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REVIEW ARTICLE

A survey on approaches for reliability-based optimization

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Abstract Reliability-based Optimization is a most appropriate and advantageous methodology for structural design. Its main feature is that it allows determining the best design solution (with respect to prescribed criteria) while explicitly considering the unavoidable effects of uncertainty. In general, the application of this methodology is numerically involved, as it implies the simultaneous solution of an optimization problem and also the use of specialized algorithms for quantifying the effects of uncertainties. In view of this fact, several approaches have been developed in the literature for applying this methodology in problems of practical interest. This contribution provides a survey on approaches for performing Reliability-based Optimization, with emphasis on the theoretical foundations and the main assumptions involved. Early approaches as well as the most recently developed methods are covered. In addition, a qualitative comparison is performed in order to provide some general guidelines on the range of applicability on the different approaches discussed in this contribution.

Keywords Reliability-based optimization · Reliability assessment · Approximate reliability methods · Advanced simulation methods

1 Introduction

Reliability-based Optimization (RBO) is a methodology that allows solving optimization problems while explicitly modeling the effects of uncertainty; these effects are accounted for by means of probabilities of occurrence and expected values. RBO constitutes a most powerful methodology for solving problems in structural design. This is due to the fact that in practical situations, one is often interested in determining the structural configuration that *optimizes* a certain predefined criterion (e.g. construction costs, benefits, etc.) while taking into account the unavoidable uncertainties in the structural performance.

Despite the evident advantages of RBO over deterministic design procedures, its application to problems of engineering interest can be quite challenging, i.e. due to high numerical costs involved in its solution. Both, optimization and reliability assessment require the repeated evaluation of the structural response for different sets of design variables and uncertain parameters; in turn, the evaluation of the structural response may require the computation of numerically involved virtual simulation models (e.g. Finite Element models). In view of this issue, several different tools have been developed for solving RBO problems efficiently. For example, the development of *approximate reliability methods* (see, e.g. Breitung 1994; Ditlevsen and Madsen 1996; Rackwitz 2001) and *advanced simulation methods* (see, e.g. Schuëller et al. 2005) allow the estimation of probabilities of failure and expected costs most efficiently. The application of meta-modeling techniques has allowed replacing numerically intensive virtual simulation models by inexpensive ones (see, e.g. Hurtado 2004; Jin et al. 2003; Papadrakakis and Lagaros 2002; Zhang and Foschi 2004). The introduction of efficient strategies and approximation concepts also play a fundamental role in yielding challenging RBO problems tractable (see, e.g. Royset and Polak 2004b; Du and Chen 2004; Chen et al. 1997). In addition, the advent of High Performance Computing (HPC) and—in particular—the application of parallel computing techniques has opened the possibility of performing

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demanding numerical simulations in reduced time, see e.g. Johnson et al. (2003), Leite and Topping (1999), Pellissetti (2009), Thierauf and Cai (1997) and Umesha et al. (2005).

In view of the scenario described above, the objective of this contribution is to presenting an overview of different techniques developed in the literature for solving RBO problems. Nonetheless, the field of optimization under uncertainties is quite vast. Therefore, issues addressed in this contribution are confined to selected topics. Thus, the emphasis is on *methods* for solving problems and not on practical applications, as in applications of RBO, it is usually necessary to deal with the particulars of the underlying physical problem, see e.g. Ellingwood (2001), Hellevik et al. (1999), Madsen et al. (1991), Moan and Song (2000) and Petryna and Krätzig (2005); similarly, aspects on parallel computing are not treated in this contribution. Moreover, this contribution considers approaches using classical probabilities only; however, it should be noted that non-classical approaches (see, e.g. Moens and Vandepitte 2005; Möller and Beer 2007) have been applied as well for problems of optimization under uncertainties, see e.g. Beer and Liebscher (2008) and De Munck et al. (2008).

This paper is organized as follows. Section 2 provides a general description of the RBO problem as well as the challenges involved by its solution. Section 3 summarizes some assumptions made throughout this contribution. Methods for solving optimization problems considering uncertainties are presented in Sections 4, 5 and 6. The different methods for RBO discussed in these Sections are categorized according to the reliability method that is applied to account for the effects of uncertainties. The reason for proposing this categorization is that the application of one particular class of methods for assessing structural reliability has a major impact in the type of RBO problems that can be analyzed; however, other classification criteria are certainly possible. Thus, Section 4 presents RBO approaches that apply simplification concepts in order to solve the associated reliability problem. Section 5 addresses approaches using approximate reliability methods while Section 6 focuses on approaches based on simulation techniques. After presenting and discussing the different methods for solving RBO problems, Section 7 provides a qualitative critical appraisal on these methods in order to provide some guidelines on their range of applicability and efficiency. Finally, Section 8 closes this contribution drawing some conclusions on the current status of methods for solving RBO problems and possible future research directions.

2 Description of the problem

RBO problems can be formulated in different ways (Moses and Kinser 1967; Enevoldsen and Sørensen 1994; Vanmarcke 1973); typical examples of such formulations include, e.g. minimization of (deterministic) construction costs under constraints including probability terms, minimization of the failure probability under fixed costs, minimization of expected life time costs considering maintenance costs and eventual failure, etc. The last formulation is of particular relevance in engineering, as it considers costs due to partial damage and structural collapse (Kupfer and Freudenthal 1977); in mathematical terms, this problem is defined as follows (Freudenthal 1956; Vanmarcke 1973; Royset et al. 2001b).

$$\min_{\boldsymbol{y}} E\left[C(\boldsymbol{y},\boldsymbol{\zeta})\right], \ \boldsymbol{y} \in \Omega_y \tag{1}$$

subject to

$$h_i(\boldsymbol{y}) \leq 0, \quad i = 1, \ldots, n_C \tag{2}$$

$$p_j(\boldsymbol{y}) \leq p_j^{tol}, \quad j = 1, \ldots, n_P \tag{3}$$

In the optimization problem above, $\boldsymbol{y}$ denotes the vector of design variables (of length n_y), which are those variables that can be selected among a certain set and that influence the performance of a structural system or trigger specific events; $\boldsymbol{\zeta}$ denotes the vector uncertain parameters; h_i are constraints of the problem (e.g. side constraints on $\boldsymbol{y}$); C is a cost function (which can eventually be a random variable depending on $\boldsymbol{\zeta}$) and $E[\cdot]$ is the expectation operator; finally, p_j denotes the probability of occurrence of the j-th event, which should be equal or smaller than a certain tolerable threshold p_j^{tol}. For the sake of simplicity, the index j is dropped in the remaining part of this publication (i.e. $n_P = 1$). However, it should be noted that in many problems n_P may be larger than 1. The terms $E\left[C(\boldsymbol{y},\boldsymbol{\zeta})\right]$ and $p(\boldsymbol{y})$ in (1) and (2), respectively, are defined by means of the multi-dimensional integrals shown below:

$$E\left[C(\boldsymbol{y},\boldsymbol{\zeta})\right] = \int_{g^*(\boldsymbol{y},\boldsymbol{\zeta})\leq 0} C(\boldsymbol{y},\boldsymbol{\zeta}) f(\boldsymbol{\zeta}/\boldsymbol{y}) d\boldsymbol{\zeta} \tag{4}$$

$$p(\boldsymbol{y}) = P\left[g(\boldsymbol{y},\boldsymbol{\zeta}) \leq 0\right] = \int_{g(\boldsymbol{y},\boldsymbol{\zeta})\leq 0} f(\boldsymbol{\zeta}/\boldsymbol{y}) d\boldsymbol{\zeta} \tag{5}$$

In (4) and (5), $f(\boldsymbol{\zeta}/\boldsymbol{y})$ is the joint probability density function associated with the vector of uncertain parameters $\boldsymbol{\zeta}$; it should be noted that the joint probability density function may depend on $\boldsymbol{y}$ in case the some of the parameters of the probability distributions (e.g. mean values) are considered as design variables. Additionally, in the aforementioned equations, $g^*(\boldsymbol{y},\boldsymbol{\zeta})$ and $g(\boldsymbol{y},\boldsymbol{\zeta})$ are two *performance functions* that are associated with the cost function and the probability of occurrence of a certain event, respectively. A performance function is a function used to model the performance objectives associated with a specific system.

It is defined such that it assumes a value smaller or equal to zero when a specific realization of the vector of uncertain parameters and a set of the design variables causes an unacceptable performance of the system; otherwise, the performance function assumes a value larger than zero. In other words, the role of the performance function is comparing the *capacity of a structure* with the *demand* (Freudenthal 1956); this comparison may involve any relevant indicator derived from a virtual simulation, e.g. stresses, displacements, stress intensity factors, etc. In order to illustrate the definition of a performance function, consider a particular structural response $r(\boldsymbol{y}, \boldsymbol{\zeta})$ and the maximum tolerable threshold level b associated with this response. Then, the performance function can be defined as follows.

$$g(\boldsymbol{y}, \boldsymbol{\zeta}) = b - r(\boldsymbol{y}, \boldsymbol{\zeta}) \tag{6}$$

It is important to note that several performance functions may be required in order to consider all possible mechanisms that may lead to an unacceptable behavior (Moses 1997). It is also important to mention that the locus of realizations of the uncertain parameters and values of the design variables for which the performance function is equal to zero is the so-called *limit state function*.

The solution of a RBO problem is numerically demanding for cases of practical interest. This is due to the fact that it is necessary to assess expected values and probabilities within the optimization algorithm (cf. (1) and (2), respectively). For a better understanding of this last point, a schematic representation of a RBO problem is presented in Fig. 1. As shown in the figure, the outer loop of a RBO problem consists of an optimization algorithm. This algorithm explores the space of design variables in order to determine the best design solution. Starting from a certain initial design ($\boldsymbol{y}^{(1)}$), a candidate optimal design ($\boldsymbol{y}^{(k)}$) is generated based on some rules which are specific to the optimization algorithm being applied. The solution of the underlying optimization problem (either using a gradient-based or gradient-free algorithm) may require several cycles of evaluations of the objective function and constraints for different

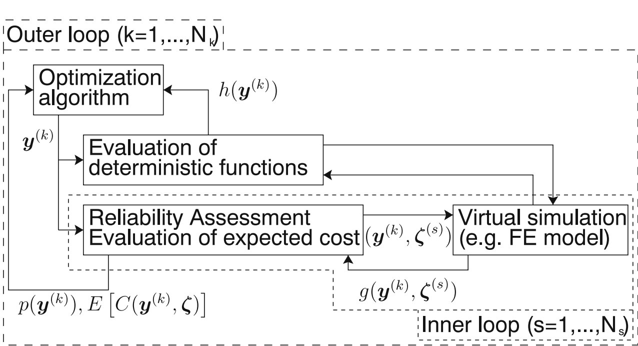


Fig. 1 Schematic representation of a RBO problem

sets of the design variables. Each of these cycles demands the computation of multi-dimensional integrals (cf. (4) and (5)). In turn, the evaluation of these integrals requires simulating the virtual model for different realizations of the vector of uncertain parameters ($\boldsymbol{\zeta}^{(s)}, s = 1 \ldots, N_S$) in order to obtain the value of the performance function(s). Additionally, the simulation of the virtual model may be quite demanding as well, e.g. in the case of a large models including non linearities, considerable computational time can be required to compute the structural response.

From the discussion above, it is clear that a RBO problem is a *double-loop problem* (Enevoldsen and Sørensen 1994; Chen et al. 1997), i.e. the reliability evaluation algorithm is nested within the optimization loop. The numerical costs associated with such formulation are usually unaffordable (except by the case of academic examples). Therefore, methods for solving RBO problems seek the introduction of simplifications or special formulations for reducing the numerical efforts.

