Consider a continuous beam over several supports carrying arbitrary loads, \( w(x) \).

Using the Moment-Area Theorem, we will analyze two adjoining spans of this beam to find the relationship between the internal moments at each of the supports and the loads applied to the beam. We will label the left, center, and right supports of this two-span segment \( L \), \( C \), and \( R \). The left span has length \( L_L \) and flexural rigidity \( EI_L \); the right span has length \( L_R \) and flexural rigidity \( EI_R \) (see figure (a)).

Applying the principle of superposition to this two-span segment, we can separate the moments caused by the applied loads from the internal moments at the supports. The two-span segment can be represented by two simply-supported spans (with zero moment at \( L \), \( C \), and \( R \)) carrying the external loads plus two simply-supported spans carrying the internal moments \( M_L \), \( M_C \), and \( M_R \) (figures (b), (c), and (d)). The applied loads are illustrated below the beam, so as not to confuse the loads with the moment diagram (shown above the beams). Note that we are being consistent with our sign convention: positive moments create positive curvature in the beam. The internal moments \( M_L \), \( M_C \), and \( M_R \) are drawn in the positive directions. The
areas under the moment diagrams due to the applied loads on the simply-supported spans (figure (b)) are $A_L$ and $A_R$; $\bar{x}_L$ represents the distance from the left support to the centroid of $A_L$, and $\bar{x}_R$ represents the distance from the right support to the centroid of $A_R$, as shown. The moment diagrams due to the unknown moments, $M_L$, $M_C$, and $M_R$ are triangular, as shown in figures (c) and (d).

Examining the elastic curve of the continuous beam (figure (e)), we recognize that the rotation of the beam at the center support, $\theta_C$, is continuous across support $C$. In other words, $\theta_C$ just to the left of point $C$ is the same as $\theta_C$ just to the right of point $C$. This continuity condition may be expressed

$$\frac{\Delta_L \tan \theta_C}{L_L} = -\frac{\Delta_R \tan \theta_C}{L_R},$$

(1)

where $\Delta_L \tan \theta_C$ is the distance from the tangent at $C$ to point $L$, and $\Delta_R \tan \theta_C$ is the distance from the tangent at $C$ to point $R$.

Using the second Moment-Area Theorem, and assuming that the flexural rigidity ($EI$) is constant within each span, we can find the terms $\Delta_L \tan \theta_C$ and $\Delta_R \tan \theta_C$ in terms of the unknown moments, $M_L$, $M_C$, and $M_R$ and the known applied loads.

$$\Delta_L \tan \theta_C = \frac{1}{EI_L} \left[ \bar{x}_L A_L + \frac{2}{3} L_L \frac{1}{2} M_C L_L + \frac{1}{3} L_L \frac{1}{2} M_L L_L \right],$$

(2)

and

$$\Delta_R \tan \theta_C = \frac{1}{EI_R} \left[ \bar{x}_R A_R + \frac{2}{3} L_R \frac{1}{2} M_C L_R + \frac{1}{3} L_R \frac{1}{2} M_R L_R \right].$$

(3)

Substituting these expressions into equation (1) and re-arranging terms, leads to the three-moment equation. \(^1\)

$$\frac{L_L}{EI_L} M_L + 2 \left( \frac{L_L}{EI_L} + \frac{L_R}{EI_R} \right) M_C + \frac{L_R}{EI_R} M_R = -\frac{6 \bar{x}_L A_L}{L_L EI_L} - \frac{6 \bar{x}_R A_R}{L_R EI_R}.$$

(4)

Note that if $EI_L = EI_R = EI$, the three-moment equation is independent of $EI$.

\(^1\) The three-moment equation was derived by Émile Clapeyron in 1857 using the differential equations of beam bending.
To apply the three-moment equation numerically, the lengths, moments of inertia, and applied loads must be specified for each span. Two commonly applied loads are point loads and uniformly distributed loads. For point loads $P_L$ and $P_R$ acting a distance $x_L$ and $x_R$ from the left and right supports, respectively, the right hand side of the three-moment equation becomes

$$-6\bar{x}_L A_L - 6\bar{x}_R A_R = -P_L \frac{x_L}{L_L EI_L} (L_L^2 - x_L^2) - P_R \frac{x_R}{L_R EI_R} (L_R^2 - x_R^2).$$  \hspace{1cm} (5)$$

For uniformly distributed loads $w_L$ and $w_R$ on the left and right spans,

$$-6\bar{x}_L A_L - 6\bar{x}_R A_R = -\frac{w_L L_L^3}{4 EI_L} - \frac{w_R L_R^3}{4 EI_R}.$$  \hspace{1cm} (6)$$

To find the internal moments at the $N + 1$ supports in a continuous beam with $N$ spans, the three-moment equation is applied to $N - 1$ adjacent pairs of spans. For example, consider the application of the three-moment equation to a four-span beam. Spans $a$, $b$, $c$, and $d$ carry uniformly distributed loads $w_a$, $w_b$, $w_c$, and $w_d$, and rest on supports 1, 2, 3, 4, and 5.

