

Solution by Pretense

LaTeX file: *pretense* — Daniel A. Graham <daniel.graham@duke.edu>, November 15, 2005

This handout describes a trick commonly used to solve for the symmetric Nash equilibrium in games of incomplete information. To illustrate this method, consider a first-price auction example in which each of two bidders independently and privately draws a valuation for the item to be auctioned from the uniform distribution on $[0, 1]$. Being a game of incomplete information, the equilibrium will take the form of a function, $b[v]$, having the interpretation that a bidder whose valuation is v will submit bid $b[v]$. To solve for this function we begin by supposing that it is strictly increasing - this will turn out to be justified - and that one of the players will use this function to submit her bid. For this to be a Nash equilibrium, then it must be the case that the other player can do no better than to use this function to generate his bid as well. If t denotes his true valuation, then he must weakly prefer submitting the bid $T = b[t]$ to any other bid.

Now here is the trick. Imagine plotting $b[v]$ with $v \in [0, 1]$ on the horizontal axis and b on the vertical. We're assuming that $b[v]$ is strictly increasing so the function increases from a minimum at $b[0]$ on the left to a maximum of $b[1]$ on the right. Would our bidder ever want to submit a bid $F < b[0]$? No, this would entail a certainty of losing and could be achieved just as effectively by bidding $F = b[0]$. Would our bidder ever want to submit a bid $F > b[1]$? No, this would entail a certainty of winning but this could be achieved at a lower cost by bidding $F = b[1]$. Since our bidder will always want to choose a bid in the range of $b[v]$, rather than thinking of our bidder choosing F we can, equivalently, think of our bidder choosing to bid as if his type were $f = b^{-1}[F]$. Thus we recast the problem of choosing the best bid to one of choosing the best type to pretend to be. That is the trick. Believe it or not, recasting the problem in this way makes it very much easier to solve.

The expected payoff to pretending to have valuation f when the true valuation is t is

$$\text{In: } \text{pay} = (t - b[f]) f$$

$$\text{Out: } f(t - b[f])$$

where $t - b[f]$ is the prize - the bidder's true valuation for the item less the price paid - and f is the probability of winning it. The first order condition for the best valuation to pretend having is

$$\text{In: } \text{fo} = D[\text{pay}, f] == 0$$

$$\text{Out: } t - b[f] - fb'[f] == 0$$

Since a Nash equilibrium requires that no-pretense is best, we obtain the condition

$$\text{In: } \text{eqn} = \text{fo}/f \rightarrow t$$

$$\text{Out: } t - b[t] - tb'[t] == 0$$

This differential equation can then be solved to obtain

$$\text{In: } \text{sol} = \text{DSolve}[\text{eqn}, b[t], t][[1]]$$

$$\text{Out: } \left\{ b[t] \rightarrow \frac{t}{2} + \frac{C[1]}{t} \right\}$$

where $C[1]$ is a constant of integration. Since the best bid for a person whose true valuation is 0 is 0, it follows that $C[1] = 0$. The Nash equilibrium, then, is for each bidder to submit a bid equal to $1/2$ of the bidder's valuation, i.e., $b[v] = v/2$. As promised, this function is strictly increasing.