1 Brittle and Ductile components

Brittle Components - do not sustain load while in a failure condition

Ductile Components - can sustain load while in a failure condition

2 Series Systems with Uncorrelated and Unequal Resistances

one component fails $\rightarrow$ entire system fails

(Examples: chain, statically determinate truss, most electronics)

For brittle or ductile components, if resistances are independent in each component, then the CDF of the system resistance, $F_{R_s}(r)$, is

$$F_{R_s}(r) = P[R_s \leq r] = 1 - P[R_s > r] = 1 - P[R_1 > r \cap R_2 > r \cap \ldots \cap R_n > r]$$

$$= 1 - (1 - F_{R_1}(r_1))(1 - F_{R_2}(r_2))\cdots(1 - F_{R_n}(r_n))$$

$$= 1 - \Pi(1 - F_{R_i}(r_i))$$

$$= 1 - (1 - F_R(r))^n \text{ (only if each component has identically distributed resistance)}$$

$$= 1 - P[ \text{all resistances} > \text{all loads}]$$

System Reliability $< \text{Component Reliability}$

Series systems with uncorrelated resistances are safest if each component is identically distributed

3 Parallel Systems with Uncorrelated and Unequal Resistances

3.1 Ductile Components

all components fail $\rightarrow$ entire system fails

$$R_s = \sum_{i=1}^{n} R_i$$

If resistances $R_i$ are independent, $\mu_{R_s} = \sum \mu_{R_i}$ and $\sigma^2_{R_s} = \sum \sigma^2_{R_i}$.

central limit theorem: $R_s$ becomes normally distributed as $n$ gets large, $R_s \sim N(\mu_{R_s}, \sigma^2_{R_s})$

If resistances $R_i$ are independent and identically distributed (i.i.d.), then $\mu_{R_s} = n\mu_{R_i}$ and $\sigma^2_{R_s} = n\sigma^2_{R_i}$, and the c.o.v. of $R_s$ is smaller than the c.o.v. of $R_i$, $V_{R_s} < V_{R_i}$.

3.2 Brittle Components

one component fails $\rightarrow$ re-distribution of loads to remaining components

If $R_1 < R_2 < \cdots < R_n$, then $R_s = \max(nR_1, (n-1)R_2, (n-2)R_3, \ldots, R_n)$.

central limit theorem: $R_s$ becomes normally distributed as $n$ gets large.
4 System Reliability Bounds for Perfectly Correlated and Uncorrelated Component Resistances

Components have one of two states: $S$ (safe) or $F$ (failed), $S \in [0, 1]$, $F \in [0, 1]$, $F = 1 - S$.

The system safety depends on component states, $S_s(S_1, S_2, \cdots, S_n)$, or, equivalently, $S_s(F_1, F_2, \cdots, F_n)$.

series system:

$$S_s = S_1 S_2 \cdots S_n = \Pi S_i$$  \hspace{1cm} (8)
$$= (1 - F_1)(1 - F_2) \cdots (1 - F_n)$$  \hspace{1cm} (9)
$$= \Pi(1 - F_i)$$  \hspace{1cm} (10)

parallel system:

$$S_s = 1 - \Pi F_i = 1 - \Pi (1 - S_i)$$  \hspace{1cm} (11)

For a single component, $i$,

$$\mathbb{E}[S_i] = 1 \cdot P[S_i = 1] + 0 \cdot P[S_i = 0] = P[S_i = 1]$$  \hspace{1cm} (12)
$$\mathbb{E}[F_i] = 1 \cdot P[F_i = 1] + 0 \cdot P[F_i = 0] = P[F_i = 1]$$  \hspace{1cm} (13)

For a system,

$$P_F = 1 - P[S_s(F_1, \cdots, F_n) = 1] = 1 - \mathbb{E}[S_s(F_i)]$$  \hspace{1cm} (14)

For a series system with independent component resistances,

$$P[S_1 = 1 \cap S_2 = 1 \cap \cdots \cap S_n = 1] = P[S_1 = 1 \cap S_2 = 1 \cap \cdots \cap S_n = 1] P[S_n = 1]$$  \hspace{1cm} (15)

$$P_F = P[S_1 = 1 \cup S_2 = 1 \cup \cdots \cup S_n = 1]$$  \hspace{1cm} (16)
$$= 1 - P[S_1 = 1 \cap S_2 = 1 \cap \cdots \cap S_n = 1]$$  \hspace{1cm} (17)
$$\leq 1 - P[S_1 = 1] P[S_2 = 1] \cdots P[S_n = 1]$$  \hspace{1cm} (18)
$$= \Pi(1 - P[F_i = 1])$$  \hspace{1cm} (20)

But, $P_F \geq \max P[F_i = 1]$, so . . .

