Dynamic Models for Yielding and Friction Hysteresis

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In materials or elements with hysteresis, for *any* monotonic or reciprocating deformation of a sufficiently large amplitude, response of stresses (forces) depend on the deformation history [11]. Such hysteretic behavior is commonly depicted as loops in graphs (2D plots) of periodic (oscillatory) output vs. periodic input. In rate-dependent hysteresis, the size and shape of the hysteresis loop changes with the rate or frequency of the input. If the loop collapses to a function (e.g., a curved line) for any input (e.g., quasi-static), then the system is *not* hysteretic [11]. Hysteresis implies a non-linear relationship between inputs and outputs: differential equation models for hysteresis must be nonlinear and convolution models for hysteresis must be nonhomogeneous. Linear visco-elastic materials are rate-dependent but are *not hysteretic* because forces and displacements are proportional in the limit of quasi-static loading. This document describes various Duhem hysteresis models [10, 16, 18, 21] which are nonlinear ordinary differential equations of the form

$$\dot{z}(t) = f(z(t), \dot{u}(t))\ g(\dot{u}(t)) \ , \qquad z(0) = z_0 \ , \tag{1}$$

where $z(t)$ is a force, $u(t)$ is a displacement, and $f(\cdot)$ and $g(\cdot)$ are functions. Usually, $g(\dot{u})$ is simply $\dot{u}$. Such differential equations relate the force, displacement, and velocity to the rate of change of force. Duhem models can be used to model the kinds of rate-independent hysteresis representative of material yielding and stick-slip friction. Consider the following ordinary differential equation,

$$\dot{z}(t) = \dot{u}(t) - |\dot{u}(t)|\, z^{\eta}(t) \ , \qquad z(0) = z_0 \ , \qquad -1 < z_0 < 1 \ . \tag{2}$$

The variable $z(t)$ represents the force normalized by the fully-plastic force or friction sliding force ($-1 < z(t) < 1 \ \forall \ t$), $u(t)$ represents a ductility ratio (the displacement divided by the *yield displacement*, defined here as the plastic force divided by the stiffness), and η is an odd positive integer.

Noting that $|\dot{u}| = \dot{u}\ \mathrm{sgn}(\dot{u})$, equation (2) may be expressed as a Duhem model with $g(\dot{u}) = \dot{u}$,

$$\dot{z} = (1 - z^{\eta}\ \mathrm{sgn}(\dot{u}))\ \ \dot{u} \ , \qquad z(0) = z_0 \ , \qquad -1 < z_0 < 1 \ . \tag{3}$$

and the slope of the force-displacement relationship is

$$\frac{dz}{du} = \frac{\dot{z}}{\dot{u}} = 1 - z^{\eta}\mathrm{sgn}(\dot{u}) \tag{4}$$

From this expression it is easy to see that:

- When the force is zero ($z = 0$), the dimensionless stiffness is 1 (the dimensional stiffness is the plastic force divided by the yield displacement).
- As the force approaches the plastic force ($z \to 1, \dot{u} > 0$ or $z \to -1, \dot{u} < 0$), dz/du approaches zero.
- When the velocity is positive, $dz/du = 1 - z^\eta$, and when the velocity is negative, $dz/du = 1 + z^\eta$.
- $dz/du \geq 0$; $\text{sgn}(\dot{z}) = \text{sgn}(\dot{u})$; and $(\dot{z})(\dot{u}) \geq 0$.

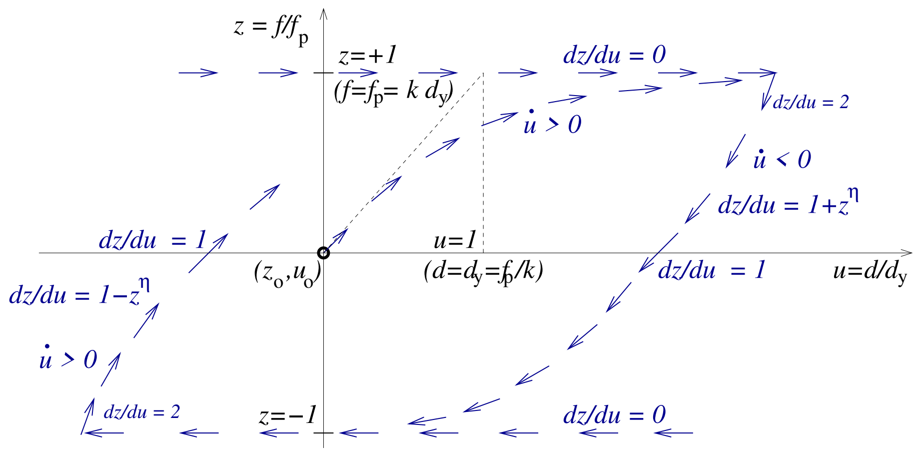

Figure 1. The vector field of dz/du for $\dot{z} = (1 - z^\eta \text{sgn}\dot{u})\dot{u}$ depends on $\text{sgn}\dot{u}$. Larger values of η result in a sharper "knee."