3 Structure of the paper and conventions

The remaining part of this contribution provides an overview on different techniques for solving RBO problems efficiently. As already mentioned in the Introduction, these techniques are categorized according to the approach that is used to solve the associated reliability problem. In particular, Section 4 presents those RBO techniques where the underlying reliability problem is solved introducing appropriate simplifications such as, e.g. characterization of uncertainty using normal or log-normal distributions (as this allows solving analytically the problem of adding two independent normal variables or multiplying two independent log-normal random variables, respectively); approximation of functions by means of linearization; assumption of independence between different failure modes; simplified mechanical models, etc. Most of the RBO techniques that are presented in Section 4 were developed *approximately* between the years 1960 and 1980 and they are termed in this contribution as *early approaches*. Due to the simplifications introduced, these techniques led to solutions which were numerically inexpensive, as the computational power was quite limited at the time these approaches were developed. Sections 5 and 6 focus on RBO approaches that apply approximate reliability methods and simulation techniques, respectively. These approaches are usually numerically more intensive than those using simplifications, although they provide much more accurate reliability estimates. They were developed starting *approximately* from the year 1980. It is important to note that some of the approaches presented Sections 5 and 6 are described in more details as—in opinion of the authors—they have constituted important milestones in the development of RBO.

For the presentation of the different approaches for solving RBO problems, it is assumed that the readership is familiar with optimization algorithms and their capabilities, advantages and limitations. A review on these algorithms is outside the scope of this contribution; for more details on these techniques, it is referred to, e.g. Arora (1989, 2007), Goldberg (1989), Haftka and Gürdal (1992), Nocedal and Wright (1999) and Spall (2003). However, it should be noted that the selection of a particular optimization algorithm can be crucial for solving a particular RBO problem. For example, gradient-based optimization methods (such as quasi-Newton methods, see e.g. Bonnans et al. (2003) and Schittkowski (1983)) can be quite efficient for determining optimal solutions, although the estimation of gradients may become an issue and also there is the possibility of finding local optima only. On the contrary, stochastic search algorithms such as Genetic Algorithms (see, e.g. Goldberg 1989), Evolution Strategies (see, e.g. Beyer and Schwefel 2002; Thierauf and Cai 1997), Simulated Annealing (see, e.g. Kirkpatrick et al. 1983), etc. may be able of finding the global optimum of a specific RBO problem; moreover, this class of algorithms does not rely on gradient information. Nonetheless, stochastic search algorithms usually require much more function evaluations than gradient-based optimization methods.

As in the case of methods for optimization, it is also assumed that the readership is familiar with the state-of-the-art of methods for structural reliability analysis. For the sake of completeness, a brief description of these methods is included in Appendix A. Moreover, throughout this contribution it is assumed that the evaluation of deterministic constraints of a RBO problem is numerically less involved than the evaluation of probabilistic constraints. Therefore, deterministic constraints are omitted in the following for the sake of brevity. In the same manner, only a single probabilistic constraint is included in the description of different methods for solving RBO problems.

Finally, it is important to note that for the application of a number of methods for structural reliability and RBO, a common assumption is that the vector of uncertain parameters is composed by independent, Gaussian standard distributed random variables. For those cases where this condition is not satisfied, it is always possible to apply a suitable mapping (e.g. Nataf's model, Liu and Der Kiureghian (1986)) in order to ensure that the vector of uncertain parameters fulfills the aforementioned requirements. In order to maintain consistency throughout this contribution, the following notation is adopted. The vector $\boldsymbol{\zeta}$ denotes the random variables associated with a particular structural reliability problem, considering both correlations and non Gaussian distributions. The vector $\boldsymbol{\xi}$ is a mapping of the vector $\boldsymbol{\zeta}$ into the independent Gaussian space of random variables, i.e. $\boldsymbol{\zeta} = T_{\xi\zeta}(\boldsymbol{\xi})$; consequently, $\boldsymbol{\xi} \sim N(\boldsymbol{\mu}_{\boldsymbol{\xi}}, \boldsymbol{\sigma}_{\boldsymbol{\xi}})$, where $\boldsymbol{\mu}_{\boldsymbol{\xi}}$ and $\boldsymbol{\sigma}_{\boldsymbol{\xi}}$ are the vectors of the mean and standard deviation, respectively. Additionally, $\boldsymbol{\theta}$ is a mapping of the vector $\boldsymbol{\zeta}$ into the independent, Gaussian standard space of random variables, i.e. $\boldsymbol{\zeta} = T_{\theta\zeta}(\boldsymbol{\theta})$. The vector $\boldsymbol{\xi}$ can also be mapped into the independent, Gaussian standard space, i.e. $\boldsymbol{\xi} = T_{\theta\xi}(\boldsymbol{\theta})$; the mapping is such that $\xi_p = \mu_{\xi_p} + \theta_p \sigma_{\xi_p}$, $p = 1, \ldots, n_\zeta$.

4 Early approaches

Some of the first approaches for solving RBO problems that were developed focused on the minimization of the weight of a structural system under a constraint referring to probability of failure (Hilton and Feigen 1960). In such approaches, the overall failure probability is calculated considering that the probability of failure of the individual components of the system is independent. The optimization problem is solved using, e.g. Lagrange multipliers (Hilton and Feigen 1960; Silvern 1963). Thus, an optimality criterion was developed where the proportion between the weights of two components of a system should be equal to the proportion of the probability of failure of the components (Silvern 1963; Switzky 1965). This criterion was extended in Murthy and Subramanian (1968), where the probability of failure of a component was approximated by an exponential function, i.e.:

$$p(y) = c_0 e^{c_1 y} \tag{7}$$

where c_0, c_1 are constants and the dimension of the vector of design variables is equal to one, i.e. $n_y = 1$. Such an assumption led to a correction on the optimality criterion described above, by introducing a non linear term related with probabilities.

In Moses and Kinser (1967), it is demonstrated that the way multiple failure modes are considered for calculating system reliability may seriously affect the optimization of the weight of a structure. In particular, it is shown that ignoring the correlation between these failure modes may cause overestimation of the probability of failure, thus leading to design solutions with increased weight. This drawback can be overcome by considering the correlation between some (but not all) the failure modes (Moses 1997); such an approximation yields more accurate estimates of the probability of failure while providing more economic final designs.

The possibility of applying Monte Carlo Simulation (MCS) for assessing reliability within the context of RBO was also discussed in the literature (see, e.g. Broding et al. 1964). In spite of the advantages of MCS for treating highly non linear problems involving non Gaussian random variables, other possibilities for assessing reliability were investigated, as the numerical costs of MCS could be unaffordable. In particular, the possibility of calculating approximate reliability using the linear perturbation method was explored in Broding et al. (1964) for minimizing the weight of a composite plate considering thermal stresses. In this approach, the safety factor associated with a particular structure is approximated by means of a first-order Taylor series expansion. In addition, the weight function is also approximated using a linear expansion. Thus, the optimization problem can be solved efficiently using the simplex method (see, e.g. Haftka and Gürdal 1992).

Besides the minimization of weight, another problem studied in the literature has been the minimization of expected costs. In Heer and Yang (1971), the minimization of the costs associated with testing and eventual degradation of a pressure vessel belonging to a spacecraft was analyzed. In Vanmarcke (1973), an approach for minimizing construction costs and eventual failure costs of a structural system was formulated. In this approach, the interaction between different failure modes is accounted for using correlation coefficients. Moreover, an efficient formulation which generated two subsets of failure events denoted as *basis* and *remainder* allowed the efficient computation of lower and upper bounds for the solution of the optimization problem.

Another approach that has been applied for the solution of RBO problems is the so-called *chance constrained programming* (see, e.g. Charnes and Cooper 1959). In this approach, the performance function is linearized around the mean value of the random variables involved in the reliability problem. Thus, the reliability can be estimated approximately by means of an explicit formula (see, e.g. Ditlevsen and Madsen 1996). This method has been applied in several studies in the field of RBO, e.g. Davidson et al. (1977), Rao (1980) and Jóźwiak (1986).

5 Solution methods applying approximate reliability techniques

This section presents different methods for solving RBO problems which apply approximate reliability techniques. These techniques include the First and Second Order Reliability Methods (FORM and SORM, respectively) and the Dimension Reduction Method (DRM). Two important parameters that are involved in the application of these techniques are the so-called *design point* $\boldsymbol{\theta}^*$ and its norm—the so-called *reliability index* β. Details on these parameters are discussed in Appendix A.

The methods for solving RBO problems presented in this section are organized in three groups: double-loop approaches, single-loop approaches and decoupling approaches. This classification has been suggested in a number of publications, see e.g. Aoues and Chateauneuf (2010), Bichon et al. (2009) and Chen et al. (1997). Nonetheless, other classification criteria could be applied as well.

5.1 Double-loop implementation

The most direct approach for solving a RBO problem is implementing a double-loop approach, i.e. estimation of the structural reliability for each set of design variables evaluated by the optimization algorithm. In case FORM is applied for reliability analysis, there are two nested optimization cycles, as the assessment of probability is equivalent to the solution of an optimization problem (in order to identify the design point). Such an approach was followed in, e.g. Nikolaidis and Burdisso (1988), where a deterministic function $C(\boldsymbol{y})$ is minimized subject to a probabilistic constraint. Thus, the optimization problem is expressed as follows.

$$\min_{\boldsymbol{y}} C(\boldsymbol{y}) \tag{8}$$

subject to

$$\beta(\boldsymbol{y}) - \beta^{tol} \geq 0 \tag{9}$$

Equation (9) corresponds to the probabilistic constraint, where $\beta(\cdot)$ is the reliability index associated with the probability of failure of the structure and β^{tol} is the minimum acceptable reliability index, which is defined as:

$$\beta^{tol} = \Phi^{-1}(1 - p^{tol}) \tag{10}$$

where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cumulative density function. The reliability index $\beta(\cdot)$ is equal to $||\boldsymbol{\theta}^*||$, where $\boldsymbol{\theta}^*$ is the solution of the following optimization problem (see also Appendix A).

$$\min_{\boldsymbol{\theta}} ||\boldsymbol{\theta}|| \tag{11}$$

subject to

$$g(\boldsymbol{y}, T_{\theta\zeta}(\boldsymbol{\theta})) \leq 0 \tag{12}$$

As the formulation of the probabilistic constraint in (9) applies the reliability index, it has been denoted in the literature as the *Reliability Index Approach* (RIA, see e.g. Tu et al. 2001).

A key issue for the efficient implementation of the double-loop approach using FORM is the computation of the sensitivity of the reliability index w.r.t. design variables in case gradient-based algorithms are employed. Sensitivity estimation was studied, e.g. in Kwak and Lee (1987), where an approach for the weight minimization under a number of probability constraints was introduced. The salient feature of this approach is that the sensitivity of the reliability index with respect to the design variables is calculated using a formula involving Lagrange multipliers related with the optimality conditions that must be fulfilled at the design point. In addition, this approach has been extended taking advantage of the Neumann expansion for solving the state equations of structural systems including random parameters (Lee and Kwak 1995).

The double-loop approach has been applied to several types of RBO problems aside those involving deterministic objective functions. For example, in Enevoldsen and Sørensen (1994), several different formulations of the RBO problem were investigated considering construction costs and costs due to eventual failure, repair, etc. In that contribution, much attention is paid to the issue on how to calculate the probabilities associated with structural systems (see, e.g. Ditlevsen 1978), which can be much more involved than the calculation of component reliability. Moreover, the issue on how to estimate probability sensitivity is discussed as well. In particular, it is pointed out that sensitivities should be calculated using semi-analytical methods in order to ensure sufficient accuracy (see, e.g. Bjerager and Krenk 1989; Enevoldsen and Sørensen 1993).

Besides the contributions described above, several other authors explored different aspects of the implementation of the double-loop approach, e.g. introduction of approximation concepts for estimating probability and its sensitivity (Reddy et al. 1994), evaluation of approaches for accounting for different failure modes in reliability analysis (Yang and Nikolaidis 1991), two-point approximations of the performance function (Grandhi and Wang 1998), etc.

