Principal Input and Output Directions and Hankel Singular Values

CEE 629. System Identification

Duke University, Fall 2017

1 Continuous-time systems in the frequency domain

In the frequency domain, the input-output relationship of a LTI system (with r inputs, m outputs, and n internal states) is represented by the m-by-r rational frequency response function matrix equation

$$y(\omega) = H(\omega)u(\omega) \ .$$

At a frequency ω a set of inputs with amplitudes $u(\omega)$ generate steady-state outputs with amplitudes $y(\omega)$. (These amplitude vectors are, in general, complex-valued, indicating magnitude and phase.)

The singular value decomposition of the transfer function matrix is

$$H(\omega) = Y(\omega) \ \Sigma(\omega) \ U^*(\omega) \tag{1}$$

where:
$U(\omega)$ is the r by r orthonormal matrix of input amplitude vectors, $U^*U = I$, and
$Y(\omega)$ is the m by m orthonormal matrix of output amplitude vectors, $Y^*Y = I$
$\Sigma(\omega)$ is the m by r diagonal matrix of singular values, $\Sigma(\omega) = \text{diag}(\sigma_1(\omega), \sigma_2(\omega), \cdots \sigma_n(\omega))$
At any frequency ω, the singular values are ordered as: $\sigma_1(\omega) \geq \sigma_2(\omega) \geq \cdots \geq \sigma_n(\omega) \geq 0$

Re-arranging the singular value decomposition of $H(s)$,

$$H(\omega)U(\omega) = Y(\omega) \ \Sigma(\omega)$$

or

$$H(\omega) \ u_i(\omega) = \sigma_i(\omega) \ y_i(\omega)$$

where $u_i(\omega)$ and $y_i(\omega)$ are the i-th columns of $U(\omega)$ and $Y(\omega)$. Since $||u_i(\omega)||_2 = 1$ and $||y_i(\omega)||_2 = 1$, the singular value $\sigma_i(\omega)$ represents the scaling from inputs with complex amplitudes $u_i(\omega)$ to outputs with amplitudes $y_i(\omega)$. Inputs amplitudes $u_1(\omega)$ results in outputs $\sigma_1 y_1(\omega)$ with the largest L_2 norm. These are called the principal input and output directions. At a frequency ω, the minimum and maximum singular values bound the L_2 norm of the response amplitudes for any set of unit input amplitudes u, ($||u||_2 = 1$),

$$\sigma_n(\omega) \leq ||y(\omega)||_2 \leq \sigma_1(\omega) \ .$$

In modal coordinates (assuming diagonalizeable dynamics),

$$H(\omega) = \bar{C}(\text{i}\omega I - \Lambda)^{-1}\bar{B} = \sum_{i=1}^{n} \bar{C} 1_i \bar{B}/(\text{i}\omega - \lambda_i) \tag{2}$$

where 1_i is a square zero matrix with a 1 on the i-th diagonal element. Elements of the modal input matrix $\bar{B}_{ir}$ represent the coupling of input r to mode i, and elements of the modal output matrix $\bar{C}_{mi}$ represent the coupling of mode i to output m. If the i-th mode is lightly damped ($0 < \zeta_i \ll 1$), it dominates the transfer function at its resonant frequency,

$$H\left(\omega_{\mathrm{n}i}\sqrt{1-\zeta_i^2}\right) \approx \bar{C} 1_i \bar{B} / (\zeta_i \omega_{\mathrm{n}i}) = c_i b_i / \left(\zeta_i \omega_{\mathrm{n}i}\right) \ ,$$

where c_i and b_i are the i-th column and row of $\bar{C}$ and $\bar{B}$, respectively. For a classically-damped system with force-input and velocity-output the modal output matrix, $\bar{C}$, is real and the modal input matrix, $\bar{B}$, is complex. However, if the damping is light, $H(\omega_{\mathrm{d}i})$ is primarily real at the damped natural frequencies ($\omega_{\mathrm{d}i} = \omega_{\mathrm{n}i}\sqrt{1-\zeta_i^2}$). In the dynamics of classically-damped structures, the values of $\bar{c}_i$ and $\bar{b}_i$ can be thought of as the values of the (real-valued) i-th mode shape at the measurement and forcing locations, respectively.

