Introduction

In this project you will experience the application of numerical optimization in the design optimization of a 3D truss. Once your truss is optimized, you will carry out a safety analysis. The project has five tasks:

1. Sketch your truss design.
2. Quantify the initial design of the truss in terms of node locations, bar locations, and bar cross sections.
3. Make use of a provided analysis code that computes truss bar forces and node displacements of the truss.
4. Make use of numerical optimization in order to minimize the weight of the truss such that the truss does not have excessive displacement and such that the bars do not buckle or yield.
5. Determine the safety of your optimized design.

Most design problems can be stated as a constrained optimization, in which optimal values for a set of design variables or design parameters, $x = [x_1, x_2, x_3, \ldots, x_n]$, minimize an objective function $f(x)$ such that a set of constraint inequalities are satisfied.

The objective function may be the cost or price of the system, the amount of material used, or an aspect of the performance of the system. The inequalities typically represent safety criteria, serviceability criteria, and other practicalities. In a structural design, inequality constraints include criteria such as (i) the stress shall not exceed the yield stress in each component, (ii) the compression force in a column shall remain less than the buckling strength of the column, and (iii) the structural deformation shall not induce discomfort to people in or on the structure.

The inequality constraints are functions of the set of design parameters. A set of $m$ inequality constraints are conventionally written as $g(x) \leq 0$, or $g_1(x) \leq 0$, $g_2(x) \leq 0$, $\cdots$, $g_m(x) \leq 0$. With this notation, a constrained optimization problem may be written:

$$\begin{align*}
\text{minimize} \quad & f(x_1, x_2, \ldots, x_n) \\
\text{subject to} \quad & g_1(x_1, x_2, \ldots, x_n) \leq 0, \\
& g_2(x_1, x_2, \ldots, x_n) \leq 0, \\
& \quad \vdots \\
& g_m(x_1, x_2, \ldots, x_n) \leq 0.
\end{align*}$$

Note that any inequality $q \leq q_{\text{max}}$ can be written $q - q_{\text{max}} \leq 0$ or $q/q_{\text{max}} - 1 \leq 0$. Similarly, an inequality $q_{\text{min}} \leq q$ can be written $1 - q/q_{\text{min}} \leq 0$.

For problems in which the objective function $f(x)$ and constraint inequalities $g(x) \leq 0$ are represented by simple mathematical expressions, linear algebra and multivariable calculus can be used to compute the optimum solution without iteration. In the vast majority of practical problems, however, the equations are much too complicated for this kind of direct approach. Instead, computer-based numerical methods are used to evaluate and compare the costs and feasibility of a series of particular trial designs. The ability to quickly compute solutions to constrained optimization problems allows designers to rapidly automate the iterative process of analysis, evaluation, and re-design.
Design Requirements

In this project each group will conceptualize, analyze, and optimize a 3D truss. The coordinates of the reaction points and the coordinates, magnitudes, and directions of the applied loads are shown in Figure 1. The bars shall be made of steel: elastic modulus, \( E = 200,000 \, \text{N/mm}^2 \), mass density of \( 7.86 \times 10^{-9} \, \text{T/mm}^3 \), and yield stress, \( S_y = 250 \, \text{N/mm}^2 \). Units of N, mm, s, and T are consistent ... \( 1 \, \text{N} = (1 \, \text{T})(1 \, \text{mm/s}^2) \).

The truss is to be built from circular tubes (outer diameter \( D_{oi} \) and a tube wall thickness \( t_i = D_{oi}/20 \)). The truss is to support vertical (downward) loads, horizontal loads in the \( y \)-direction, and its own self-weight. The truss is to be supported at nodes 1,2,3, and 4 at the coordinates shown on the \( x - y \) plane. The truss is to be loaded at nodes 5,6,7, and 8 at the coordinates shown. You may design the truss using the eight nodes provided, or you may add additional nodes in order to make use of shorter bars. The design shall specify:

- the \((x_n, y_n, z_n)\) coordinates of the additional nodes you may add;
- the bar locations in terms of the joints the bars connect; and
- the diameter of each bar \((D_{oi})\).