The approaches described above formulate probability constraints using the so-called RIA (see (9)). However, an alternative means for expressing a probability constraint is using the so-called inverse FORM (iFORM) approach (Der Kiureghian et al. 1994; Li and Foschi 1998; Winterstein et al. 1994), which is also denoted as the *Performance Measure Approach* (Tu et al. 2001). Using the iFORM approach, the RBO problem is formulated as shown below.

$$\min_{\boldsymbol{y}} C(\boldsymbol{y}) \tag{13}$$

subject to

$$g\left(\boldsymbol{y}, T_{\theta\zeta}\left(\boldsymbol{\theta}^{iF}\right)\right) \geq 0 \tag{14}$$

In (14), $\boldsymbol{\theta}^{iF}$ is a realization of the uncertain parameters derived from the iFORM analysis; $\boldsymbol{\theta}^{iF}$ is the solution of the following optimization problem.

$$\min_{\boldsymbol{\theta}} g\left(\boldsymbol{y}, T_{\theta\zeta}\left(\boldsymbol{\theta}\right)\right) \tag{15}$$

subject to

$$||\boldsymbol{\theta}|| = \beta^{tol} \tag{16}$$

The equality constraint of the optimization problem in (15) and (16) *imposes* the prescribed tolerable failure probability by setting the norm of $\boldsymbol{\theta}$ equal to β^{tol}. It has been indicated in the literature (see, e.g. Lee et al. 2002; Ramu et al. 2006) that iFORM is numerically more stable than RIA. This is due to the fact that it is much simpler to solve an optimization problem with an equality constraint (see (15) and (16)) than solving a problem comprising an involved inequality constraint (see (11) and (12)) (Youn et al. 2003). Moreover, iFORM is much more amenable than RIA for treating inactive probabilistic constraints (Tu et al. 2001). For the solution of the optimization problem related with iFORM, different methods have been applied, taking advantage on the convexity or concavity of the performance function, see e.g. Youn et al. (2003, 2005). A schematic representation of the solution of a RBO problem considering a double-loop approach and iFORM is shown in Fig. 2. In this figure, the segmented lines represent the contour levels of the cost function; moreover, it is assumed that $\boldsymbol{\sigma}_{\xi} = \mathbf{1}$ and that $\boldsymbol{y}$ corresponds to the mean value of the vector of random variables, i.e. $\boldsymbol{y} = \boldsymbol{\mu}_{\xi}$. Starting from a design $\boldsymbol{y}^{(1)}$, the iFORM analysis is carried out; it is found that the probabilistic constraint is not active because $g\left(\boldsymbol{y}, T_{\theta\zeta}\left(\boldsymbol{\theta}^{iF,(1)}\right)\right) > 0$. Thus, the optimization algorithm explores a new candidate optimum design $\boldsymbol{y}^{(2)}$ at which iFORM is carried out again. Finally, the design optimal design $\boldsymbol{y}^{(3)}$ is determined; at this design, the optimality conditions are fulfilled and $g\left(\boldsymbol{y}, T_{\theta\zeta}\left(\boldsymbol{\theta}^{iF,(3)}\right)\right) = 0$.

It is most interest to note that the so-called iFORM approach and its application within RBO is closely related with semi-infinite programming techniques, as shown in Royset et al. (2001a). In that contribution, the RBO problem is replaced by an approximate, deterministic one using

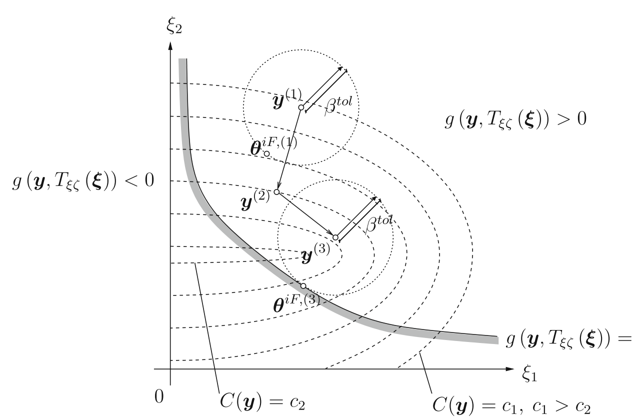


Fig. 2 Schematic representation of the double-loop approach applying iFORM (considering $\boldsymbol{\sigma}_{\xi} = \mathbf{1}$)

semi-infinite programming. That is, the probabilistic constraint is replaced by an infinite number of deterministic constraints; such a replacement is based on the theory of FORM. Thus, the RBO problem can be solved most efficiently using specialized algorithms for semi-infinite programming problems (Polak 1997). Two types of problems are studied: the minimization of a deterministic function subjected to a number of reliability constraints and the maximization of the reliability under deterministic constraints; in both cases, both, component and series system reliability are considered. In subsequent contributions, the semi-infinite approximation of the RBO problem was extended to cover the case of minimization of an expected cost function (Royset et al. 2001b, 2006). It should be noted that the approaches introduced in Royset et al. (2001a, b, 2006) can be applied using not only FORM, but also SORM, MCS or any other appropriate reliability technique.

Aside the application of classical approximate reliability methods, the so-called Dimension Reduction Method (DRM) (Rahman and Xu 2004; Xu and Rahman 2004) has also been applied within the context of RBO using a double-loop approach. For example, in Rahman and Wei (2008), the RBO problem is solved using gradient-based optimization, where the sensitivity of the probability is calculated using semi-analytical expressions based on DRM reliability analysis. Another example is the approach developed in Lee et al. (2008), where the DRM is applied within the context of inverse reliability analysis (as in iFORM) for solving RBO problems.

5.2 Converting the double-loop into a single-loop

An approach for avoiding the so-called double loop in RBO was proposed in Chen et al. (1997). The problem treated in that contribution refers to the minimization of the structural weight under a number of probabilistic constraints. The design variables correspond to the mean value of all uncertain parameters present in the problem, i.e. $\boldsymbol{y} = \boldsymbol{\mu}_{\xi}$. This particular structure of the problem allows replacing the probabilistic constraint with an approximate, deterministic constraint; the latter constraint is formulated in terms of the associated performance function $g(\cdot, \cdot)$ and depends exclusively on the value of the design variables. Thus, the original double-loop problem is converted into a single-loop problem. In mathematical terms, the RBO problem is formulated as follows.

$$\min_{\boldsymbol{y}} C(\boldsymbol{y}) \tag{17}$$

subject to

$$g\left(\boldsymbol{y}, T_{\xi\zeta}\left(\boldsymbol{\xi}^{(k)}\right)\right) \geq 0 \tag{18}$$

where:

$$\boldsymbol{\xi}^{(k)} = \left\langle y_1 - \beta^{tol}\sigma_{\xi_1}\alpha_1^{(k-1)}, \ldots, y_{n_\zeta} - \beta^{tol}\sigma_{\xi_{n_\zeta}}\alpha_{n_\zeta}^{(k-1)}\right\rangle^T \tag{19}$$

$$\boldsymbol{\alpha}^{(k)} = \frac{\nabla_{\boldsymbol{\theta}}\left(g\left(\boldsymbol{y}, T_{\xi\zeta}\left(\boldsymbol{\xi}\right)\right)\right)\Big|_{\boldsymbol{\xi}=\boldsymbol{\xi}^{(k)}}}{\left\|\nabla_{\boldsymbol{\theta}}\left(g\left(\boldsymbol{y}, T_{\xi\zeta}\left(\boldsymbol{\xi}\right)\right)\right)\Big|_{\boldsymbol{\xi}=\boldsymbol{\xi}^{(k)}}\right\|} \tag{20}$$

The strategy for solving the RBO problem according to the method described above is shown schematically in Fig. 3. In this figure, the segmented lines represent the contour levels of the cost function; moreover, it is assumed that $\boldsymbol{\sigma}_{\xi} = \mathbf{1}$. Starting from an initial candidate $\boldsymbol{\xi}^{(1)}$, the unit vector $\boldsymbol{\alpha}^{(1)}$ (cf. (20)) is calculated. Then, the optimization problem in (17) and (18) is solved w.r.t. to $\boldsymbol{y}$, keeping $\boldsymbol{\alpha}^{(1)}$ constant; as shown in the figure, the distance between $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{y}$ is kept—by construction—equal to β^{tol} (cf. (19)). Once the optimization is finished, $\boldsymbol{\xi}^{(2)}$ is calculated based on the optimum $\boldsymbol{y}^{(2)}$. Thus, $\boldsymbol{\alpha}^{(2)}$ can be determined and optimization is performed again, starting from the design $\boldsymbol{y}^{(2)'}$ (in the figure, the distance between $\boldsymbol{y}^{(2)'}$ and $\boldsymbol{\xi}^{(2)}$ is—by construction—equal to β^{tol}); this leads to the optimal design $\boldsymbol{y}^{(3)}$. In this way, the steps described above are repeated until fulfilling a prescribed convergence criterion.

The key issue in the approach described above is that the unit direction $\boldsymbol{\alpha}$ is kept constant within each iteration. This is equivalent to assume that the direction of the design point vector associated with the performance function remains constant despite eventual changes in the value of $\boldsymbol{y}$. In this way—and applying the theory of FORM (see Appendix A)—it is possible to construct the constraint in (18). This allows breaking the inner loop related with reliability analysis.

Although the method presented in Chen et al. (1997) can be quite advantageous, it should be noted that its efficiency and accuracy can be affected by several factors. For example, the selection of a particular starting point $\boldsymbol{\xi}^{(1)}$ for the algorithm may affect the efficiency considerably (Yang and

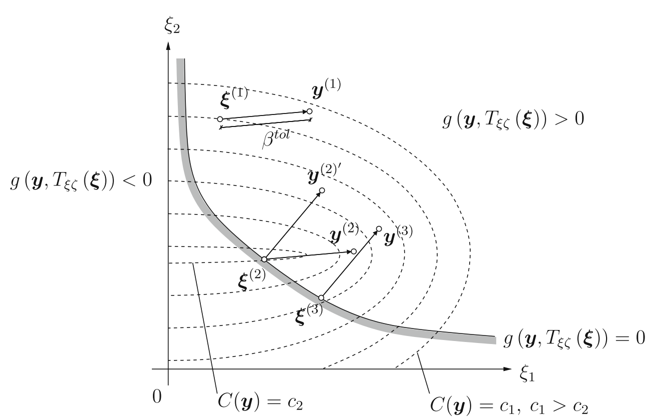


Fig. 3 Schematic representation of the single-loop approach proposed in Chen et al. (1997) (considering $\boldsymbol{\sigma}_{\xi} = \mathbf{1}$)

Gu 2004). Moreover, in those cases where the performance function is non linear, the application of this algorithm may not be appropriate, as the FORM hypothesis may not be representative of the actual reliability problem.

Another important class of approaches that allows avoiding the so-called double loop are those that take advantage of the Karush–Kuhn–Tucker (KKT) optimality conditions (see, e.g. Bonnans et al. 2003) and Lagrange multipliers. For example, in Kuschel and Rackwitz (1997), the KKT conditions related with the design point identification are incorporated in the formulation of the RBO problem. In this way, the inner reliability loop is avoided. This allows a simultaneous convergence w.r.t. the design variables and the design point location. In spite of the evident advantage that a single optimization loop provides, the approach requires the computation of second order derivatives. Moreover, a recent benchmark study (Aoues and Chateauneuf 2010) indicates that this approach may suffer instability problems. More recently, in Agarwal et al. (2007), an approach similar to the one introduced in Kuschel and Rackwitz (1997) was developed, with the difference that the probabilistic constraints are formulated using the iFORM approach and that the calculation of second order sensitivities is avoided by applying a quasi-Newton method (see, e.g. Bonnans et al. 2003). Another example of the application of the KKT conditions can be found in Kharmanda et al. (2002) and Mohsine et al. (2006), where a *hybrid formulation* is applied for solving the RBO problem, i.e. the objective function is expressed as the product between the objective function and the reliability of the structure. In Kaymaz and Marti (2007) and Marti and Kaymaz (2006), the KKT conditions were employed to formulate RBO problems related with the design of elasto-plastic mechanical structures. Finally, an approach that uses the KKT conditions and that is very similar to the one introduced in Chen et al. (1997) was proposed in Liang et al. (2008); the distinctive characteristic of the latter approach is that the unit direction $\boldsymbol{\alpha}$ is calculated exactly for each design explored by the optimizer.