The principal input and output directions at the resonant frequencies can be approximated by the singular value decomposition of the (m-by-r) matrix $\bar{c}_i\bar{b}_i$, with the associated amplification being the largest singular value of $\bar{c}_i\bar{b}_i$ divided by $\zeta_i\omega_{\mathrm{n}i}$. (Keep in mind here that $\bar{c}_i$ is the i-th column of the output matrix *in modal coordinates* and $\bar{b}_i$ is the i-th row of the input matrix *in modal coordinates*.) The dyad $\bar{c}_i\bar{b}_i$ is rank-one; it has one non-zero singular value equal to $(||\bar{c}_i||\ ||\bar{b}_i||)$. So at resonant frequencies,

$$\sigma_1\left(\omega_{\mathrm{n}i}\sqrt{1-\zeta_i^2}\right) \approx \sigma^{(i)} = \left(||\bar{c}_i||\ ||\bar{b}_i||\right) / (\zeta_i \omega_{\mathrm{n}i}) \ . \tag{3}$$

The variable $\sigma^{(i)}$ is called the *Hankel singular value* of the i-th mode. It is an approximation of the largest singular value of the transfer function matrix at the i-th resonant frequency and is expressed in terms of the continuous-time representation of the input to the i-th mode, the output at the i-th mode, and the natural frequency and damping ratio of the i-th mode. The maximum singular value of the transfer function matrix at other frequencies is very nearly the sum of the transfer function magnitudes, $\sigma_1(\omega) \approx \sum_{i,j} |H_{ij}(\omega)|$. The amplification from inputs to outputs at mode i can be increased by moving sensor and actuator locations to extrema of the i-th mode.

2 Discrete-time systems in the time domain

Now consider the response of a LTI discrete-time system (having r inputs, m outputs, and n internal states) to a unit impulse $u(0) = 1$. At times $k \in [1, 2, \ldots]$, the output response to a unit impulse at $t = 0$ is

$$H(k) = CA^{k-1}B \text{ ... } \left[\; CB, \;\; CAB, \;\; CA^2B, \;\; CA^3B, \;\; \cdots \; \right] .$$

Likewise, the output response at times $k \in [1, 2, \ldots]$ to a unit impulse $u(-1) = 1$ are

$$H(k+1) = CA^{k}B \text{ ... } \left[\; CAB, \;\; CA^2B, \;\; CA^3B, \;\; CA^4B, \;\; \cdots \; \right] .$$

and the outputs at times $k \in [1, 2, ...]$ to a unit impulse $u(-2) = 1$ are

$$H(k+2) = CA^{k+1}B \text{ ... } \left[\; CA^2B, \;\; CA^3B, \;\; CA^4B, \;\; CA^5B, \;\; \cdots \; \right] ,$$

and the outputs at times $k \in [1, 2, ...]$ to a unit impulse $u(-3) = 1$ are

$$H(k+3) = CA^{k+2}B \text{ ... } \left[\; CA^3B, \;\; CA^4B, \;\; CA^5B, \;\; CA^6B, \;\; \cdots \; \right] ,$$

and the outputs at times $k \in [1, 2, ...]$ to a unit impulse $u(-4) = 1$ are

$$H(k+4) = CA^{k+3}B \text{ ... } \left[\; CA^4B, \;\; CA^5B, \;\; CA^6B, \;\; CA^7B, \;\; \cdots \; \right] ,$$

and so on, and so on. The sequence of m-by-r matrices $[CB, \; CAB, \; CA^2B, \; CA^3B, \; \cdots]$ are called *Markov parameters* of the system, and are independent of the state-space realization.