![Figure 1. Configuration of the reaction nodes and load nodes. The vertical (downward, \(-z\)) forces have a value of 10 kN. (40 kN total). The horizontal \((+y)\) forces have a value of 2 kN. (8 kN total).]

The volume of material used shall be minimized such that the following criteria are satisfied:

- Safety against yielding shall be satisfied in each bar with a safety factor of 1.2 on the loads and a safety factor of 0.9 on the yield strength of the bars; and
- Safety against buckling shall be satisfied in each bar with safety factor of 1.2 on the loads and a safety factor of 0.8 on the buckling strength of the bars.
Task 1: Design Conceptualization

Think of a truss that inter-connects the reaction nodes with the load nodes. The trusses optimized in this project will be analyzed for their bar forces and node displacements using MATLAB software provided in class. In general, stable trusses form some tessellations of triangles.

Conceptualize a simple design that will meet the criteria listed in the design requirements.

Task 2: Optimization-oriented design analysis (with an example)

The nominal truss configuration

Sketch the system to be designed with an $x, y, z$ coordinate system.

1. Try to identify any potential collapse mechanisms due to square panels, aligned nodes, etc. If you suspect a potential collapse mechanism, add bars or draw a different truss.

2. Once steps 1 and 2 are ok, number the nodes (joints) and number the elements (bars).

3. List the set of unknown design parameters in the problem. In this example the design parameters are the coordinates of nodes 3 and 4, and the diameter of the bars.

$$p = \{x_3, y_3, x_4, y_4, D_o\}$$

4. Write a matrix for the bar connectivity (listing the pair of nodes that each bar connects, sorted by the bar number) ... bar 1 links nodes 1 and 4, bar 2 links nodes 1 and 3, etc.

$$C = \begin{bmatrix}
1 & 4 \\
1 & 3 \\
2 & 3 \\
4 & 3 \\
3 & 5 \\
4 & 5
\end{bmatrix}$$

Optimization-oriented analysis

From a sketch of the truss to be analyzed for optimization, in which some of the node locations are unknown, and given a set of numerical trial values for the design parameters, for example,

$$p = \{x_3, y_3, x_4, y_4, D_o\} = \{13, 8, 12, 14, 0.1\},$$

we may:
1. solve for the inner bar diameters, given an equation like $D_i = 0.9D_o$

2. solve for the cross section area of each bar $A_b = \pi(D_o^2 - D_i^2)/4$

3. solve for the cross section second moment of area of each bar $I_b = \pi(D_o^4 - D_i^4)/64$

4. solve for the total volume of material used in the truss $V = \sum A_b L_b$

5. solve for the bar tensions of each bar $T_b$ from the provided MATLAB analysis method

6. given a value for the material elastic modulus, the buckling strength is $P_{cr} = \pi^2 EI_b/L_b^2$

7. given a value for the material yield strength, the yield force is $T_{yield} = \sigma_y A_b$

8. write inequalities expressing the safety criteria for bar buckling and bar yielding and the geometric criteria for bar lengths in terms of the bar tensions $T_b$, the cross section properties $A_b$ and $I_b$, the bar lengths $L_i$, the material yield strength, $S_y$, the cross section properties $A_b$ and $I_b$, the bar lengths $L_i$, the material yield strength, $S_y$, the cross section properties $A_b$ and $I_b$, the bar lengths $L_i$, the material yield strength, $S_y$, the elastic modulus $E$, and safety factors, $\phi_L$ for loads, $\phi_S$ for yield stress, and $\phi_B$ for critical buckling loads. Write the inequality in a nondimensionalized comparison to zero. For example, the inequality

$$\phi_L \cdot L \leq \phi_R \cdot R$$

can be expressed as a comparison to zero

$$\phi_L \cdot L - \phi_R \cdot R \leq 0,$$

or as a non-dimensionalized comparison to zero

$$\frac{\phi_L \cdot L}{\phi_R \cdot R} - 1 \leq 0.$$  

In other words, we may compute a design objective (i.e., the cost to be minimized) such as the volume of material in the truss, and a set of constraints to ensure that the bars do not yield or buckle.