5.3 Decoupling approach

The implementation of a decoupling approach implies that information from the reliability analysis stage is extracted and used at the optimization stage in order to improve numerical efficiency. In this way, the so-called double loop problem associated with RBO is avoided, i.e. it may not be necessary to perform a full reliability analysis each time a new point in the space of the design variables is explored by the optimization algorithm.

One of the first decoupling approaches was introduced in Li and Yang (1994), where the RBO problem is formulated as a linear programming problem; the key step in this approach is the construction of a linear approximation of the reliability index using information on sensitivities, i.e.:

$$\beta(\boldsymbol{y}) = \beta\left(\boldsymbol{y}^{(k)}\right) + \sum_{l=1}^{n_y} \frac{\partial \beta(\boldsymbol{y})}{\partial y_l}\bigg|_{\boldsymbol{y}=\boldsymbol{y}^{(k)}} \left(y_l - y_l^{(k)}\right) \tag{21}$$

where $\boldsymbol{y}^{(k)}$ is the k-th candidate optimal design. In Tu et al. (2001), the idea of constructing a linear approximation of the probability was explored as well; however, in this approach, the special treatment of active and inactive constraints allows improving the overall efficiency. Besides the approaches described previously, other alternatives have been investigated as well, e.g. consideration of both linear and reciprocal approximations of the reliability index (Chandu and Grandhi 1995), introduction of approximations using Benders cuts (Mínguez and Castillo 2009), calculation of the sensitivity of the reliability index using Lagrange multipliers and the (approximate) Hessian matrix associated with the design point identification problem (Agarwal and Renaud 2006), application of Sequential Linear Programming and identification of active constraints (Chan et al. 2006, 2007), application of recursion formulas for estimating the design point and its sensitivity (Cheng et al. 2006), etc.

The approaches described above use information on sensitivity for constructing approximations of the probability. A different approach which does not rely on sensitivities is the so-called *Sequential Optimization and Reliability Assessment* (SORA) method (Du and Chen 2004). In order to describe this approach, assume that all the design variables correspond to the mean values of the Gaussian distributed random variables, i.e. $\boldsymbol{y} = \boldsymbol{\mu}_\xi$. This assumption is introduced here for the sake of simplicity. The optimization problem that is solved in SORA is shown below.

$$\min_{\boldsymbol{y}} C(\boldsymbol{y}) \tag{22}$$

subject to

$$g\left(\boldsymbol{y}, T_{\xi\zeta}\left(\boldsymbol{y} - \boldsymbol{s}^{(k)}\right)\right) \geq 0 \tag{23}$$

In (23), $\boldsymbol{s}^{(k)}$ is a vector which is equal to:

$$\boldsymbol{s}^{(k+1)} = \begin{cases} \boldsymbol{0} & k = 1 \\ \boldsymbol{y}^{(k)} - T_{\theta\xi}\left(\boldsymbol{\theta}^{iF,(k)}\right) & k \geq 2 \end{cases} \tag{24}$$

where $\boldsymbol{y}^{(k)}$ is the solution of the optimization problem in (22) and (23) for the k-th iteration and $\boldsymbol{\theta}^{iF,(k)}$ is the solution of the iFORM optimization problem (cf. (15) and (16)), considering $\boldsymbol{y}^{(k)}$ as the mean value of the uncertain parameters. The way SORA proceeds is as follows. The first iteration (i.e. $k = 1$) solves the RBO problem considering the performance function as a regular deterministic constraint. This will lead to an optimum design $\boldsymbol{y}^{(1)}$. This design will be—for problems of practical interest—unfeasible. Then, the iFORM problem is solved in order to determine $\boldsymbol{\theta}^{iF,(1)}$, which is a realization of the uncertain parameters lying in the failure domain with Euclidean norm equal to β^{tol} (see (15) and (16)). After solving the iFORM problem, the parameter $\boldsymbol{s}^{(2)}$ is determined. The purpose of this parameter is introducing a shifting in the deterministic constraint in (23) such that actual probabilistic constraint is enforced. Thus, when the optimization problem considering $\boldsymbol{s}^{(2)}$ is solved, an improved design (from the point of view of reliability) will be obtained. By repeating this procedure a number of times, it is possible to determine the sought optimum.

6 The use of simulation techniques in reliability-based optimization

This section presents techniques for solving RBO problems that apply simulation methods for assessing reliability. These techniques are organized in three groups: application of meta-models, decoupling approach and direct integration with optimization algorithms. As in the case of the previous Section, other classification criteria could be certainly applied as well.

Before discussing in detail the different RBO techniques applying simulation methods, an important issue must be addressed. As it is well known, simulation methods produce reliability estimates that are subject to a certain degree of variability. That is, for analyzing the same problem of structural reliability, two independent runs of a simulation technique would most likely produce different estimates of the associated probability. Although in principle such variability can be controlled by increasing the number of samples of the uncertain parameters that are drawn when applying a simulation method, this alternative can become numerically demanding. An alternative means for coping with the variability of the reliability estimates generated through simulation techniques is the application of common random numbers (CRN) and *smoothing* of indicator functions (Taflanidis and Beck 2008a). The application of CRN implies that the same stream of random numbers is used for evaluating the reliability associated with two different sets of design variables $\boldsymbol{y}^{(k_1)}$ and $\boldsymbol{y}^{(k_2)}$. The use of a smooth indicator function implies that instead of using a binary criterion to classify whether or not a particular realization of the uncertain parameters is included in the probability integral (cf. (5)), a relaxed criterion is applied. It should be noted that these strategies do not eliminate the variability of the estimates. Instead, they introduce a *consistent* estimation error. That is, although the estimates (of, e.g. reliability) for two different sets of the design variables still contain error, these estimates are still comparable between them. For more details on the application of this approach, it is referred to, e.g. Taflanidis and Beck (2008a, b).

6.1 Application of meta-models for assessing reliability using simulation techniques

The evaluation of virtual simulation models (based on, e.g. finite elements, boundary elements, etc.) may be computationally expensive, specially when analyzing large structures with a high degree of refinement in the discretization and possibly including non linearities. Within the context of RBO, the simulation of such models may lead to computation times which are unaffordable. As a means to reduce computational efforts, the virtual simulation model can be approximated with a *meta-model*. The advantage of using a meta-model in this context is that the numerical effort associated with its evaluation is usually negligible. In this manner, the reliability analyses using simulation techniques that are performed for solving the RBO problem can be carried out at low numerical costs.

In order to ensure that the meta-model is accurate, a proper *training* should be carried out, where the parameters of the meta-model are adjusted based on simulations of the full model. A key issue to be defined at the training phase is which data points should be used to perform the calibration and how many of these points are required. The points to be selected for performing the calibration can be selected using an appropriate design of experiments scheme (see, e.g. Cox and Reid 2000), such as factorial designs, latin hypercube designs, etc. Additionally, the training points can also be selected adaptively according to the specific needs of the problem at hand. Concerning the number of points to be selected, this can be chosen based on some prescribed convergence criterion. For example, in Bichon et al. (2008), meta-models for reliability analysis are trained using the so-called *Efficient Global Optimization* procedure, which allows selecting the samples to train the meta-model adaptively in order to ensure accuracy in the vicinity of the limit state function. Another example of adaptive selection of training points is the approach introduced in Basudhar and Missoum (2008); in that approach, points for training a Support Vector Machine (SVM) are selected such that they improve the quality of the meta-model.

Once a meta-model has been calibrated, the associated RBO problem can be solved using virtually any appropriate optimization algorithm and simulation technique, as the meta-model is very inexpensive to evaluate. In this context, the construction of a meta-model is not restricted to a virtual simulation model. A meta-model can also replace, e.g. spectral quantities (vibration frequencies and modes), performance functions, etc. It should also be noted that for solving a RBO problem, meta-models do not need to be global. That is, it may be easier and more efficient to construct several meta-models that are valid over subdomains of the variables involved in a particular problem.

Meta-models can be applied in two different forms when solving a particular RBO problem. In the first one, the meta-model is used to represent directly a performance function, i.e. given a certain realization of the uncertain parameters and design variables, the meta-model produces a numerical value that approximates the one that would be obtained by evaluating the performance function. In the second one, the meta-model is used as a *classification* tool, i.e. the meta-model determines whether or not a certain realization of the uncertain parameters and design variables causes an acceptable or unacceptable behavior (i.e. the associated value of the performance function is larger than zero or smaller than zero, respectively) without actually computing an approximate value of the performance function.

Meta-models as a means for approximating directly the performance function have been used thoroughly in the literature. A typical example of this class of meta-models is the response surface (RS) methodology. The application of RS techniques in context with reliability analysis has been investigated in, e.g. Bucher and Bourgund (1990) and Rajashekhar and Ellingwood (1993). The RS methodology has also been applied for solving RBO problems efficiently. For example, in Foschi et al. (2002), the performance function is replaced with an incomplete quadratic RS; then, FORM and Importance Sampling techniques are applied in order to assess the reliability. A similar approach is implemented in Agarwal and Renaud (2004), where the performance function is replaced with a quadratic RS. This meta-model is constructed using information extracted from approximate reliability analysis; once the RS model is calibrated, direct Monte Carlo Simulation (MCS, see Appendix A) is carried out with reduced computational efforts. The RS methodology has been applied not only to replace the performance function but also intermediate responses. For example, in Jensen (2005), the spectral quantities associated with a structure are approximated by means of linear response surfaces. This allows reducing considerably the number of eigenvalue/ eigenvector decompositions required to solve RBO problems involving linear structures subject to stochastic dynamic loadings.

Although the RS methodology has been widely used, there are several other techniques that are a viable alternative for reducing numerical costs within the context of RBO. For example, the possibility of approximating numerically demanding FE models with Artificial Neural Networks (ANN) been investigated in, e.g. Papadrakakis et al. (2005), Papadrakakis and Lagaros (2002); in these contributions, the RBO problem has been solved using Monte Carlo Simulation (MCS) for reliability analysis and evolution strategies (see, e.g. Beyer and Schwefel 2002) for optimization. In Zhang and Foschi (2004), ANN were applied for the optimization of dynamical systems. Another strategy for RBO where meta-models approximate the performance function was recently introduced in Bichon et al. (2009). In this approach, the performance function is replaced with a Gaussian process (GP) meta-model, which allows assessing probability efficiently. The meta-model is incorporated at different levels of the RBO problem, leading to double-loop, single-loop or decoupled solution strategies. In addition to the applications described above, meta-models have also been used as classification tools for solving RBO problems. For example, Support Vector Machines (SVM) have shown to be a feasible means for determining whether or not particular realizations of $(\boldsymbol{y}, \boldsymbol{\zeta})$ lead to failure. An important feature of SVM is their flexibility and adaptability for approximating the exact model when compared to the RS approach (Hurtado 2004, 2007). Within the context of RBO, SVM have been applied in Basudhar et al. (2008), Basudhar and Missoum (2008) and Basudhar and Missoum (2009); in these contributions, it has been shown that SVM can be used to deal with involved limit state functions that might even be discontinuous. In addition to SVM, Convex sets are another alternative for constructing a meta-model that works as a classification tool. In Missoum et al. (2007), it is proposed to approximate the limit state functions related to the failure event by means of a convex hull; such an approximation allows to apply direct MCS at low numerical costs, rendering a feasible means for performing RBO.