In general, the output responses at future times $k \in [1, 2, \ldots]$, due to a sequence of K past impulses $[u(0), \; u(-1), \; u(-2), \; u(-3), \; u(-4), \; \cdots , \; u(-K)]$ can be computed as a convolution sum or as a matrix-vector product.

$$\begin{aligned}
y(j) &= \sum_{k=0}^{-K} H(j-k)u(k) \\
\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \\ \vdots \\ y(M) \end{bmatrix} &= \begin{bmatrix} CB & CAB & CA^2B & CA^3B & \cdots & CA^{R-1}B \\ CAB & CA^2B & CA^3B & CA^4B & \cdots & CA^{R-0}B \\ CA^2B & CA^3B & CA^4B & CA^5B & \cdots & CA^{R+1}B \\ CA^3B & CA^4B & CA^5B & CA^6B & \cdots & CA^{R+2}B \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ CA^{M-1}B & CA^{M}B & CA^{M+1}B & CA^{M+2}B & \cdots & CA^{M+R-2}B \end{bmatrix} \begin{bmatrix} u(0) \\ u(-1) \\ u(-2) \\ u(-3) \\ \vdots \\ u(-R) \end{bmatrix} \\
y_{(1,M)} &= \mathsf{H}_{(M,R)} \; u_{(0,-R)} \; .
\end{aligned}$$

The block-symmetric matrix $\mathsf{H}_{(M,R)}$ is a *Hankel matrix* of Markov parameters; it is a matrix representation of the input-output relationship for the system in the discrete-time domain.

Note that

$$\mathsf{H}_{(M,R)} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{M-1} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{R-1}B \end{bmatrix} = \mathcal{P}_M \ \mathcal{Q}_R \tag{4}$$

$$= \begin{bmatrix} CTT^{-1} \\ CTT^{-1}A \\ CTT^{-1}A^2 \\ \vdots \\ CTT^{-1}A^{M-1} \end{bmatrix} TT^{-1} \begin{bmatrix} TT^{-1}B & ATT^{-1}B & A^2TT^{-1}B & \cdots & A^{R-1}TT^{-1}B \end{bmatrix}$$

$$= \begin{bmatrix} \bar{C} \\ \bar{C}\Lambda \\ \bar{C}\Lambda^2 \\ \bar{C}\Lambda^3 \\ \vdots \\ \bar{C}\Lambda^{M-1} \end{bmatrix} \begin{bmatrix} \bar{B} & \Lambda\bar{B} & \Lambda^2\bar{B} & \Lambda^3\bar{B} & \cdots & \Lambda^{R-1}\bar{B} \end{bmatrix} = \bar{\mathcal{P}}_M \ \bar{\mathcal{Q}}_R \tag{5}$$

where $AT = T\Lambda$ and $x(k) = Tq(k)$ and the over-bar indicates input and output matrices in modal coordinates. The matrix $\mathcal{P}_M$ is an *observability matrix*, the matrix $\mathcal{Q}_R$ is a *controllability matrix*, Λ is a diagonal matrix of the eigenvalues of the discrete-time dynamics matrix, A, and $\bar{B}$ and $\bar{C}$ are the input and output matrices in of the discrete-time system in modal coordinates. For controllable and observable systems, the matrices $\mathcal{P}$ and $\mathcal{Q}$ are rank-n, and $\mathsf{H}_{M,R}$ is therefore also rank-n.