Given a computer-aided analysis that will calculate all of the items in the list above, we may apply an optimization method, (e.g., the Nelder-Mead method) in order solve the design optimization that minimizes the amount of material used while ensuring that the optimized system is safe.

Cool.

**Task 3: Write a MATLAB function to analyze the truss**

Write a MATLAB function called `ABC_truss_analysis.m` of the form

```
[cost, constraints, random_values] = ABC_truss_analysis(param, constants)
```

where you replace `ABC` with the initials of the people in your group. This function:

1. uses units of Newtons, millimeters, Tons, and seconds (and N/mm², etc.)

2. breaks out the vector `constants` into individual scalar constants; This vector provides values for constants, e.g.: $E$, $S_y$, $P_1$, $P_2$, $\phi_L$, $\phi_S$, and $\phi_B$;

3. breaks out the parameter vector `param` into individual design variables;

4. defines the the truss node coordinates, sorted according to your node numbering scheme;

5. defines a matrix $C$ (TEN) for the truss bar connectivity;

6. computes values for cross section area, $A$, and moment of inertia, $I$, for each bar in the truss;

7. uses `truss_3dg.m` to solve for the vector of truss bar tensions, $T$
8. calculates the total volume of steel of the truss (as a *scalar* value);
9. calculates the yield strength of each bar, $T_y$ in Newtons;
10. calculates the buckling strength of each bar, $P_{cr}$ in Newtons;
11. calculates the (column) vector of safety constraints (yielding and buckling, including factors of safety) and geometric constraints (bar lengths). For a truss with $B$ bars, the first $B$ constraints should be for safety against yielding, the next $B$ constraints should be for safety against buckling, and the last $B$ constraints should be for bar length.
12. saves the values of $P_1$, $P_2$, and $S_y$ as a column vector called *random_values*; and
13. plots the truss using the provided MATLAB function

The downloadable .m-function *two_bar_analysis.m* gives a template and example for a two-bar truss. and the downloadable .m-function *truss_3dg.m* and example *truss_3dg_test.m* show how a 3D truss can be analyzed and plotted.

**Task 4: Constrained optimization using NMAopt and SQPopt**

Use our optimization .m-functions *NMAopt* and *SQPopt* to optimize your truss.

The methods implemented in these optimization functions keep track of the parameters, the cost function, and the constraints. The algorithms use this information to determine what the next guess at the optimal value of the parameters should be, based on the information it has developed by trying several different values of the parameters. As information about the problem accumulates with each successive iteration, the trial parameter values get closer and closer to the true set of optimal parameters (at least in theory). The optimization functions *NMAopt* and *oldSPQ*, will make use of the function *ABC_truss_analysis* that you wrote for Task 3. Your function *ABC_truss_analysis* will be passed to *NMAopt* and *SQPopt* in single quotes, to indicate that the variable is a function name. The initial guess of the optimal parameters is specified in the MATLAB vector *p_init*. The lower and upper bounds of the parameters are given in the MATLAB vectors *p_min* and *p_max*. You will need to specify the values for the vectors *p_init*, *p_min*, and *p_max*. Set the max and min values of the joint coordinates to limit the scope of where the joints could be located. If you allow the joints to be just about anywhere, you could get a very strange looking truss, or a truss that is not stable. On the other hand, make sure that the max values for the bar diameters are not too small to start with. If you don’t allow the bars to be big enough, you may not get a feasible solution. Think about the physical size of the problem to guide your decision about good values for the max and min parameter values. You can relax these parameter bounds once the optimization is running well. The initial parameter values do not need to satisfy the constraints, but it helps if they do. The optimal parameter values are returned in the vector *p_opt*.