6.2 Decoupling

As already discussed in Section 5.3, a feasible means for solving RBO problems efficiently is the application of a decoupling approach. Within the context of RBO considering simulation techniques for reliability assessment, the possibility of constructing an approximate representation of the probabilities as an explicit function of the design variables around an expansion point has been widely investigated. This allows solving the optimization problem most efficiently, as the outer optimization loop (that explores the space of the design variables) is decoupled from the inner reliability loop. Thus, the key issue in such an approach is the construction of an approximate representation of the probabilities. Early efforts demonstrated that a probability function may be approximately represented by means of an exponential function (Murthy and Subramanian 1968; Lind 1976; Kanda and Ellingwood 1991). This approach was employed in the context of RBO in Gasser and Schuëller (1997), in order to generate a *global* approximation of the failure probabilities as an explicit function of the design variables; in this context, *global* refers to the fact that the approximation was assumed to be valid over the whole domain of the design variables. The exponential approximation is constructed by selecting some predefined interpolation points in the space of the design variables, at which the failure probability is calculated by means of simulation; then, an exponential function is adjusted to the data collected at the interpolation points in a least square sense. The argument of the exponential function is a polynomial of second or higher order, i.e.:

$$p(\boldsymbol{y}) = e^{(pol(\boldsymbol{y}))} \tag{25}$$

where $pol(\cdot)$ is a polynomial depending on $\boldsymbol{y}$. The approach introduced in Gasser and Schuëller (1997) was further extended in Jensen and Catalan (2007) and Jensen (2005), where it was shown that the construction of *local* approximations of the failure probabilities could be advantageous. This is due to the fact that the argument of the exponential function can be selected as a polynomial of lower order than in the case of the global approximation. The approach using local approximations can be incorporated in a sequential approximate optimization framework (see, e.g. Haftka and Gürdal 1992; Jacobs et al. 2004) in order to solve the target RBO problem. That is, after identifying a candidate optimal design located within a given subdomain of the design variables, a new approximation of the probability is constructed around this candidate and the optimization w.r.t. the design variables is repeated (within a new subdomain of the design variables). A schematic representation on how this approach operates is shown in Fig. 4. In this figure, the RBO problem is represented in the space of the design variables. The continuous line indicates a contour level of the probability function; the segmented line, contour levels of the objective function and the dash-dotted line, sub-domains $\Omega_y^{(k)}$

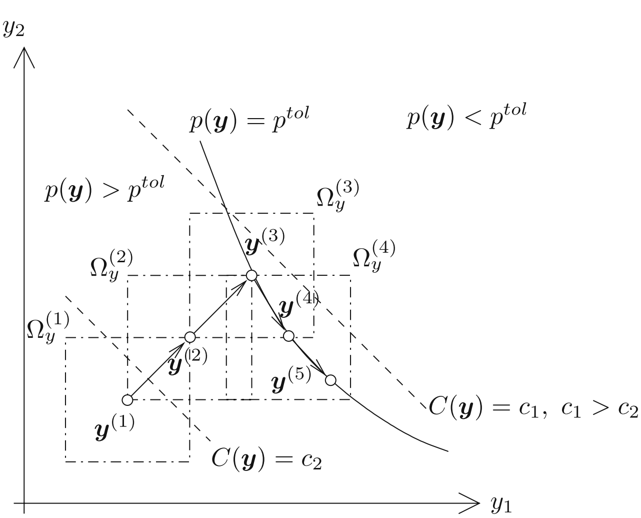


Fig. 4 Schematic representation of sequential approximate optimization for RBO

for performing optimization, where the approximate representation of the probability is constructed. The dots denote candidate optimal designs. As depicted in the figure, the optimization procedure starts from an unfeasible design; as iterations progress, the candidate design is improved until finding the optimal solution at the fifth iteration.

The major disadvantage of the approaches based on global and local approximations of the failure probability described above is that the number of reliability analyses required to adjust the exponential approximation grows rapidly with the number of design variables, i.e. at least linearly with n_y.

An alternative technique for constructing an approximation of the failure probabilities was proposed in Au (2005). The key issue in this technique is associating a so-called *instrumental variability* with the (deterministic) design variables; then, the sought approximation can be obtained using Bayes' theorem in conjunction with histograms representing the probability distribution of the design variables conditioned on the failure event; in this context, it is important to note that a *single* reliability analysis suffices for obtaining all required information. This technique was further developed in Ching and Hsieh (2007a, b), where the aforementioned histograms were replaced by probability density functions determined using the maximum entropy principle (Jaynes 1968; Ormoneit and White 1999), in order to construct a global approximation of the failure probabilities in the space of the design variables. An approach based on calculating the area and first moments of area of the failure probability function over a sub-domain of Ω_y was proposed in Jensen et al. (2008) for constructing a local approximation of $p(\boldsymbol{y})$ using a single reliability analysis; it can be shown that such an approach is actually equivalent to the one proposed in Ching and Hsieh (2007b), although they rely on a different theoretical background. More recently, in Koutsourelakis (2008), an approach which is also based on associating an *instrumental variability* with the (deterministic) design variables and Bayes' theorem was presented; the information extracted from the samples of the design variables at different stages of the reliability analysis is employed to construct a global approximation of $p(\boldsymbol{y})$ using probabilistic classifiers. However, it was also shown in that contribution that the construction of approximate representations of the probability using a single reliability analysis may be restricted to a low number of design variables, e.g. three or four.

An alternative strategy for constructing an approximate representation of the probability function is using information on sensitivity, e.g.:

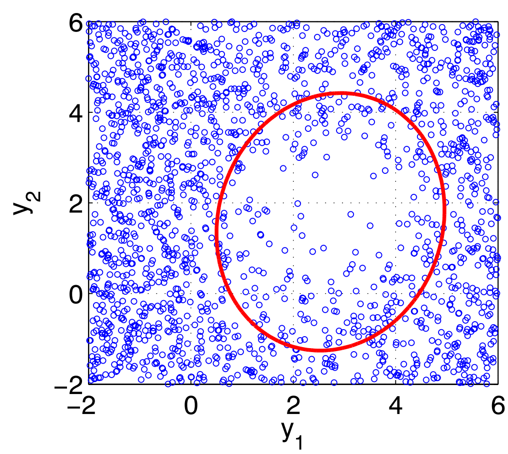
$$p(\boldsymbol{y}) = p\left(\boldsymbol{y}^{(k)}\right) + \sum_{l=1}^{n_y} \frac{\partial p(\boldsymbol{y})}{\partial y_l}\bigg|_{\boldsymbol{y}=\boldsymbol{y}^{(k)}} \left(y_l - y_l^{(k)}\right) \tag{26}$$

where $\boldsymbol{y}^{(k)}$ is the expansion point. Such an approach was adopted in Zou and Mahadevan (2006), where the problem of weight minimization under probabilistic constraints was analyzed considering that the vector of design variables corresponds to the mean value of the uncertain parameters. The required sensitivities at the expansion point are calculated using an approach proposed in Wu (1994). The salient feature of this approach is that the required sensitivities can be calculated using the same samples generated for estimating the reliability. Another approach using information on sensitivity was introduced in Valdebenito and Schuëller (2010). In that approach, the probability of failure is approximated using an exponential function. The sensitivity is calculated with a novel algorithm which is applicable in cases where the number of uncertain parameters is very large, e.g. n_ζ in the order of hundreds or even thousands.

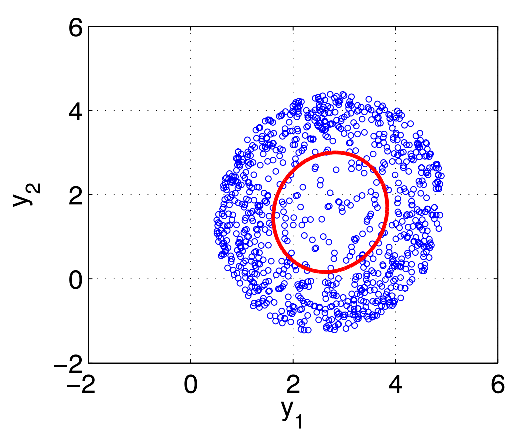
6.3 Direct integration with optimization algorithms

In the previous section, the possibility of constructing an approximate representation of the probability functions using information on the sensitivity was analyzed. However, the information on the probability sensitivity could be used directly within a gradient-based optimization algorithm in order to solve the RBO problem efficiently. For example, in Royset and Polak (2004a, b), the issue as to how to compute the gradient of the probabilities is addressed by means of simulation. In particular, an algorithm for estimating $\partial p(\boldsymbol{y})/\partial y_l$ using either Monte Carlo simulation (MCS) or Importance Sampling (IS) is proposed. This algorithm requires solving the equation $g\left(\boldsymbol{y}, T_{\theta\zeta}\left(\boldsymbol{\theta}\right)\right) = 0$ for one component of $\boldsymbol{\theta}$, either analytically or numerically. The information on the sensitivities is then used within an efficient optimization scheme in order to determine an optimal solution for the RBO problem. In Jensen et al. (2009), another approach for solving RBO problems based on sensitivity information is presented. In this approach, sensitivities of the probability are computed using the algorithm developed in Valdebenito and Schuëller (2010). The sensitivity information is then integrated within an optimization algorithm based on feasible directions. The efficiency of the procedure is increased by performing line search using a polynomial approximation of the probability along the search direction; this approximation is constructed using information on both the probability estimate and its directional derivative, following a procedure developed in van Keulen and Vervenne (2004).

Another approach that integrates directly a simulation technique with optimization is the so-called Stochastic Subset Optimization (SSO) method (Taflanidis and Beck 2008a, b). In this technique, the capabilities of exploring the space of the uncertain parameters for reliability assessment of Subset Simulation (Au and Beck 2001)—which is an advanced simulation method—are exploited for exploring the space of the design variables at the same time. This allows then the evaluation of the structural reliability and identification of the optimal solution of the RBO problem simultaneously. SSO operates by generating a pool of samples of the uncertain parameters and design variables (note that an *instrumental* variability is associated with the design variables, see Au 2005). In this way, it is possible to identify a subset of the design variables which, on the average, improves the value of the objective function. By repeating this procedure a number of times, it is possible to determine at each step a smaller subset of the design variables which in turn improves the value of the objective function. At the end, this subset will be sufficiently small to identify directly the optimum solution of the optimization problem or it will provide sufficient information in order to launch another optimization algorithm, such as the Stochastic Perturbation Simultaneous Approximation (SPSA) algorithm (see, e.g. Spall 2003). A schematic representation on how SSO proceeds is shown in Fig. 5. In this figure, it is considered that



(a) First iteration of SSO



(b) Second iteration of SSO

Fig. 5 Schematic representation of two iterations of SSO

$\Omega_y = [-2, 6] \times [-2, 6]$. In the first iteration, samples of the design variables are generated over Ω_y; then, a region (in this case, an ellipse marked with continuous line) with the lowest density of samples is identified, as this region improves—on the average—the value of the objective function. In the second iteration, a new region (smaller than the previous one) is identified again. In this manner, SSO can converge towards the optimal solution of a RBO problem.

7 Comparison of different methods

The spectrum of algorithms for solving RBO problems is quite wide. Several methods have been developed under specific assumptions or tailored to specific problems; very few approaches can be applied in a *black box* fashion to any arbitrary problem in structural design. Therefore, an objective comparison of the different methods based on quantitative results is—in opinion of the authors—currently not feasible. In view of this fact, this section attempts to provide a qualitative comparison between different methods for solving RBO problems. Nonetheless, it should be noted that a few benchmark studies covering part of the spectrum of algorithms for RBO are available, see e.g. Aoues and Chateauneuf (2010) and Yang and Gu (2004). Such studies are certainly a very valuable tool for comparing some of the algorithms available.

The discussion and comparison of different approaches for RBO is carried out on the basis of five main issues which are presented in detail below.

Dimensionality of the vector of design variables A large part of the RBO algorithms developed so far addresses problems involving very few design variables. In fact, few contributions consider problems involving more than 10 design variables; a notable exception is Grandhi and Wang (1998). Naturally, the fact that only a low number of design variables is considered is a direct consequence of the difficulties in addressing uncertainties. That is, the multi-dimensional integrals associated with the evaluation of probabilities and expected costs impose a major challenge for performing function evaluations and sensitivity analysis. Therefore, the application of RBO in problems involving structural systems with a large number of decision variables can be quite challenging. In such cases, a feasible approach is implementing a *screening strategy* (Sues et al. 2001) in order to determine the most influential variables and reduce the dimensionality of the design variable vector.