4.1 Series System Bounds

<table>
<thead>
<tr>
<th>perfect correlation</th>
<th>no correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max (P[F_i = 1])$</td>
<td>$P_F$ $\leq$ $1 - \Pi (1 - P[F_i = 1])$</td>
</tr>
<tr>
<td>$\min \beta_c$</td>
<td>$\beta_s$ $\geq$ $-\Phi^{-1}(1 - \Pi (1 - \Phi(-\beta_c)))$</td>
</tr>
<tr>
<td>$\beta_s = \min \beta_c$</td>
<td>$\beta_s &lt; \beta_c$</td>
</tr>
</tbody>
</table>

least safe component determines system reliability

4.2 Parallel System Bounds

<table>
<thead>
<tr>
<th>perfect correlation</th>
<th>no correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi P[F_i = 1]$</td>
<td>$\leq$ $P_F$ $\leq$ $\min (P[F_i = 1])$</td>
</tr>
<tr>
<td>$-\Phi^{-1}(\Pi(\Phi(-\beta_c)))$</td>
<td>$\geq$ $\beta_s$ $\geq$ $\max \beta_c$</td>
</tr>
<tr>
<td>$\beta_s &gt; \beta_c$</td>
<td>$\beta_s = \max \beta_c$</td>
</tr>
</tbody>
</table>

system fails if all components fail

safest component determines system reliability
5 Reliability of Systems with Equally-Correlated Components and Equal Component Reliability

Consider a system with \( n \) components having resistances \( R_1, R_2, \ldots, R_n \) where \( R_i \) are normally distributed random variables and the correlation coefficient between \( R_i \) and \( R_j \) is \( \rho \) for any \( i, j, i \neq j \).

Assume loads are deterministic and constant in time and all components are designed in such a way that each components has the same reliability index \( \beta_c \), \( \beta_c = (\mu_{R_i} - Q_i)/\sigma_{R_i} \), or \( \mu_{R_i} = Q_i + \beta_c \sigma_{R_i} \); \( V_i = \sigma_{R_i}/\mu_{R_i} \); and \( \mu_{R_i} = Q_i/(1 - \beta_c V_i) \).

5.1 Series systems with equal component correlation and reliability

\[
P_F = 1 - \int_{-\infty}^{\infty} \left[ \Phi \left( \frac{\beta_c + \sqrt{\rho}z}{\sqrt{1-\rho}} \right) \right]^n \phi(z) \, dz. \quad \text{(Stuart’s formula)} \tag{21}
\]

The greater the number of components in the system, the lower the system reliability.

5.2 Parallel systems with equal component correlation and and reliability

For \( n \) ductile components with equally distributed resistances, \( R_i \sim f(\mu_{R_i}, \sigma_{R_i}), R_s = \sum R_i = nR \),

\[
\mu_{R_s} = \sum_{i=1}^{n} \mu_{R_i} = n \mu_R, \tag{22}
\]

and

\[
\sigma_{R_s}^2 = \sum_{i=1}^{n} \sigma_{R_i}^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho_{ij} \sigma_{R_i} \sigma_{R_j} \tag{23}
\]

\[
= \sum_{i=1}^{n} \sigma_{R_i}^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho \sigma_R^2 \tag{24}
\]

\[
= n\sigma_R^2 + 2 \rho \sigma_R^2 \frac{n(n-1)}{2} \tag{25}
\]

\[
= n\sigma_R^2 (1 + \rho(n-1)) \tag{26}
\]

For each component, \( \beta_c = (\mu_{R_i} - Q_i)/\sigma_{R_i} \), so \( Q_i = \mu_{R_i} - n\beta_c \sigma_{R_i} \), and the total load on a equally-distributed parallel system with component reliability \( \beta_c \) is \( nQ_i = n\mu_{R_i} - n\beta_c \sigma_{R_i} \).