The sequence of free responses computed by equation (4) can be represented in modal coordinates, $y(k) = \bar{C}q(k)$. The portion of the modal response in mode i is $\bar{c}_i q_i(k)$ where $\bar{c}_i$ is the i-th column of $\bar{C}$ and $q_i(k)$ is the i-th modal coordinate at time instant k. Components of the observability and controllability matrices in modal coordinates, $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$, can be expanded as

$$\bar{C}\Lambda^\eta = \sum_{i=1}^{n} \bar{c}_i \lambda_i^\eta$$

$$\Lambda^\nu \bar{B} = \sum_{i=1}^{n} \lambda_i^\nu \bar{b}_i$$

where $\bar{c}_i$ is the i-th column of $\bar{C}$ and $\bar{b}_i$ is the i-th row of $\bar{B}$. A component of the Hankel matrix is therefore a product of these sums, and since the modal state sequence $q_i(k)$ couples only only to the modal input vector $\bar{b}_i$ and modal output vector $\bar{c}_i$, the product of sums may be written as a sum of products,

$$\bar{C}\Lambda^\eta\Lambda^\nu\bar{B} = \left(\sum_{i=1}^{n} \bar{c}_i\lambda_i^\eta\right)\left(\sum_{i=1}^{n} \lambda_i^\nu\bar{b}_i\right) = \sum_{i=1}^{n} \lambda_i^{\eta+\nu}\bar{c}_i\bar{b}_i$$

Extending this sum of products to the full Hankel matrix,

$$\mathsf{H}_{(M,R)} = \sum_{i=1}^{n} \left[\begin{array}{c} \bar{c}_i \\ \bar{c}_i \lambda_i \\ \bar{c}_i \lambda_i^2 \\ \bar{c}_i \lambda_i^3 \\ \vdots \\ \bar{c}_i \lambda_i^{M-1} \end{array} \right] \left[\begin{array}{cccccc} \bar{b}_i & \lambda_i \bar{b}_i & \lambda_i^2 \bar{b}_i & \lambda_i^3 \bar{b}_i & \cdots & \bar{b}_i \lambda_i^{R-1} \end{array} \right] = \sum_{i=1}^{n} \left[\bar{\mathcal{P}}_{Mi} \ \bar{\mathcal{Q}}_{Ri} \right] \tag{6}$$

The vectors $\bar{\mathcal{P}}_{Mi}$ and $\bar{\mathcal{Q}}_{Ri}$ are the i-th column and row of $\bar{\mathcal{P}}_M$ and $\bar{\mathcal{Q}}_R$, respectively. The portion of the Hankel matrix corresponding to the free response of the i-th mode is a rank-one matrix, and its sole non-zero singular value is

$$\begin{aligned} \sigma^{(i)} &= \left\| \begin{array}{c} \bar{c}_i \\ \bar{c}_i \lambda_i \\ \bar{c}_i \lambda_i^2 \\ \bar{c}_i \lambda_i^3 \\ \vdots \\ \bar{c}_i \lambda_i^{M-1} \end{array} \right\| \left\| \begin{array}{cccccc} \bar{b}_i & \lambda_i \bar{b}_i & \lambda_i^2 \bar{b}_i & \lambda_i^3 \bar{b}_i & \cdots & \bar{b}_i \lambda_i^{R-1} \end{array} \right\| = ||\bar{\mathcal{P}}_{Mi}|| \ ||\bar{\mathcal{Q}}_{Ri}|| \qquad (7) \\ &= ||\bar{c}_i|| \left(1 + |\lambda_i|^2 + |\lambda_i|^4 + |\lambda_i|^6 + \cdots \right)^{1/2} \ ||\bar{b}_i|| \left(1 + |\lambda_i|^2 + |\lambda_i|^4 + |\lambda_i|^6 + \cdots \right)^{1/2} \\ &= ||\bar{c}_i|| \ ||\bar{b}_i|| \ \left(1 + |\lambda_i|^2 + |\lambda_i|^4 + |\lambda_i|^6 + \cdots \right) \\ &= ||\bar{c}_i|| \ ||\bar{b}_i|| \ / \ \left(1 - |\lambda_i|^2\right) \end{aligned}$$

where, in the last steps, the sequences are extended to the limits $R \to \infty$ and $M \to \infty$. Eigenvalues of the discrete-time dynamics, λ_i, are related to the natural frequency, damping ratio, and time step, Δt, through