Dimensionality of the uncertain parameters The number of uncertain parameters required to characterize a particular model will be highly variable. While in a given problem, very few random variables might be required, in others hundreds or even thousands might be necessary. For example, large models may require a high number of random variables in order to capture the effects of uncertainty, see e.g. Pellissetti et al. (2006). Another example is the representation of stochastic processes and random fields, see e.g. Katafygiotis and Wang (2009). The number of random variables involved in a RBO problem may become a challenging issue in case approximate reliability methods are used. Two reasons justify this assertion. In the first place, several methods rely on optimization for determining the design point. The identification of this point can be quite challenging, as an increasing number of dimensions will imply—in most cases—an increase in the number of evaluations of the performance function. Secondly, approximate reliability methods may not be applicable when the dimensionality of $\boldsymbol{\zeta}$ is high, as discussed in Katafygiotis and Zuev (2008) and Valdebenito et al. (2010). On the contrary, RBO approaches using simulation methods are much better suited for treating problems with a large number of random variables. In fact, several methods of this class have been developed especially for treating such problems.

Application of meta-models The application of meta-models always constitutes a very attractive approach for solving RBO problems. As the meta-model is inexpensive to evaluate, it is possible to try several different solution approaches, perform exhaustive sensitivity analyses, etc. Moreover, certain types of meta-models—such as soft computing techniques—are capable of capturing involved input-output relations of a virtual simulation model. However, a major issue in applying meta-models is that their training can be challenging for cases where the input vector is of high dimension.

Component reliability vs. system reliability Partial failure in a structural system may cause loss of serviceability. Such type of failure can be originated due to, e.g. failure of a particular component. However, structural systems are often designed such that they possess a high level of redundancy. Thus, structural collapse will occur most likely due to the combined effect of several different failure modes. These arguments indicate that both component and system reliability are very important when formulating and solving a RBO problem. In spite of the relevance of both types of reliability, RBO considering system reliability has received little attention compared to component reliability, see e.g. Aoues and Chateauneuf (2008) and Pu et al. (1997). In the case of RBO methods based on approximate reliability methods, almost all contributions refer to component reliability. Although efforts have been devoted for considering system reliability (see e.g. Enevoldsen and Sørensen 1993; Royset et al. 2001a), the accuracy of the assessment of the probability becomes a serious issue. Approximate formulas for assessing reliability in these cases (see, e.g. Ditlevsen 1978) usually assume linear or weakly non linear performance functions. RBO methods based on simulation can be more appropriate when considering system reliability. In particular, RBO methods applying *black box* simulation techniques (such as Subset Simulation, Au and Beck (2001)) are able to handle even thousands of non linear performance functions simultaneously.

Overall efficiency As already discussed above, a fair comparison of RBO methods is quite challenging, as different approaches treat different types of problems. For example, while some approaches treat small structural systems under static loads, other approaches address non linear systems subject to stochastic loading. Or while some approaches are restricted to analyzing a few failure modes, others can consider simultaneously thousands of possible failure modes. However, some general guidelines can be established. RBO methods based on approximate reliability methods can be very efficient for problems involving component reliability, a low number of uncertain parameters and linear or mild non linear performance functions. In such cases, the basic hypotheses of approximate reliability methods will be most likely fulfilled. In addition, the identification of the design point and assessment of the probabilistic constraints can be performed most efficiently using well-established techniques, see e.g. Youn et al. (2003) and Liu and Der Kiureghian (1991). Results reported in the literature (see e.g. Du and Chen 2004; Chen et al. 1997; Agarwal et al. 2007) indicate that for such cases, the total number of function evaluations (e.g. FE simulations) required for solving a RBO problem may be between two or three orders of magnitude. For those problems involving system reliability, a large number of random variables and non linear performance functions, RBO methods based on simulation constitute a natural choice for solution. According to results in the literature (see, e.g. Ching and Hsieh 2007a; Taflanidis and Beck 2008b; Jensen et al. 2009), the number of function evaluations required for solving a RBO problem may vary between three and five orders of magnitude.

8 Conclusions

This contribution has addressed nearly 50 years of developments in the field of RBO. The progress achieved in this field during this period has been considerable. While early efforts focused on simplified analysis and explicit formulas, modern approaches are capable of addressing complex problems involving realistic models and several failure criteria most efficiently. The progress achieved during this period is the product of a combination of factors such as increased computational power, advances in virtual simulation, efficient strategies for assessing reliability and a better understanding of the RBO problem. In particular, the latter feature has allowed the introduction of, e.g. approximation concepts, simplifications of the double-loop approach, efficient sensitivity estimation, etc.

The spectrum of approaches for RBO is quite broad. Some approaches are highly specialized and can treat certain classes of problems most efficiently. For example, single-loop approaches are most appropriate for treating component reliability, linear or mildly non linear performance functions and small or medium sized problems from the point of view of reliability. However, this high efficiency comes at the price of a narrowing field of application. Other approaches, such as Stochastic Subset Optimization, are capable of treating problems involving a large number of random variables and failure criteria considering non linear performance functions. Nonetheless, the price of generality is higher numerical costs. Correspondingly, general guidelines have been discussed in this contribution in order to select which class of algorithm might be more appropriate for a particular RBO problem.

The considerable development of algorithms for RBO indicates that optimization under uncertainty is not any longer the subject of academic examples, but a well developed methodology that can be applied in realistic engineering problems. It is expected that this tendency will accentuate even more in the near future due to the progress in the application of high performance computing. In particular, parallel computing allows reducing computation times considerably.

In spite of all the advances that have been achieved so far in the field of optimization under uncertainties, there are still many open issues to be addressed in the future. In the opinion of the authors, three main research directions can be envisioned. The first one refers to the improvement of current strategies for solving RBO problems, particularly with respect to aspects of numerical efficiency and robustness. As optimization under uncertainties is more involved than its deterministic counterpart, the efficiency of strategies for RBO will always be a crucial issue. Probably, the most promising strategies for RBO from the point of view of numerical costs are those that integrate partially or totally the reliability assessment step and the optimization algorithm. In addition, strategies for solving RBO problems should also improve their robustness, i.e. these strategies should be applicable to a wide spectrum of problems that can be found in engineering.

The second research direction that can be identified within the context of RBO is application of parallel computing. As the solution of a problem can be time consuming, parallel computing becomes a necessity in order to render the application of RBO feasible. Efforts in this direction should be focused towards identifying the real potential of parallel computing and the most effective strategies for applying it in a problem. In this context, it is important to note that parallel computing techniques can be applied at different levels, e.g. at the virtual simulation level, reliability analysis or optimization.

The third research direction that can be identified is the translation of tools developed within the field of RBO towards the engineering community. That is, the tools for RBO should be put in such way that they are accessible in engineering practice. For achieving this purpose, a key issue is the implementation of appropriate software tools that enable access to these procedures in a user-friendly, automatized environment. Although at first sight the implementation of appropriate software may seem to be the task of programmers, researchers on the field of RBO should play a major role in defining how the aforementioned software tools should be developed. In the opinion of the authors, this is the *only* feasible means for spreading all the knowledge and methods that have been developed within the field of RBO.

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Appendix A: Methods for reliability analysis

This appendix presents a brief overview on methods that have been developed in order to compute the integral associated with probability. These methods can be broadly classified into two categories (Schuëller et al. 2004): *approximate methods* and *simulation methods*.

A.1 Approximate reliability methods

The key concept of approximate reliability methods is introducing an asymptotic approximation of the limit state function (LSF), i.e. $g(\boldsymbol{y}, \boldsymbol{\theta}) = 0$, using a Taylor series. Although this approximation of the limit state function can be regarded as a meta-model, its scope is different from the meta-models discussed in Section 6.1. This is due to the fact that in approximate reliability methods, the objective of generating a Taylor series is replacing an unknown probability integral by a known one. The approximation using a Taylor series is constructed around the so-called *design point*. For defining the design point, assume that the vector $\boldsymbol{\theta}$ is composed by independent, Gaussian standard distributed random variables. Thus, the design point (which is denoted as $\boldsymbol{\theta}^*$) can be defined using two equivalent criteria (Freudenthal 1956). According to the *geometrical criterion*, the design point is the realization in the standard normal space which lies on the LSF ($g(\boldsymbol{y}, \boldsymbol{\theta}) = 0$) with the minimum Euclidean norm (β) with respect to the origin; this is shown schematically in Fig. 6. According to the *probabilistic interpretation*, the design point is the failure point with highest probability density. This means, it is the point that maximizes $f(\boldsymbol{\theta})$ subject to $g(\boldsymbol{y}, \boldsymbol{\theta}) \leq 0$, where $f(\cdot)$ is the standard normal probability density function in $\mathcal{R}^{n_\theta}$ (see Fig. 6). It should be noted that the norm of the design point ($\beta = \left|\left|\boldsymbol{\theta}^*\right|\right|$) has been denoted in the literature as *reliability index*.

From the discussion above, it is clear that the identification of the design point is also an optimization problem, as it is necessary either to minimize the Euclidean norm or maximize the probability density function. For details on how to determine the design point, it is referred to, e.g. Liu and Der Kiureghian (1991), Au (2006), Koo et al. (2005) and Wu et al. (1990).

Once the design point has been determined, the integral associated with the probability of failure can be approximated using the First or Second Order Reliability Method (FORM and SORM, respectively). In the case of FORM, the LSF is replaced by a first order Taylor expansion centered around the design point. In the case of SORM, the LSF is replaced with an incomplete second order Taylor expansion (also centered around the design point). A more detailed explanation of FORM and SORM is outside the scope of this paper; for more details on these reliability techniques, it is referred to, e.g. Rackwitz (2001). However, it is important to note that no estimator of the error introduced when approximating the probability integral using FORM and/or SORM is available. Moreover, these methods may not always be applicable, e.g. in cases where the performance function is highly non linear and/or the dimensionality of $\boldsymbol{\theta}$ is high (Katafygiotis and Zuev 2008; Valdebenito et al. 2010).

Besides FORM and SORM, another technique that can be classified as an approximate reliability method is the so-called Dimension Reduction Method (DRM), which was introduced in the field of structural reliability analysis in (Rahman and Xu 2004; Xu and Rahman 2004). The key idea of this approach is approximating the original performance

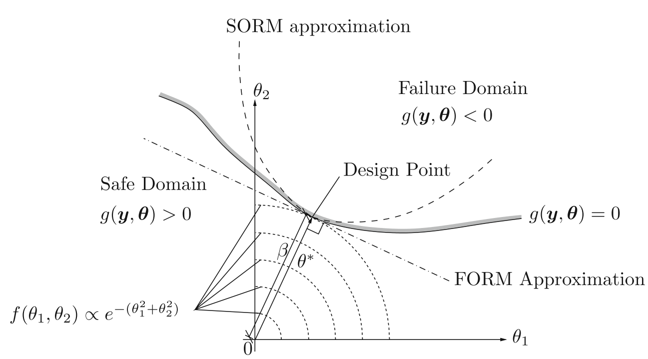

Fig. 6 Schematic representation of the design point and the FORM/SORM approximations in the standard normal space

function—with an associated n_θ-dimensional domain—as a summation of a number of simpler functions, where each of the domains of the latter functions has lower dimensionality. This approximate representation of the performance function can then be used to perform reliability analysis at reduced numerical costs using, e.g. uni-dimensional numerical integration (Rahman and Wei 2008), an appropriate response surface (Rahman and Wei 2006), etc.

A.2 Simulation methods

Simulation methods estimate the value of the probability integral by generating samples of the uncertain parameters according to some prescribed rule. The most widely known method of this class is Monte Carlo Simulation (MCS) (Metropolis and Ulam 1949). This method is based on generating N_S samples of $\boldsymbol{\theta}$ which are distributed according to $f(\boldsymbol{\theta})$. Then, the failure probability can be estimated as:

$$p \approx \hat{p} = \frac{1}{N_S}\sum_{i=1}^{N_S} I\left(\boldsymbol{y}, \boldsymbol{\theta}^{(s)}\right), \quad \boldsymbol{\theta}^{(s)} \sim f(\boldsymbol{\theta}) \tag{27}$$

where $I(\cdot)$ is an indicator function which is equal to one in case $g\left(\boldsymbol{y}, \boldsymbol{\theta}^{(s)}\right) \leq 0$ and zero, otherwise. The error in the estimator of the failure probability can be estimated by means of the coefficient of variation δ_{MC}, i.e. $\delta_{MC} = \sqrt{(1-\hat{p})/(N_S\hat{p})}$.