The system reliability index is therefore given by

\[
\beta_s = \frac{\mu_{R_s} - Q_s}{\sigma_{R_s}} = \frac{n\mu_R - (n\mu_R - n\beta_c \sigma_R)}{\sqrt{n\sigma_R^2(1 + \rho(n-1))}} = \beta_c \sqrt{\frac{n}{1 + \rho(n-1)}} \tag{28}
\]
6 Reliability of Systems with Unequally-Correlated Components and Equal Component Reliability

6.1 Parallel systems

Consider a system with \( n \) components with identically distributed resistances \( R_i \sim f(\mu_R, \sigma_R) \) carrying a deterministic load \( Q \) constant in time. The component reliability, \( \beta_c \), is the same for each component. Given the matrix of correlation coefficients, \( \rho_{ij} \), \( (\rho_{ij} = \rho_{ji}, \text{and } \rho_{ii} = 1) \), the reliability index for the system is

\[
\beta_s = \frac{\mathbb{E}[R_s] - Q}{\sigma_{R_s}}
\]

(29)

where \( \mathbb{E}[R_s] = \mu_{R_s} = n\mu_R \) and

\[
\sigma_{R_s}^2 = n\sigma_R^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho_{ij} \sigma_R \sigma_{R_j}
\]

(30)

\[
= n\sigma_R^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho_{ij} \sigma_R \sigma_{R_j}
\]

(31)

\[
= \sigma_R^2 \left( n + \sum_{i,j=1 \atop i \neq j}^{n} \rho_{ij} \right)
\]

(32)

from which

\[
\beta_s = \frac{n\mu_R - (n\mu_R - \sigma_R\beta_c n)}{\sigma_{R_s}} = \frac{\sigma_R \beta_c n}{\sigma_R \sqrt{n + \sum_{i,j=1 \atop i \neq j}^{n} \rho_{ij}}} = \beta_c \sqrt{n \left( 1 + \frac{1}{n} \sum_{i,j=1 \atop i \neq j}^{n} \rho_{ij} \right)}
\]

(34)

where \( \bar{\rho} \) is the average of all off-diagonal elements \( \rho_{ij} \)

\[
\bar{\rho} = \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \rho_{ij}
\]

(35)

6.2 Series systems

The failure probability of a series system with \( n \) components can be approximated by equation (21) in which the uniform component resistance correlation \( \rho \) is replaced by the average correlation value, \( \bar{\rho} \), from equation (35).

This approximation is good for smaller values of \( n \). A much better approximation is obtained from

\[
P_F = P_F(\bar{\rho}) - P_{F_2}(\bar{\rho}) - P_{F_2}(\rho_{\max})
\]

(36)

where \( P_F(\bar{\rho}) \) is equation (21) evaluated with \( \bar{\rho} \) from equation (35); \( P_{F_2}(\bar{\rho}) \) is equation (21) evaluated for \( n = 2 \) and \( \bar{\rho} \) from equation (35) (any two elements may be used); and \( P_{F_2}(\rho_{\max}) \) is equation (21) evaluated for \( n = 2 \) and \( \rho_{\max} = \max(\rho_{ij}), \forall i \neq j \).

6.3 Summary for systems with equal component reliability

<table>
<thead>
<tr>
<th>Series Systems</th>
<th>Parallel Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>uncorrelated component resistance</td>
<td>( \beta_s &lt; \beta_c )</td>
</tr>
<tr>
<td>correlated component resistance</td>
<td>( \beta_s = \beta_c )</td>
</tr>
<tr>
<td>uncorrelated component resistance</td>
<td>( \beta_s = \beta_c )</td>
</tr>
</tbody>
</table>

4
7 Problems

1. Consider chains with two links carrying a deterministic load of 1.1 kN. The strength of each link, $R_c$, is uniformly distributed between 1 and 3 kN. Derive the CDF of the strength of the system, $F_{R_s}(r)$ for the three regions $r < 1$, $1 \leq r < 3$ and $3 \leq r$, and plot $F_{R_s}(r)$. What is the reliability index of a link? What is the reliability index of the chain? What is the probability of failure of a link? What is the probability of failure of the chain? Find the system failure probability and the system reliability index of such chains with 1, 2, 3, 5, and 10 links.

2. Consider a system of three ductile components. Components 1 and 2 are in parallel, and this parallel sub-system is in series with component 3. The strengths of components 1 and 2 ($R_1$ and $R_2$) are normally distributed with mean of 5 kN and a standard deviation of 1 kN. The strength of component 3 is uniformly distributed between 8 kN and 12 kN. The component strengths are statistically independent. Derive the CDF of the system strength, $F_{R_s}(r)$.

3. A bridge-truss is made of three panels with ductile cross-bracing in each panel. This truss is internally statically indeterminate. Assume that only the cross bracing is susceptible to failure. Express this system as a combination of parallel and series cross-brace components.

4. The component reliabilities, $\beta_c$, of ductile components of a two-component series system is given by $\beta_c = 3$. Determine bounds for the system failure probability.

