Envelopes*

LaTeX file: *EnvelopesPuzzle* — Daniel A. Graham <daniel.graham@duke.edu>, June 16, 2005

An honest but mischievous father tells his two risk-neutral sons, Andy and Bill, that he has placed 10^n dollars in one envelope and 10^{n+1} dollars in another envelope, where n was chosen with equal probability among the integers between 1 and 6. The sons completely believe their father. He then randomly hands each son one of the envelopes and the sons privately open their envelopes.

Andy finds $10^4 = 10{,}000$ dollars in his envelope and concludes that the other envelope contains either \$1,000 or \$100,000 with equal probability and thus has an expected value of \$50,500.

Bill finds $10^3 = 1{,}000$ dollars in his envelope and concludes that the other envelope has \$100 or \$10,000 with equal probability and thus has an expected value of \$5,500.

Case A. Suppose the father privately asks each son whether or not he would be willing to pay \$1 to switch envelopes with the other son. Both sons say yes. Suppose the father reports this to each son and repeats the question. Both sons again say yes, the father reports this fact and repeats the question a third time. Again both say yes but if the father reports this and asks a fourth time, then Bill will say yes but Andy will say no. Why?

Case B. Suppose instead that the father tells each of his sons that he can pay \$1 to switch envelopes with the understanding that the switch will take place if both sons say yes. Otherwise each son will keep his own envelope. The father will keep any dollar(s) either way. Both will immediately say no. Why?

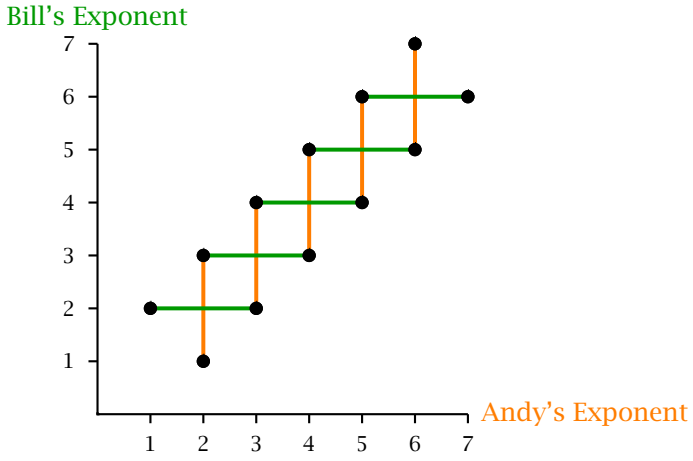

Figure 1: Interim Knowledge

The twelve possible states for this problem are illustrated in Figure 1. State $(4, 3)$, for example, corresponds to the state in which Andy's envelope contains 10^4 dollars and Bill's envelope contains 10^3 dollars. In this figure, states connected by vertical lines correspond to interim[1] possibility sets for Andy

*Based on the discussion in John Geanakoplos, "Common Knowledge", *Journal of Economic Perspectives*, 6 No. 4 (Fall), 1992, 53–82.

[1]Following the usual terminology, *ex ante* refers to the situation before anyone receives any private information, *interim* refers to the situation in which the participants have received their private information and *ex post* to the situation that results from information revealed in the subsequent play of the game.

and states connected by horizontal lines correspond to possibility sets for Bill. At $(4,3)$, for example, Andy's possibility set is $\{(4,3),(4,5)\}$ and Bill's possibility set is $\{(2,3),(4,3)\}$.

The interim common knowledge partition has two possibility sets which correspond to the two connected graphs in Figure 1 — those states in which Andy's exponent is an even number (and Bill's is odd) and those states in which Bill's exponent is an even number (and Andy's is odd). At $(4,3)$, for example, Andy can think that Bill's exponent is 5 and thus that Bill can think that Andy's exponent is 6 and so forth.

Case A

What do the sons learn when the father reports after asking the question the first time that both are willing to switch. Since neither would have been willing to switch had their envelope contained 10^7 = 10,000,000 dollars, each learns that neither $(6,7)$ nor $(7,6)$ is the true state. The effect is illustrated in Figure 2

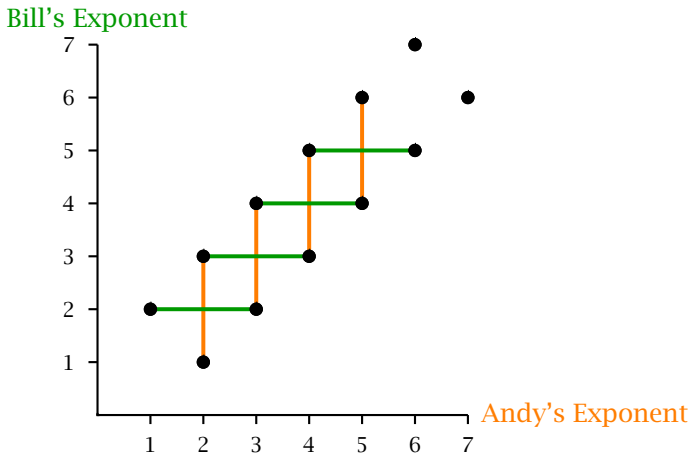

Figure 2: First Yes

When it is reported that both sons have said yes a second time, each learns that neither has 10^6 = 1,000,000 dollars and thus that neither $(5,6)$ nor $(6,5)$ is the true state. When the father reports that both sons have said yes a third time, each learns that neither $(4,5)$ nor $(5,4)$ is the true state. The situation at this stage is illustrated in Figure 3. At this stage, Bill still thinks that both $(2,3)$ and $(4,3)$ are possible and thus answers yes. Andy, on the other hand, knows that the true state is $(4,3)$ and thus answers no.

Case B

For Andy to be willing to ante \$1, he would have to think that Bill is also willing to ante \$1 and that Bill thinks that Andy is willing to ante and so forth. It would, in short, have to be common knowledge that both are willing to ante and thus common knowledge that both expect to gain. Can it be common knowledge at $(4,3)$ that both expect to gain? No. This would require that Andy think it possible that Bill's exponent is 5 and that Andy thinks Bill thinks it possible that Andy's is 6 and that Andy thinks Bill thinks Andy thinks it possible that Bill's is 7. Both Andy can't think it possible that Bill's is 7 because Bill would not expect to gain if this were the case. Thus it cannot be common knowledge that both expect to gain and both must therefore immediately decline their father's offer.

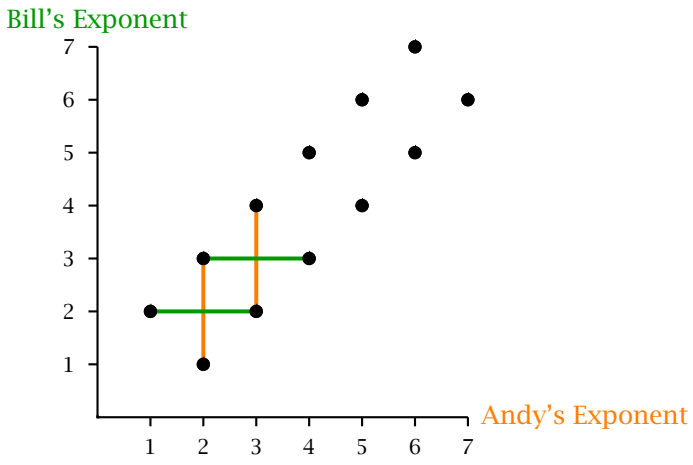

Figure 3: Third Yes