The MCS method is a general simulation technique, i.e. it is applicable to linear and non linear problems indifferently. Moreover, its efficiency is independent of the number of random variables involved in the problem under analysis. However, its major drawback is that for calculating low failure probabilities, a large number of samples (proportional to $1/p$) is required for generating a reliable estimator, i.e. with sufficient accuracy (or, equivalently, a low coefficient of variation). Hence, the numerical costs involved in estimating probabilities of rare occurrence of failure events may be extremely high and even prohibitive, especially when a structural system is modeled using large FE models. In view of this shortcoming, the so-called *advanced simulation methods* have been developed, which allow estimating low failure probabilities with increased efficiency if compared with MCS.

Advanced simulation methods are also based on generating samples of the uncertain parameters. However, specific sampling procedures are followed in order to increase the efficiency. An important characteristic of several advanced simulation methods is that they are specially designed for addressing reliability problems involving a large number of uncertain parameters (Katafygiotis and Zuev 2008; Valdebenito et al. 2010). Some examples of these advanced simulation methods are Importance Sampling (Schuëller and Stix 1987), Line Sampling (Schuëller et al. 2003), Subset Simulation (Au and Beck 2001), Domain Decomposition Method (Katafygiotis and Cheung 2006), Auxiliary Domain Method (Katafygiotis et al. 2007), Linked Importance Sampling (Katafygiotis and Zuev 2007; Neal 2005), Horseracing Simulation method (Katafygiotis and Zuev 2009; Zuev 2009), etc.

References

Agarwal H, Renaud J (2004) Reliability based design optimization using response surfaces in application to multidisciplinary systems. Eng Optim 36(3):291–311
Agarwal H, Renaud J (2006) New decoupled framework for reliability-based design optimization. AIAA J 44(7):1524–1531
Agarwal H, Mozumder C, Renaud J, Watson L (2007) An inverse-measure-based unilevel architecture for reliability-based design optimization. Struct Multidisc Optim 33(3):217–227
Aoues Y, Chateauneuf A (2008) Reliability-based optimization of structural systems by adaptive target safety—application to RC frames. Struct Saf 30(3):144–161
Aoues Y, Chateauneuf A (2010) Benchmark study of numerical methods for reliability-based design optimization. Struct Multidisc Optim 41(2):277–294
Arora J (1989) Introduction to optimum design. McGraw-Hill, New York
Arora J (ed) (2007) Optimization of structural and mechanical systems. World Scientific, Singapore
Au S (2005) Reliability-based design sensitivity by efficient simulation. Comput Struct 83(14):1048–1061
Au S (2006) Critical excitation of SDOF elasto-plastic systems. J Sound Vib 296(4–5):714–733
Au S, Beck J (2001) Estimation of small failure probabilities in high dimensions by subset simulation. Probab Eng Mech 16(4): 263–277
Basudhar A, Missoum S (2008) Adaptive explicit decision functions for probabilistic design and optimization using support vector machines. Comput Struct 86(19–20):1904–1917
Basudhar A, Missoum S (2009) A sampling-based approach for probabilistic design with random fields. Comput Methods Appl Mech Eng 198(47–48):3647–3655
Basudhar A, Missoum S, Sanchez A (2008) Limit state function identification using support vector machines for discontinuous responses and disjoint failure domains. Probab Eng Mech 23(1):1–11
Beer M, Liebscher M (2008) Designing robust structures—a nonlinear simulation based approach. Comput Struct 86(10):1102–1122
Beyer HG, Schwefel HP (2002) Evolution strategies—a comprehensive introduction. Nat Comput 1(1):3–52
Bichon B, Eldred M, Swiler L, Mahadevan S, McFarland J (2008) Efficient global reliability analysis for nonlinear implicit performance functions. AIAA J 46(10):2459–2468
Bichon B, Mahadevan S, Eldred M (2009) Reliability-based design optimization using efficient global reliability analysis. In: 50th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference. Palm Springs, California, AIAA 2009-2261
Bjerager P, Krenk S (1989) Parametric sensitivity in first order reliability theory. J Eng Mech 115(7):1577–1582
Bonnans JF, Gilbert J, Lemaréchal C, Sagastizábal C (2003) Numerical optimization. Springer, Heidelberg
Breitung K (1994) Asymptotic approximations for probability integrals. In: Lecture notes in mathematics, vol 1592. Springer, Berlin

Broding W, Diederich F, Parker P (1964) Structural optimization and design based on a reliability design criterion. J Spacecr Rockets 1(1):56–61

Bucher C, Bourgund U (1990) A fast and efficient response surface approach for structural reliability problems. Struct Saf 7(1):57–66

Chan KY, Skerlos S, Papalambros P (2006) Monotonicity and active set strategies in probabilistic design optimization. J Mech Des 128(4):893–900

Chan KY, Skerlos S, Papalambros P (2007) An adaptive sequential linear programming algorithm for optimal design problems with probabilistic constraints. J Mech Des 129(2):140–149

Chandu S, Grandhi R (1995) General purpose procedure for reliability based structural optimization under parametric uncertainties. Adv Eng Softw 23(1):7–14

Charnes A, Cooper WW (1959) Chance-constrained programming. Manage Sci 6(1):73–79

Chen X, Hasselman T, Neill D (1997) Reliability-based structural design optimization for practical applications. In: Proceedings of the 38th AIAA structures, structural dynamics, and materials conference, Florida

Cheng G, Xu L, Jiang L (2006) A sequential approximate programming strategy for reliability-based structural optimization. Comput Struct 84(21):1353–1367

Ching J, Hsieh Y (2007a) Approximate reliability-based optimization using a three-step approach based on subset simulation. J Eng Mech 133(4):481–493

Ching J, Hsieh Y (2007b) Local estimation of failure probability function and its confidence interval with maximum entropy principle. Probab Eng Mech 22(1):39–49

Cox D, Reid N (2000) The theory of the design of experiments. Chapman & Hall/CRC, Boca Raton

Davidson J, Felton L, Mart G (1977) Optimum design of structures with random parameters. Comput Struct 7(3):481–486

De Munck M, Moens D, Desmet W, Vandepitte D (2008) A response surface based optimisation algorithm for the calculation of fuzzy envelope FRFs of models with uncertain properties. Comput Struct 86(10):1080–1092

Der Kiureghian A, Zhang Y, Li CC (1994) Inverse reliability problem. J Eng Mech 120(5):1154–1159

Ditlevsen O (1978) Narrow reliability bounds for structural systems. Tech. Rep. 145, DCAMM (The Danish Center for Applied Mathematics and Mechanics)

Ditlevsen O, Madsen H (1996) Structural reliability methods. Wiley, Hoboken

Du X, Chen W (2004) Sequential optimization and reliability assessment method for efficient probabilistic design. J Mech Des 126(2):225–233

Ellingwood B (2001) Earthquake risk assessment of building structures. Reliab Eng Syst Saf 74(3):251–262

Enevoldsen I, Sørensen J (1993) Reliability-based optimization of series systems of parallel systems. J Struct Eng 119(4):1069–1084

Enevoldsen I, Sørensen J (1994) Reliability-based optimization in structural engineering. Struct Saf 15(3):169–196

Foschi R, Li H, Zhang J (2002) Reliability and performance-based design: a computational approach and applications. Struct Saf 24(2–4):205–218

Freudenthal A (1956) Safety and the probability of structural failure. ASCE Trans 121:1337–1397

Gasser M, Schuëller G (1997) Reliability-based optimization of structural systems. Math Methods Oper Res 46(3):287–307

Goldberg D (1989) Genetic algorithms in search, optimization, and machine learning. Addison Wesley, Reading

Grandhi R, Wang L (1998) Reliability-based structural optimization using improved two-point adaptive nonlinear approximations. Finite Elem Anal Des 29(1):35–48

Haftka R, Gürdal Z (1992) Elements of structural optimization, 3rd edn. Kluwer, Dordrecht

Heer E, Yang J (1971) Optimization of structures based on fracture mechanics and reliability criteria. AIAA J 9(4):621–628

Hellevik S, Langen I, Sørensen J (1999) Cost optimal reliability based inspection and replacement planning of piping subjected to CO^2 corrosion. Int J Pressure Vessels and Piping 76(8):527–538

Hilton H, Feigen M (1960) Minimum weight analysis based on structural reliability. J Eerosp Sci 27(9):641–652

Hurtado J (2004) An examination of methods for approximating implicit limit state functions from the viewpoint of statistical learning theory. Struct Saf 26(3):271–293

Hurtado J (2007) Filtered importance sampling with support vector margin: a powerful method for structural reliability analysis. Struct Saf 29(1):2–15

Jacobs J, Etman L, van Keulen F, Rooda J (2004) Framework for sequential approximate optimization. Struct Multidisc Optim 27(5):384–400

Jaynes E (1968) Prior probabilities. IEEE Trans Syst Sci Cybern 4(3):227–241

Jensen H (2005) Design and sensitivity analysis of dynamical systems subjected to stochastic loading. Comput Struct 83:1062–1075

Jensen H, Catalan M (2007) On the effects of non-linear elements in the reliability-based optimal design of stochastic dynamical systems. Int J Non-Linear Mech 42(5):802–816

Jensen H, Valdebenito M, Schuëller G (2008) An efficient reliability-based optimization scheme for uncertain linear systems subject to general Gaussian excitation. Comput Methods Appl Mech Eng 194(1):72–87

Jensen H, Valdebenito M, Schuëller G, Kusanovic D (2009) Reliability-based optimization of stochastic systems using line search. Comput Methods Appl Mech Eng 198(49–52):3915–3924

Jin R, Du X, Chen W (2003) The use of metamodeling techniques for optimization under uncertainty. Struct Multidisc Optim 25(2): 99–116

Johnson E, Proppe C, Spencer B Jr, Bergman L, Székely G, Schuëller G (2003) Parallel processing in computational stochastic dynamics. Probab Eng Mech 18(1):37–60

Jóźwiak S (1986) Minimum weight design of structures with random parameters. Comput Struct 23(4):481–485

Kanda J, Ellingwood B (1991) Formulation of load factors based on optimum reliability. Struct Saf 9(3):197–210

Katafygiotis L, Cheung S (2006) Domain decomposition method for calculating the failure probability of linear dynamic systems subjected to Gaussian stochastic loads. J Eng Mech 132(5): 475–486

Katafygiotis L, Wang J (2009) Reliability analysis of wind-excited structures using domain decomposition method and line sampling. J Struct Eng Mech 32(1):37–51

Katafygiotis L, Zuev K (2007) Estimation of small failure probabilities in high dimensions by adaptive linked importance sampling. In: Papadrakakis M, Charmpis D, Lagaros N, Tsompanakis Y (eds) ECCOMAS thematic conference on computational methods in structural dynamics and earthquake engineering (COMPDYN). Rethymno, Crete

Katafygiotis L, Zuev K (2008) Geometric insight into the challenges of solving high-dimensional reliability problems. Probab Eng Mech 23(2–3):208–218

Katafygiotis L, Zuev K (2009) Horseracing simulation algorithm for evaluation of small failure probabilities. In: Papadrakakis M, Kojic M, Papadopoulos V (eds) 2nd South-East European conference on computational mechanics (SEECCM 2009). Rhodes, Greece

Katafygiotis L, Moan T, Cheung S (2007) Auxiliary domain method for solving multi-objective dynamic reliability problems for nonlinear structures. Struct Eng Mech 25(3):347–363

Kaymaz I, Marti K (2007) Reliability-based design optimization for elastoplastic mechanical structures. Comput Struct 85(10): 615–625

van Keulen F, Vervenne K (2004) Gradient-enhanced response surface building. Struct Multidisc Optim 27(5):337–351

Kharmanda G, Mohamed A, Lemaire M (2002) Efficient reliability-based design optimization using a hybrid space with application to finite element analysis. Struct Multidisc Optim 24(3):233–245

