The Simple Mathematics of Optimization

There are two main approaches to solving the optimization problems that arise in Economics: the method of Substitution and the method of Lagrangian Multipliers. The method of Substitution is stressed in this class. While this handout illustrates the substitution method in the context of the consumer choice problem, the method will be applicable to optimization problems throughout the course.

The Standard Consumer Choice Problem

The standard optimization problem is to maximize a utility function subject to a budget constraint. More precisely, the problem is to choose quantities of two commodities denoted by \( x_1 \) and \( x_2 \), respectively, in order to maximize a function, \( u(x_1, x_2) \) subject to the requirement that the chosen quantities satisfy the budget constraint.

In: budget = \( p_1 x_1 + p_2 x_2 = m \);

The values of \( x_1 \) and \( x_2 \) are to be chosen to solve the maximization problem. Their values will thus be determined "inside the model" and they are, accordingly, called endogenous variables. The prices of the commodities, \( p_1 \) and \( p_2 \), and the person's income, \( m \), are determined "outside the model" and are called exogenous variables. The exogenous variables can be thought of as constants.

The Substitution Method

This approach first asks a question and then, depending upon the answer, proceeds through a series of specific steps.

Question: Will the budget constraint hold as an equality in the optimal solution? The answer to this is "yes" either if the partial derivative of the utility function wrt \( x_1 \) or if the partial derivative of the utility function wrt \( x_2 \) is positive. Why? If either of these marginal utilities is positive then the optimal choice could not lie below the budget constraint since utility could be made greater by increasing the consumption of which ever commodity had a positive marginal utility. And, as noted by Sherlock Holmes, "When you have eliminated the impossible, whatever remains must be the truth."

Suppose, for example, that the utility function is

\[ \text{utility} = x_1 x_2; \]

Then since the partial derivatives

In: \( D[\text{utility}, x_1] \)

Out: \( x_2 \)

and

In: \( D[\text{utility}, x_2] \)
are both positive when $x_1$ and $x_2$ are positive, it follows that the budget constraint must hold as an equality.

If the budget constraint holds as an equality, the specific steps to find the optimal solution are as follows:

1. Solve the budget constraint for either of the endogenous variables, $x_1$ or $x_2$. Supposing you decide to solve for $x_2$, you would get

$$x_2 = \frac{m - p_1 x_1}{p_2}$$

as the amount of $x_2$ that can just be afforded with the given amount of $x_1$. This, of course, is just the equation of the budget constraint in slope-intercept form.

2. Substitute this result into the utility function. The resulting expression shows how utility varies with $x_1$ as one moves along the budget constraint. Continuing the earlier example, this would give

$$u_{budget} = \frac{x_1 (m - p_1 x_1)}{p_2}$$

Note that solving the budget constraint for $x_2$ and substituting the result into the utility function eliminates $x_2$ from the utility function. Now let’s plot some indifference curves for $u$, the budget constraint and $u_{budget}$ for some specific values of prices and income to see where these steps are taking us.

In: values = \{p_1 \rightarrow 2, p_2 \rightarrow 3, m \rightarrow 60\};

In: Needs["Graphics'ImplicitPlot"]

In: budget/values

Out: $2x_1 + 3x_2 == 60$

In: ImplicitPlot[\{utility == 90, utility==120, utility == 150, 2x_1 + 3x_2 == 60\}, \{x_1, 0, 30\}, \{x_2, 0, 30\}, PlotStyle \rightarrow \{Hue[0.6], Hue[0.6], Hue[0.6], Hue[0.0]\}, AxesLabel \rightarrow \{"x_1", "x_2"\}, PlotLabel \rightarrow "Figure 1";]

Figure 1
Note that the tangency of the highest indifference curve to the budget constraint in Figure 1 corresponds exactly to the "top of the hill" in Figure 2. The top of the hill is, of course, where the first derivative (slope) is equal to zero. This observation leads directly to step 3:

3. **Differentiate** the utility function with respect to the remaining endogenous variable (one endogenous variable was eliminated in step 2), set the result equal to zero and solve for the optimal value of the remaining endogenous variable. For our example

\[
D\{\text{ubudget}, x_1\} = 0
\]

This gives the optimal value of the remaining endogenous variable, \(x_1\), in terms only of the exogenous variables.