5. Compute and plot equation (21) as $P_F$ vs. $\rho$ for $0 < \rho < 1$, for $n=1, 2, 4, \text{ and } 10$, and $\beta_c = 3.0$. Make a second plot for $\beta_c = 2.0$. For $\beta_s = -\Phi^{-1}(P_F)$, plot $\beta_s/\beta_c$ vs. $\rho$ for $0 < \rho < 1$ and $\beta_c = 3.0$. Make another plot for $\beta_c = 2.0$. Describe the meaning of the plots.

6. Compute and plot equation (28) as $\beta_s/\beta_c$ vs. $\rho$ for $0 < \rho < 1$ and $n=1, 2, 5, \text{ and } 10$.

7. Consider a statically determinate truss with seven bars. Let $\beta_c = 3$ for all bars and $\rho = 0.85$ between bars. Compute $P_F$ for a bar and for the truss.

8. For the bridge truss described in problem 3, consider each panel as a two-component parallel system and the truss as a series system with three panels. Assume that the cross-braces have equal component correlation, $\rho = 0.8$, and reliability, $\beta_c = 2.85$. Compute the reliability index of each panel, $\beta_p$.

The panel strengths therefore have equal correlation $\rho_p$ and equal panel reliability $\beta_p$. Derive an expression for the correlation between the strengths of two panels, $\rho_p$, using the hints below:

If $R_{d1}$ and $R_{d2}$ are the strengths of two diagonals, note that $\nabla(R_{d1}, R_{d2}) = \rho \sigma_{R_d}^2 = \mathbb{E}[R_{d1}R_{d2}] - \mu_{R_d}^2$.

If $R_{p1}$ and $R_{p2}$ are the strengths of two panels and $R_{p1} = R_{d1} + R_{d2}$ and $R_{p2} = R_{d3} + R_{d4}$, show that $\mathbb{E}[R_{p1}R_{p2}] = 4(\rho \sigma_{R_d}^2 + \mu_{R_d}^2)$. Also, note that $\nabla(R_{p1}, R_{p2}) = \rho_p \sigma_{R_p}^2 = \mathbb{E}[R_{p1}R_{p2}] - \mu_{R_p}^2$. Use these expressions to derive a relation for $\rho_p$ in terms of $\rho$.

Compute the reliability index of the truss as series system of three panels with equal reliability $\beta_p$ and correlation $\rho_p$. 
9. Consider a system of five parallel ductile components with a component strength correlation matrix of:

\[
\rho = \begin{bmatrix}
1 & 0.5 & 0.2 & 0.1 & 0.0 \\
0.5 & 1 & 0.5 & 0.2 & 0.1 \\
0.2 & 0.5 & 1 & 0.5 & 0.2 \\
0.1 & 0.2 & 0.5 & 1 & 0.5 \\
0.0 & 0.1 & 0.2 & 0.5 & 1
\end{bmatrix}
\]  \hspace{1cm} (37)

Assume \( \beta_c = 3.5 \), and all component strengths are normally distributed. What is the system reliability index? What is the system reliability index of all correlations are 0.0? What is the system reliability index of all correlations are 1.0?

10. Consider a system of five series ductile components with the same component strength correlation matrix as given in problem 9, and with \( \beta_c = 3.5 \). Compute the system failure probability using \( \bar{\rho} \) from equation (35) in equation (21). Compute the system failure probability from equation (36) What is the system reliability index of all correlations are 0.0? What is the system reliability index of all correlations are 1.0?

11. Consider a parallel system of \( n \) ductile components with identically distributed component strengths \( R_i \sim f(\mu_R, \sigma_R) \). If the component strengths are perfectly correlated show that \( \sigma_{R_s} = n\sigma_R \).

Use this fact to show that the limits on the coefficient of variation of the system strength, \( V_{R_s} \), has the bounds \( V_R/\sqrt{n} \leq V_{R_s} \leq V_R \) where \( V_R = \sigma_R/\mu_R \).

Does the lower bound correspond to the perfectly-correlated case or the uncorrelated case?

12. For the system of problem 2, the system load is static and deterministic with a value \( Q = 3 \) kN, and the strength of component 3 is lognormally distributed with \( \mu_{R_3} = 9 \) kN and \( \sigma_{R_3} = 1 \) kN.

Determine \( P_F \) of the system when \( R_1, R_2, \) and \( R_3 \) are uncorrelated.

Determine \( P_F \) when \( R_1, R_2, \) and \( R_3 \) are perfectly correlated.

Determine \( P_F \) when \( R_1 \) and \( R_2, \) are perfectly correlated, but \( R_3 \) is uncorrelated with \( R_1 \) and \( R_2 \).