3.2.4 4-derivative

$$\partial^\mu = \left( \frac{1}{c} \frac{\partial}{\partial \tau}, -\nabla \right)$$
3.2.5 4-Potential

\[ A^\mu = \left( \frac{V}{c}, \mathbf{A} \right) \]

3.3 Maxwell’s equations

The Maxwell gauge condition becomes

\[ \partial_\mu A^\mu = 0 \]

We can also write Maxwell’s equations as

\[ (\partial_\nu \partial^\nu) A^\mu = \mu_0 J^\mu \]