4. **Substitute** the optimal value for the given endogenous variable into the result of step 1 to get the optimal value for the other endogenous variable

\[
D\{\text{affordable}_2/.\text{best}_1\}
\]

The expressions for \(\text{best}_1\) and \(\text{best}_2\) give the optimal quantities of the endogenous variables, \(x_1\) and \(x_2\), as functions of the exogenous variables. This is called the reduced form of the model and is regarded as perhaps the most useful form of the model. [In the context of consumer choice, the reduced form is also known as the consumer’s demand functions.] Just stick in the values of the exogenous variables on the right hand side and out pop the predicted (optimal) values of the endogenous variables on the left hand side. The predicted (optimal) quantities for the specific values identified above, for example, are

\[
\{x_1 \rightarrow 15\}, \{x_2 \rightarrow 10\}
\]
The Case of Perfect Substitutes

The substitution method even gives insights into cases in which steps 3 and 4 won’t work. For perfect substitutes, the utility function is

\[
\text{utility} = x_1 + x_2;
\]

Since both partial derivatives are positive

\[
\{D[\text{utility}, x_1], D[\text{utility}, x_2]\}
\]

the budget constraint must hold as an equality in the optimal solution. Proceeding as before, we solve the budget constraint and substitute into the utility function to obtain

\[
x_1 \cdot \frac{m - p_1 x_1}{p_2}
\]

Now plot the budget (red) constraint for the specific values defined above and some representative indifference curves (blue):

\[
\text{ImplicitPlot}[[\text{utility} == 20, \text{utility} == 30, \text{utility} == 40, 2x_1 + 3x_2 == 60], \{x_1, 0, 30\}, \{x_2, 0, 30\}, \text{PlotStyle} \rightarrow \{\text{Hue}[0.6], \text{Hue}[0.6], \text{Hue}[0.6], \text{Hue}[0.0]\}, \text{AxesLabel} \rightarrow \{"x_1", "x_2"\}, \text{PlotLabel} \rightarrow "\text{Figure 3}"];
\]

and note that there is a corner solution for this problem in which \(x_1=30\) and \(x_2=0\). This is confirmed by plotting the result of \(\text{ubudget}\)

\[
\text{uspecific} = \frac{1}{3}(60 - 2x_1) + x_1
\]

\[
\text{Plot}[x_1 + (60 - 2x_1)/3, \{x_1, 0, 30\}, \text{AxesLabel} \rightarrow \{"x_1", "\text{util}"\}, \text{PlotLabel} \rightarrow "\text{Figure 4}"];
Trying to solve for a value of $x_1$ that makes the derivative (slope) of this function equal to zero won’t get us anywhere since the derivative is always positive. But the fact that the derivative is positive suggests that the solution is to consume as much $x_1$ as possible. More generally, the fact that

\[ \text{Out: } 1 - \frac{p_1}{p_2} \]

implies that the derivative is positive (and the solution is to consume only $x_1$) when $p_1 < p_2$ and that the derivative is negative (and the solution is to consume no $x_1$) when $p_1 > p_2$.

The Case of Perfect Complements

The substitution method gives insights for the case of perfect complements even though steps 3 and 4 once again won’t work. Here utility is given by

\[ \text{In: utility = Min}[x_1, x_2]; \]

The marginal utilities for this case are more complicated. When $x_1 < x_2$, this function is equal to $x_1$ which means that the partial derivative wrt $x_1$ is equal to 1 and the partial derivative wrt $x_2$ is equal to 0. Alternatively, when $x_1 > x_2$, this function is equal to $x_2$, the partial derivative wrt $x_1$ is equal to 0 and the partial derivative wrt $x_2$ is equal to 1. When $x_1 = x_2$ neither partial derivative is defined. Still, as the plot below illustrates, in the optimal solution the budget constraint will hold as an equality.

\[ \text{In: ImplicitPlot}[[\text{utility} == 8, \text{utility} == 12, \text{utility} == 16, 2x_1 + 3x_2 == 60], \{x_1, 0, 30\}, \{x_2, 0, 30\}, \text{PlotPoints} \to 100, \text{PlotStyle} \to \{\text{Hue}[0.6], \text{Hue}[0.6], \text{Hue}[0.6], \text{Hue}[0.0]\}, \text{AxesLabel} \to \{"x_1", "x_2"\}, \text{PlotLabel} \to "\text{Figure 5}"]; \]
Since the budget constraint is binding we can proceed with Step 1

in: `ubudget = utility/.affordable_2`

Out: `Min[x_1, (m - p_1 x_1) / p_2]`

in: `uspecific = ubudget/.values`

Out: `Min[1/3 (60 - 2x_1), x_1]`

in: `Plot[Min[x_1, (60 - 2x_1)/3], {x_1, 0, 30}, AxesLabel -> {"x_1", "util"}, PlotLabel -> "Figure 6"]`

The derivative of this function is never equal to zero, but it is apparent, nevertheless, that the maximum occurs at the kink where

in: `kink = x_1 == (60 - 2x_1)/3;`

or where

in: `Solve[kink, x_1][[1]]`

Out: `{x_1 -> 12}`

More generally the solution occurs where

in: `kink = x_1 == x_2/.affordable_2`
Out: $x_1 = \frac{m - p_1 x_1}{p_2}$

or where

in: Solve[kink, x_1][[1]]

Out: \{x_1 \rightarrow \frac{m}{p_1 + p_2}\}