Kirkpatrick S, Gelatt C, Vecchi M (1983) Optimization by simulated annealing. Science 220(4598):671–680

Koo H, Der Kiureghian A, Fujimura K (2005) Design-point excitation for non-linear random vibrations. Probab Eng Mech 20(2): 134–147

Koutsourelakis P (2008) Design of complex systems in the presence of large uncertainties: a statistical approach. Comput Methods Appl Mech Eng 197(49–50):4092–4103

Kupfer H, Freudenthal A (1977) Structural optimization and risk control. In: Kupfer H, Shinozuka M, Schuëller G (eds) Proceedings of the 2nd International Conference on Structural Safety and Reliability (ICOSSAR'77), Werner, Düsseldorf, Munich, pp 627–639

Kuschel N, Rackwitz R (1997) Two basic problems in reliability-based structural optimization. Math Methods Oper Res 46(3):309–333

Kwak B, Lee T (1987) Sensitivity analysis for reliability-based optimization using an AFOSM method. Comput Struct 27(3): 399–406

Lee I, Choi K, Du L, Gorsich D (2008) Inverse analysis method using MPP-based dimension reduction for reliability-based design optimization of nonlinear and multi-dimensional systems. Comput Methods Appl Mech Eng 198(1):14–27

Lee J, Kwak B (1995) Reliability-based structural optimal design using the Neumann expansion technique. Comput Struct 55(2):287–296

Lee JO, Yang YS, Ruy WS (2002) A comparative study on reliability-index and target-performance-based probabilistic structural design optimization. Comput Struct 80(3–4):257–269

Leite J, Topping B (1999) Parallel simulated annealing for structural optimization. Comput Struct 73(1–5):545–564

Li H, Foschi R (1998) An inverse reliability method and its application. Struct Saf 20(3):257–270

Li W, Yang L (1994) An effective optimization procedure based on structural reliability. Comput Struct 52(5):1061–1067

Liang J, Mourelatos Z, Tu J (2008) A single-loop method for reliability-based design optimisation. Int J Prod Dev 5(1–2):76–92

Lind N (1976) Approximate analysis and economics of structures. ASCE J Struct Div 102(ST6):1177–1196

Liu P, Der Kiureghian A (1986) Multivariate distribution models with prescribed marginals and covariances. Probab Eng Mech 1(2):105–112

Liu PL, Der Kiureghian A (1991) Optimization algorithms for structural reliability. Struct Saf 9(3):161–177

Madsen H, Torhaug R, Cramer E (1991) Probability-based cost benefit analysis of fatigue design, inspection and maintenance. In: Marine Structural Inspection, Maintenance and Monitoring Symposium. Society of Naval Architects and Marine Engineers, Arlington, Virginia

Marti K, Kaymaz I (2006) Reliability analysis for elastoplastic mechanical structures under stochastic uncertainty. ZAMM 86(5):358–384

Metropolis N, Ulam S (1949) The Monte Carlo method. J Am Stat Assoc 44(247):335–341

Mínguez R, Castillo E (2009) Reliability-based optimization in engineering using decomposition techniques and FORMS. Struct Saf 31(3):214–223

Missoum S, Ramub P, Haftka R (2007) A convex hull approach for the reliability-based design optimization of nonlinear transient dynamic problems. Comput Methods Appl Mech Eng 196(29–30):2895–2906

Moan T, Song R (2000) Implications of inspection updating on system fatigue reliability of offshore structures. J Offshore Mech Arct Eng 122(3):173–180

Moens D, Vandepitte D (2005) A survey of non-probabilistic uncertainty treatment in finite element analysis. Comput Methods Appl Mech Eng 194(12–16):1527–1555

Mohsine A, Kharmanda G, El-Hami A (2006) Improved hybrid method as a robust tool for reliability-based design optimization. Struct Multidisc Optim 32(3):203–213

Möller B, Beer M (2007) Engineering computation under uncertainty—capabilities of non-traditional models. Comput Struct 86(10):1024–1041

Moses F (1997) Problems and prospects of reliability-based optimization. Eng Struct 19(4):293–301

Moses F, Kinser D (1967) Optimum structural design with failure probability constraints. AIAA J 5(6):1152–1158

Murthy P, Subramanian G (1968) Minimum weight analysis based on structural reliability. AIAA J 6(10):2037–2039

Neal R (2005) Estimating ratios of normalizing constants using linked importance sampling. Tech. Rep. No. 0511, Dept. of Statistics, University of Toronto

Nikolaidis E, Burdisso R (1988) Reliability based optimization: a safety index approach. Comput Struct 28(6):781–788

Nocedal J, Wright S (1999) Numerical optimization. Springer, New York

Ormoneit D, White H (1999) An efficient algorithm to compute maximum entropy densities. Econom Rev 18:127–140

Papadrakakis M, Lagaros N (2002) Reliability-based structural optimization using neural networks and Monte Carlo simulation. Comput Methods Appl Mech Eng 191(32):3491–3507

Papadrakakis M, Lagaros N, Plevris V (2005) Design optimization of steel structures considering uncertainties. Eng Struct 27(9): 1408–1418

Pellissetti M (2009) Parallel processing in structural reliability. J Struct Eng Mech 32(1):95–126

Pellissetti MF, Schuëller GI, Pradlwarter HJ, Calvi A, Fransen S, Klein M (2006) Reliability analysis of spacecraft structures under static and dynamic loading. Comput Struct 84(21):1313–1325

Petryna Y, Krätzig W (2005) Computational framework for long-term reliability analysis of RC structures. Comput Methods Appl Mech Eng 194(12–16):1619–1639

Polak E (1997) Optimization: algorithms and consistent approximations. Springer, New York

Pu Y, Das P, Faulkner D (1997) A strategy for reliability-based optimization. Eng Struct 19(3):276–282

Rackwitz R (2001) Reliability analysis—a review and some perspectives. Struct Saf 23(4):365–395

Rahman S, Wei D (2006) A univariate approximation at most probable point for higher-order reliability analysis. Int J Solids Struct 43(9):2820–2839

Rahman S, Wei D (2008) Design sensitivity and reliability-based structural optimization by univariate decomposition. Struct Multidisc Optim 35(3):245–261

Rahman S, Xu H (2004) A univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics. Probab Eng Mech 19(4):393–408

Rajashekhar M, Ellingwood B (1993) A new look at the response surface approach for reliability analysis. Struct Saf 12(3): 205–220

Ramu P, Qu X, Youn B, Haftka R, Choi K (2006) Inverse reliability measures and reliability-based design optimization. Int J Reliab Saf 1(1–2):187–205

Rao S (1980) Structural optimization by chance constrained programming techniques. Comput Struct 12(6):777–781

Reddy M, Grandhi R, Hopkins D (1994) Reliability based structural optimization: a simplified safety index approach. Comput Struct 53(6):1407–1418

Royset J, Polak E (2004a) Implementable algorithm for stochastic optimization using sample average approximations. J Optim Theory Appl 122(1):157–184

Royset J, Polak E (2004b) Reliability-based optimal design using sample average approximations. Probab Eng Mech 19(4):331–343

Royset J, Der Kiureghian A, Polak E (2001a) Reliability-based optimal design of series structural systems. J Eng Mech 127(6):607–614

Royset J, Der Kiureghian A, Polak E (2001b) Reliability-based optimal structural design by the decoupling approach. Reliab Eng Syst Saf 73(3):213–221

Royset J, Der Kiureghian A, Polak E (2006) Optimal design with probabilistic objective and constraints. J Eng Mech 132(1):107–118

Schittkowski K (1983) On the convergence of a sequential quadratic programming method with an augmented Lagrangian line search function. Math Operforsch Stat Ser Optim 14:1–20

Schuëller G, Stix R (1987) A critical appraisal of methods to determine failure probabilities. Struct Saf 4(4):293–309

Schuëller G, Pradlwarter H, Koutsourelakis P (2003) A comparative study of reliability estimation procedures for high dimensions using FE analysis. In: Turkiyyah G (ed) Electronic proceedings of the 16th ASCE engineering mechanics conference (CD-ROM). University of Washington, Seattle, USA

Schuëller G, Pradlwarter H, Koutsourelakis P (2004) A critical appraisal of reliability estimation procedures for high dimensions. Probab Eng Mech 19(4):463–474

Schuëller GI, Pradlwarter HJ, Beck J, Au S, Katafygiotis L, Ghanem R (2005) Benchmark study on reliability estimation in higher dimensions of structural systems—an overview. In: Soize C, Schuëller GI (eds) Structural dynamics EURODYN 2005—Proceedings of the 6th international conference on structural dynamics. Millpress, Rotterdam, pp 717–722

Silvern D (1963) Optimization of system reliability. AIAA J 1(12):2872–2873

Spall J (2003) Introduction to stochastic search and optimization. Estimation, simulation and control. Wiley, Hoboken

Sues R, Cesare M, Pageau S, Wu JYT (2001) Reliability-based optimization considering manufacturing and operational uncertainties. J Aerosp Eng 14(4):166–174

Switzky H (1965) Minimum weight design with structural reliability. J Aircr 2(3):228–232

Taflanidis A, Beck J (2008a) An efficient framework for optimal robust stochastic system design using stochastic simulation. Comput Methods Appl Mech Eng 198(1):88–101

Taflanidis A, Beck J (2008b) Stochastic subset optimization for optimal reliability problems. Probab Eng Mech 23(2–3):324–338

Thierauf G, Cai J (1997) Parallel evolution strategy for solving structural optimization. Eng Struct 19(4):318–324

Tu J, Choi K, Park Y (2001) Design potential method for robust system parameter design. AIAA J 39(4):667–677

Umesha P, Venuraju M, Hartmann D, Leimbach K (2005) Optimal design of truss structures using parallel computing. Struct Multidisc Optim 29(4):285–297

Valdebenito M, Schuëller G (2010) Efficient strategies for reliability-based optimization involving non linear, dynamical structures. Comput Struct (in press)

Valdebenito M, Pradlwarter H, Schuëller G (2010) The role of the design point for calculating failure probabilities in view of dimensionality and structural non linearities. Struct Saf 32(2): 101–111

Vanmarcke E (1973) Matrix formulation of reliability analysis and reliability-based design. Comput Struct 3(4):757–770

Winterstein S, Ude T, Cornell C, Bjerager P, Haver S (1994) Environmental parameters for extreme response: inverse FORM with omission factors. In: Schuëller G, Shinozuka M, Yao J (eds) Proceedings of the 6th international conference on structural safety and reliability (ICOSSAR'93). A.A. Balkema, Rotterdam, pp 551–557

Wu Y (1994) Computational methods for efficient structural reliability and reliability sensitivity analysis. AIAA J 32(8):1717–1723

Wu YT, Millwater H, Cruse TA (1990) Advanced probabilistic structural analysis method for implicit performance functions. AIAA J 28(9):1663–1669

Xu H, Rahman S (2004) A generalized dimension-reduction method for multi-dimensional integration in stochastic mechanics. Int J Numer Methods Eng 61(12):1992–2019

Yang JS, Nikolaidis E (1991) Design of aircraft wings subjected to gust loads—a safety index based approach. AIAA J 29(5):804–812

Yang R, Gu L (2004) Experience with approximate reliability-based optimization methods. Struct Multidisc Optim 26(1–2):152–159

Youn B, Choi K, Park Y (2003) Hybrid analysis method for reliability-based design optimization. J Mech Des 125(2):221–232

Youn B, Choi K, Du L (2005) Adaptive probability analysis using an enhanced hybrid mean value method. Struct Multidisc Optim 29(2):134–148

Zhang J, Foschi R (2004) Performance-based design and seismic reliability analysis using designed experiments and neural networks. Probab Eng Mech 19(3):259–267

Zou T, Mahadevan S (2006) A direct decoupling approach for efficient reliability-based design optimization. Struct Multidisc Optim 31(3):190–200

Zuev K (2009) Advanced stochastic simulation methods for solving high-dimensional reliability problems. Ph.D. thesis, The Hong Kong University of Science and Technology