Extension 1 - even values of the exponent η

Replacing z^η in equation (3) with $|z|^\eta \text{sgn}(z)$ and noting that $\text{sgn}(a)\text{sgn}(b) = \text{sgn}(ab)$,

$$\dot{z} = (1 - |z|^\eta \ \text{sgn}(\dot{u}z)) \ \dot{u} \ , \tag{5}$$

allows the exponent η to be any positive value.

Extension 2 - purely dissipative

Replacing $\text{sgn}(\dot{u}z)$ in equation (5) with $(\beta\text{sgn}(\dot{u}z) + \gamma)$

$$\dot{z} = (1 - |z|^\eta \ (\beta \ \text{sgn}(\dot{u}z) + \gamma)) \ \dot{u} \ , \tag{6}$$

allows for a wide range of hysteretic forms, as shown in figure 2. If $\beta + \gamma = 1$, then $-1 < z < 1$. If $\eta > 0$, $\gamma > 0$ and $-\gamma < \beta < \gamma$ the model respects the Second Law of Thermodynamics [1, 12, 15] (that dissipated energy cannot be recovered).

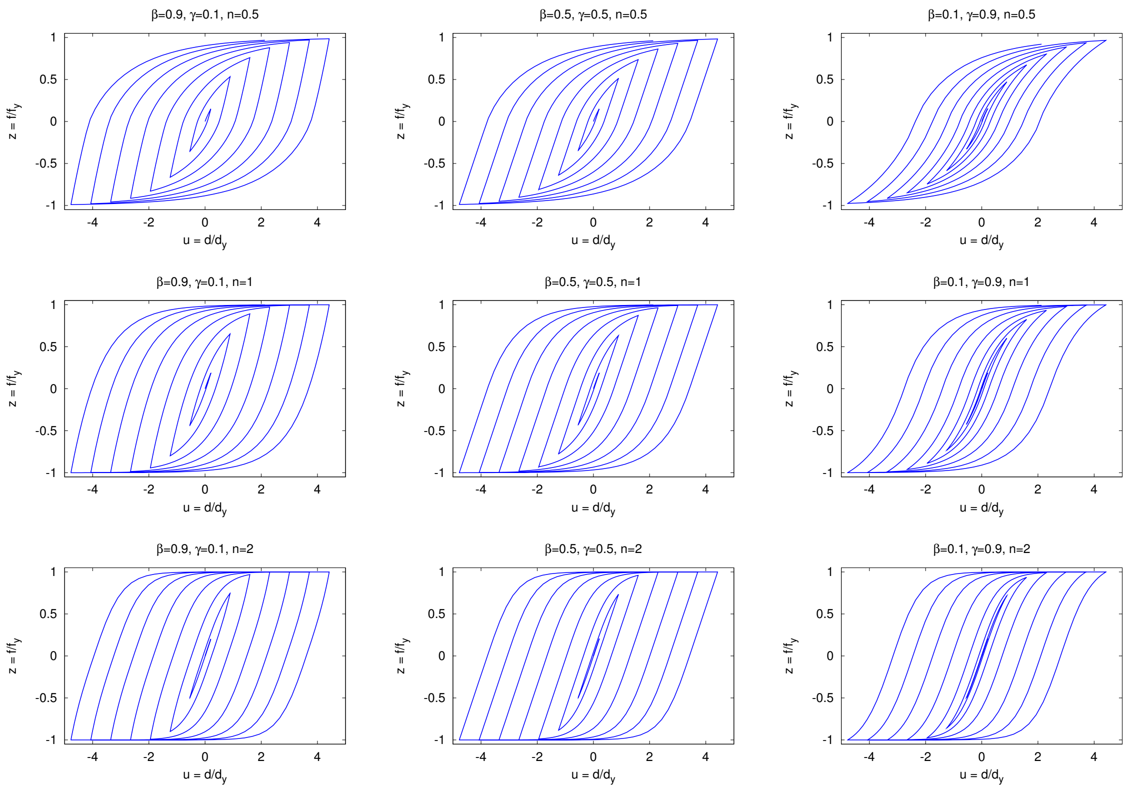

Figure 2. Dependence of hysteretic shape on β, γ, and η.

