ENERGY METHODS AND CASTIGLIANO’S THEOREMS
CE 130 — Structural Design and Optimization
Spring, 2006

- Strain Energy: $U = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} \, dV$
- External Work: $W_E = U = \int F \, dD$; Complementary Work: $W_E^* = U^* = \int D \, dF$
- Superposition: $N = N_o + \sum v_i F_i; M = M_o + \sum m_i F_i; V = V_o + \sum v_i F_i$; etc.
- Castigliano’s First Theorem: $F_i = \frac{\partial U}{\partial D_i}$
- Castigliano’s Second Theorem: $D_i = \frac{\partial U^*}{\partial F_i}$
- Linear Elastic Systems: $U = U^*$

Mechanical Loads

**Axial**

$U = \frac{1}{2} \int_l N^2 \frac{1}{EA} \, dl = \frac{1}{2} \sum N_i^2 \frac{1}{EA} \quad n_i = \frac{\partial N}{\partial F_i} \quad D_i = \int_l \frac{N_i}{EA} \, dl = \sum N_i L \frac{1}{EA}$

**Bending**

$U = \frac{1}{2} \int_l M^2 \frac{1}{EI} \, dl \quad m_i = \frac{\partial M}{\partial F_i} \quad D_i = \int_l \frac{M m_i}{EI} \, dl$

**Shear**

$U = \frac{1}{2} \int_l V^2 \frac{1}{G(A/\alpha)} \, dl \quad v_i = \frac{\partial V}{\partial F_i} \quad D_i = \int_l \frac{V v_i}{G(A/\alpha)} \, dl$

**Torsion**

$U = \frac{1}{2} \int_l T^2 \frac{1}{GJ} \, dl \quad t_i = \frac{\partial T}{\partial F_i} \quad D_i = \int_l \frac{T v_i}{GJ} \, dl$

Temperature Loads

**Axial**

$U = \sum N \alpha \Delta T L \quad \frac{\partial U}{\partial F_i} = \sum \frac{\partial N}{\partial F_i} \alpha \Delta T L$

**Bending**

$U = \int_l M \alpha \left[ \frac{\Delta T_h - \Delta T_i}{h} \right] \, dl \quad \frac{\partial U}{\partial F_i} = \int_l \frac{\partial M}{\partial F_i} \alpha \left[ \frac{\Delta T_h - \Delta T_i}{h} \right] \, dl$

Statically Indeterminate Structures and Superposition

1. Remove $I$ redundants, $R_i, i = 1, \ldots, I$, where $I$ is the degree of indeterminacy.
2. Solve for the internal forces, $M_o, N_o, V_o$, in the resulting statically determinate structure (without the redundants), due to the real applied loads.
3. Now, remove all of the real applied loads, and apply $I$ unit virtual loads to the structure in the direction of the redundants, one at a time.
4. Solve for $I$ sets of internal forces, $m_i, n_i, v_i$, in each of the $I$ different statically determinate systems.
5. Apply superposition for moments, axial forces, and shears.

$$M = M_o + \sum_{i=1}^{I} R_i m_i \quad N = N_o + \sum_{i=1}^{I} R_i n_i \quad V = V_o + \sum_{i=1}^{I} R_i v_i$$

6. Write $I$ statements of Castigliano’s Theorem, one for each virtual system, and enforce compatibility with respect to support settlement, and relative positions, and solve for the redundants, $R_i$. 