$$\lambda_i = e^{-\zeta_i \omega_{\mathrm{n}i} \Delta t} \left(\cos \omega_{\mathrm{n}i} \sqrt{1 - \zeta_i^2} \Delta t + \mathrm{i} \sin \omega_{\mathrm{n}i} \sqrt{1 - \zeta_i^2} \Delta t \right)$$

so $|\lambda_i|^2 = e^{-2\zeta_i \omega_{\mathrm{n}i} \Delta t}$, and

$$\sigma^{(i)} = 2 \ ||\bar{c}_i|| \ ||\bar{b}_i|| \ / \ \left(1 - e^{-2\zeta_i \omega_{\mathrm{n}i} \Delta t}\right) \ . \tag{8}$$

The variable $\sigma^{(i)}$ is the *Hankel singular value* for the i-th mode expressed in terms of the discrete-time representation of the input to the i-th mode, the output of the i-th mode, and the natural frequency and damping ratio of the i-th mode. Computing $\sigma^{(i)}$ using equation (8) requires only a small fraction of the computational time required for a singular value decomposition of $\mathsf{H}_{M,R}$, and does not require the computation or storage of $\mathsf{H}_{M,R}$, $\mathcal{P}_M$, or $\mathcal{Q}_R$. With the scaling factor of 2, $\sigma^{(i)}$ in equation (8) numerically matches $\sigma^{(i)}$ in equation (3) to within numerical round-off. (Note again that B in equation (3) is the input matrix in modal coordinates for a *continuous-time* system, whereas $\bar{B}$ in equation (8) is the input matrix in modal coordinates for the associated *discrete-time* system.)

The identification of linear systems from finite-duration discrete-time data using methods such as the eigensystem realization algorithm involves the singular value decomposition of Hankel matrices of Markov parameters. In lightly-damped systems, the eigenvalues of A, the columns of C, and the rows of B occur in complex-conjugate pairs, and it is reasonable to expect that the singular values of H would also occur in pairs. When the singular values of the Hankel matrix of such systems do not occur in pairs it is can be due to an inadequate sampling of the Markov parameters or noise in the measured data. As a general rule of thumb, the sample interval Δt should be one tenth the shortest natural period and there should be at least a few cycles of the longest period represented in the sequence of Markov parameters. The number of Markov parameters in a row/column of H should therefore be about $4\pi/(\omega_{\mathrm{n}i}\Delta t)$. If these rules are observed, the product of vector norms $||\mathcal{P}_i||||\mathcal{Q}_i||$ are very nearly the same as the singular values of the (finite-dimensional) Hankel matrix, $\mathsf{H}_{M,R}$. For *very* long sequences of Markov parameters, $(20\pi/(\omega_1\Delta t)$, representing dozens of fundamental periods of response) $||\mathcal{P}_i||||\mathcal{Q}_i||$ approaches the Hankel singular values. Note, however, that raising the discrete-time dynamics matrix to a very high power (Λ^k for $k \gg n$) can incur round-off error.

3 Summary

In MIMO systems, the maximum singular values of transfer function matrices show how strongly inputs can couple to outputs. The maximum singular values of the transfer function matrix at resonant frequencies are the strengths of these couplings for each mode. The left- and right- singular vectors give the associated input and output amplitudes.

Maximum singular values of the matrix of frequency response functions evaluated at the resonant frequencies, $\sigma^{(i)}$, are the same as the singular values of an infinitely-large Hankel matrix of Markov parameters, $\sigma^{(i)}$. These are called *Hankel singular values*. Hankel singular values are system properties that indicate the strength of modal coupling from inputs to outputs for lightly-damped systems. Hankel singular values are easily computed from realizations in modal coordinates.

As the finite dimensional Hankel matrix (equation (4)) gets large in dimension, its singular values approach the Hankel singular values (equations (3) and (8)) but become prohibitively expensive to compute, and can suffer from round-off errors.