Extension 3 - scaling

Replacing the value 1 in equation (6) with a positive-valued scaling parameter A,

$$\dot{z} = (A - |z|^\eta \; (\beta \; \text{sgn}(\dot{u}z) + \gamma)) \; \dot{u} \; . \tag{7}$$

If $\beta + \gamma = A$ then $-A < z < A$. This is the "Bouc-Wen" model for hysteresis [5, 6, 15, 27] and is a Duhem hysteresis model with $g(\dot{u}) = \dot{u}$ [18, 25, 26].

If $A \gg 1$, (7) can be stiff. It is preferable to use (6) with $\alpha + \beta = 1$ and scale the force via $f(t) = f_{\mathsf{p}} z(t)$.

Extension 4 - isotropic biaxial hysteresis

Isotropic biaxial hysteretic behavior may be modeled in orthogonal directions x and y by coupling the hysteretic variables and velocities [13, 23].

$$\dot{z} = A \; \dot{u} - a(z, \dot{u}) \; z \tag{8}$$

where $z = [z_x, z_y]$, $\dot{u} = [\dot{u}_x, \dot{u}_y]$, A is the scaling parameter, usually set to 1, and

$$a(z, \dot{u}) = [\beta(|z_x \dot{u}_x| + |z_y \dot{u}_y|) + \gamma(z_x \dot{u}_x + z_y \dot{u}_y)] \; (z_x^2 + z_y^2)^{(\eta-2)/2} \tag{9}$$

This is a Duhem hysteresis model with $g(\dot{u}) = 0$.

Extension 5 - material hardening

Hysteretic behavior with material hardening (a non-zero post-yield stiffness) may be modeled modeled by combining equation (6) or (8) with

$$f(t) = f_{\mathsf{p}}((1-\kappa)z(t) + \kappa u(t)) \ , \tag{10}$$

where f_{p} is a plastic force level, u is the displacement divided by the yield displacement (the ductility), and κ is the ratio of the post yield stiffness to the pre-yield stiffness. When using (10), set $A = \beta + \gamma = 1$.

Extension 6 - degrading strength and stiffness

To model the accumulation of damage [4], strength f_{p} and other model parameters may be linked to a damage accumulation index $\mathcal{D}$, where $\dot{\mathcal{D}} \approx |\dot{u} - \dot{z}|$ and

$$f_{\mathsf{p}}(t) = \frac{f_{\mathsf{p}}(0)}{1 + \mathcal{D}(t)/d_0} \ , \tag{11}$$

where d_0 is a positive constant.

Extension 7 - stick-slip friction

Friction models [2] are similar to models for yielding hysteresis. For example, the Dahl friction model [8, 9], $\dot{z} = (\ 1 - z \ \mathrm{sgn}(\dot{u})\ )^{\eta} \ \dot{u}$ is a Duhem hysteresis model and is equivalent to (5) with $\eta = 1$.

The "LuGre" friction model [7], developed in a collaboration of researchers from universities in Lund and Grenoble, is a friction model which captures the Stribeck ("stick-slip") effect [2]. The following is a re-parameterization of the LuGre model as it is presented in [18, 25, 26], and is consistent with the the well-conditioned and purely dissipative dynamic hysteresis model (6).

$$\dot{z}(t) = (1 - |z|^{\eta} \ (\beta \ \mathrm{sgn}(\dot{u}z) + \gamma) \ q(\dot{u})) \ \dot{u} \tag{12}$$

$$q(\dot{u}) = \frac{q_0}{1 + e^{-(\dot{u}/v_{\mathsf{s}})^2}} \tag{13}$$

$$f(t) = f_{\mathsf{c}} \ q_0^{1/\eta} \ (z(t) + T_{\mathsf{s}} \dot{z}(t)) \tag{14}$$

Generally, the static friction scaling parameter $q_0 \approx 1$, the stiction decay time $T_{\mathsf{s}} \approx 0.05$ s, and the Stribeck velocity $v_{\mathsf{s}} \approx 0.001$/s. As the stiction decay time T_{s} goes to zero, the static friction drops to the Coulomb friction f_{c}. Observations of the behavior of this model indicates that the coefficient g_0 affects the stiffness more than the peak static friction and T_{s} affects the peak static friction more than the stiffness.

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