Write a MATLAB script called *ABC_truss_opt.m* where you replace *ABC* with the initials of the people in your group. This script:

1. sets values of the constants in the problem, and puts these constants into a vector. For example, in this problem some of the constants (and their values) are: $P_1 = 10 \text{ kN}$, $P_2 = 2 \text{ kN}$, $E = 200,000 \text{ N/mm}^2$, $S_y = 250 \text{ N/mm}^2$, $\phi_L = 1.2$, $\phi_S = 0.9$, and $\phi_B = 0.8$; Put the variable *draw_truss* in last element of the *constants* vector. If *draw_truss* == 1 then a truss is drawn for you. This indicates if the analysis program should (take the time to) draw the truss.
2. specifies upper- and lower- limits for the parameter values, *p_min* and *p_max*, and specifies a reasonable initial guess for the parameter values *p_init*. 


3. gets a roughly-optimal design using NMAopt and then refine the design using SQPopt such as:

```matlab
[ cost, constraints ] = truss_analysis(p_init, constants) % initial design
if (input(‘ OK to continue? [y]/n : ’, ’s’) == ’n’) return; end
constants(end) = draw_truss;
% use the Nelder Mead method with "loose" tolerances to get a rough design
options = [ 1 0.10 0.10 0.001 1000 5 0.5 ];
[p_opt, f_opt, g_opt, cvg_hst] = ...
    NMAopt(’truss_analysis’, p_init, p_min, p_max, options, constants);
plot_cvg_hst( cvg_hst, p_opt, 10 );
constants(end) = 1;
% if: plot the truss shapes, 0: don’t
if (input(‘ OK to continue? [y]/n : ’, ’s’) == ’n’) return; end
[ cost, constraints ] = truss_analysis(p_opt, constants) % optimized design
% constrain the feasible parameters space to be near the NMA-optimal solution
p_init = p_opt;
p_max = p_init + 0.1*abs(p_init);
p_min = p_init - 0.1*abs(p_init);
constants(end) = draw_truss;
% use a gradient method with "tight" tolerances to get a precise solution
options = [ 1 0.001 0.001 0.001 1000 ];
[p_opt, f_opt, g_opt, cvg_hst, Hess, lambda] = ...
    SQPopt(’truss_analysis’, p_init, p_min, p_max, options, constants);
```

The downloadable .m-function `two_bar_opt.m` gives a template and example for a two-bar truss.

The Lagrange multipliers will be in the vector `lambda` of length $(m + 2n)$. The first $m$ Lagrange multipliers correspond to the $m$ constraint equations. The next $n$ Lagrange multipliers correspond to the $p_{\text{min}} \leq p$ constraints, and the last $n$ Lagrange multipliers correspond to the $p \leq p_{\text{max}}$ constraints.

1. What are the optimal parameter values for your design? Print the MATLAB figure of your optimized truss.
2. What is the optimal cost in cubic millimeters? (Remember to specify units.)
3. Do these “optimal” parameter values and the optimal cost depend upon the initial guess?
4. Which of the constraints are being enforced at the optimum? What do each of these constraints represent, physically?
5. In general, are the Lagrange multipliers for buckling larger than, or smaller than the Lagrange multipliers for yielding?
6. Which of the constraints has the largest Lagrange multiplier? $\max(\text{lambda})$
7. Which of the constraints has the smallest positive (non-zero) Lagrange multiplier? $\min(\text{lambda})$.
Points

(a) Figure 1 shows eight nodes, four of the nodes are reaction nodes, four of the nodes carry external loads.
   The \((x,y,z)\) coordinates of these eight nodes are fixed. They are not design parameters. The location
   of these nodes may not be modified in the design optimization.

(b) Each reaction shown in Figure 1 is fixed, in all three axes.

(c) Bars connecting two fixed nodes can not stretch or compress and therefore do not carry tension or
   compression. There’s no need or purpose for bars connecting fixed joints.

(d) Your truss must be stable. In a stable truss the number of bars plus the number of reaction components
   is equal to or greater than three times the number of joints \((B + R \geq 3J)\). In a stable 3D truss, every
   (non-reaction) node has at least three bars connected to it, in three independent directions.
   In an unstable truss, joints can move without stretching or compressing bars. If the analysis of
   the truss in Task 3 gives an error ... 
   **Warning:** Matrix is singular, close to singular or badly scaled. Results may be inaccurate. \(RCOND = NaN\).
   (and you have no other errors) then your truss is unstable. One way to fix this is to re-arrange the
   bars so that all the panels are triangular. You may need to add bars. Alternatively, you may need to
   increase the bar diameters in your initial design.

(e) If your truss has 4 joints in addition to the joints with prescribed coordinates (the reaction joints you
   use and the loaded joints), then the design parameters will will include the twelve values of the \((x,y,z)\)
   locations of these four joints, and the remaining design parameters will give the diameters of the bars.
   ...Lines 18-19 of two_bar_analysis.m would change to something like ...

```plaintext
x09 = param(1); % x-coordinate of joint 9
y09 = param(2); % y-coordinate of joint 9
z09 = param(3); % z-coordinate of joint 9
x10 = param(4); % x-coordinate of joint 10
y10 = param(5); % y-coordinate of joint 10
z10 = param(6); % z-coordinate of joint 10
x11 = param(7); % x-coordinate of joint 11
y11 = param(8); % y-coordinate of joint 11
z11 = param(9); % z-coordinate of joint 11
x12 = param(10); % x-coordinate of joint 12
y12 = param(11); % y-coordinate of joint 12
z12 = param(12); % z-coordinate of joint 12
Do = param(13); % outer diameters of all bars
```

(f) The bars are circular tubes. The outer bar diameter of each bar may be a separate design parameters.
   The inner diameter could 90% of the outer diameter \(D_t = 0.9D_o\)

(g) It is helpful to analyze and plot the truss using the set of initial parameter values before starting
   the optimization. Make sure the plot looks like a reasonable initial guess before proceeding with the
   analysis.

(h) Do not edit any of the .m-functions downloaded from the course webpages, e.g., NMAopt.m, avg_cov_func.m.

(i) Once you see that the plot generated by the MATLAB analysis looks right, you may set draw_truss=0;
   in your function ABC_truss_opt.m to get the optimization to run faster, or you may leave draw_truss=1;
   to see how the optimization modifies your truss from iteration to iteration.

(j) For the limits on the joint coordinates, \(p_{min}\) and \(p_{max}\) should be different for every joint coordinate.
   To start with, don’t make these ranges very large. If an optimization is limited by any \(p_{min}\) value or
   \(p_{max}\) value, you can expand the range for those values and try re-optimizing.

(k) Answers to questions 5 and 6 of Task 2 can be found toward the end of two_bar_analysis.m.
Project Report Contents

1. A print-out of the truss configuration figure, showing labels for node numbers.

2. A printout of your .m-files ABC_truss_3dg.m, ABC_truss_analysis.m, and ABC_truss_opt.m
   % Every MATLAB file that you write in this course should be well-commented.
   Immediately beneath the line
     function [results] = my_function(variables)
   you should include comments that tell a user about your .m-file ... what it does, how to use it, your
   name, your e-mail, and the date.

   Your ABC_truss_analysis.m must carry out all thirteen steps listed in Task 3 without error.
   Send me an email with the final version of your three .m-files as attachments.

3. The optimal design from Task 4 including answers to the seven questions from Task 4.

4. A printout of the truss configuration corresponding to your optimized design, and showing truss bar
   forces.

5. Any additional pointers that would have been helpful for this project.

Grading

35 points: Correctness of your analysis and computer code.

10 points: The optimized cost of your truss, in comparison with other group projects.

10 points: The failure probability of your truss, in comparison with other group projects.

30 points: A complete report (including all five items above).

15 points: Quality of the report:
   (typed, 11pt or 12 pt, 1.5 space, 1 inch margins, page numbers, figure numbers, figure captions )