The three-moment equation is applied to spans $a - b$, spans $b - c$, and spans $c - d$.

$$\frac{L_a}{EI_a} M_1 + 2 \left( \frac{L_a}{EI_a} + \frac{L_b}{EI_b} \right) M_2 + \frac{L_b}{EI_b} M_3 = -\frac{w_a L_a^3}{4 EI_a} - \frac{w_b L_b^3}{4 EI_b}$$

$$\frac{L_b}{EI_b} M_2 + 2 \left( \frac{L_b}{EI_b} + \frac{L_c}{EI_c} \right) M_3 + \frac{L_c}{EI_c} M_4 = -\frac{w_b L_b^3}{4 EI_b} - \frac{w_c L_c^3}{4 EI_c}$$

$$\frac{L_c}{EI_c} M_3 + 2 \left( \frac{L_c}{EI_c} + \frac{L_d}{EI_d} \right) M_4 + \frac{L_d}{EI_d} M_5 = -\frac{w_c L_c^3}{4 EI_c} - \frac{w_d L_d^3}{4 EI_d}$$
We know that \( M_1 = 0 \) and \( M_5 = 0 \) because they are at the ends of the span. Applying these end-moment conditions to the three three-moment equations and casting the equations into matrix form,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{L_a}{EI_a} & 2 \left( \frac{L_a}{EI_a} + \frac{L_b}{EI_b} \right) & \frac{L_b}{EI_b} & 0 & 0 \\
0 & \frac{L_b}{EI_b} & 2 \left( \frac{L_b}{EI_b} + \frac{L_c}{EI_c} \right) & \frac{L_c}{EI_c} & 0 \\
0 & 0 & \frac{L_c}{EI_c} & 2 \left( \frac{L_c}{EI_c} + \frac{L_d}{EI_d} \right) & \frac{L_d}{EI_d} \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5
\end{bmatrix} =
\begin{bmatrix}
0 \\
-\frac{w_a L_a^3}{4EI_a} - \frac{w_b L_b^3}{4EI_b} \\
-\frac{w_b L_b^3}{4EI_b} - \frac{w_c L_c^3}{4EI_c} \\
-\frac{w_c L_c^3}{4EI_c} - \frac{w_d L_d^3}{4EI_d} \\
0
\end{bmatrix}.
\]

(7)

The \( 5 \times 5 \) matrix on the left hand side of equation (7) is called a flexibility matrix and is tri-diagonal and symmetric. This equation can be written symbolically as \( F \mathbf{m} = \mathbf{d} \). By examining the general form of this expression, we can write a matrix representation of the three-moment equation for arbitrarily many spans. If a numbering convention is adopted in which support \( j \) lies between span \( j - 1 \) and span \( j \), the three non-zero elements in row \( j \) of matrix \( F \) are given by

\[
F_{j,j-1} = \frac{L_{j-1}}{EI_{j-1}},
\]

(8)

\[
F_{j,j} = 2 \left( \frac{L_{j-1}}{EI_{j-1}} + \frac{L_j}{EI_j} \right),
\]

(9)

\[
F_{j,j+1} = \frac{L_j}{EI_j}.
\]

(10)

For the case of uniformly distributed loads, row \( j \) of vector \( \mathbf{d} \) is

\[
d_j = -\frac{w_{j-1} L_{j-1}^3}{4EI_{j-1}} - \frac{w_j L_j^3}{4EI_j}.
\]

(11)

The moments at the supports are computed by solving the system of equations \( F \mathbf{m} = \mathbf{d} \) for the vector \( \mathbf{m} \).
Once the internal moments are found, the reactions at the supports can be computed from static equilibrium.

\[ R_j = \frac{1}{2} w_{j-1} L_{j-1} + \frac{1}{2} w_j L_j + P_L \frac{x_L}{L_{j-1}} + P_R \frac{x_R}{L_j} - \frac{M_j}{L_j} - \frac{M_{j-1}}{L_{j-1}} + \frac{M_{j+1}}{L_j} \]  

(12)

where the first two terms on the right hand side correspond to a uniformly distributed load and the next two terms correspond to interior point loads.

Having computed the reactions and internal moments, we can find the shear and moment diagrams from equilibrium equations. For example, consider span \( j \) between support \( j \) and support \( j + 1 \). The internal shear force at support \( j \) in span \( j \) is

\[ V_{j,j} = \frac{M_j - M_{j+1}}{L_j} - \frac{1}{2} w_j L_j - P_k \frac{x_k}{L_j}, \]  

(13)

and the internal shear force at support \( j + 1 \) in span \( j \) is

\[ V_{j+1,j} = \frac{M_j - M_{j+1}}{L_j} + \frac{1}{2} w_j L_j + P_k \left( 1 - \frac{x_k}{L_j} \right). \]  

(14)
The term $w_jL_j/2$ in these two equations corresponds to a uniformly distributed load, $P_k$ is a point load within span $j$, and $x_k$ is the distance from the right end of span $j$ to the point load $P_k$.

Beam rotations at the supports may be computed from equations (1), (2), and (3). The slope of the beam at support $j$ is $\tan \theta_j$. From the second Moment-Area Theorem,

$$\tan \theta_j = -\frac{1}{EI_j} \left[ \frac{1}{24} w_jL_j^3 + \frac{1}{6} P_k \frac{x_k}{L_j} (L_j^2 - x_k^2) + \frac{1}{3} M_jL_j + \frac{1}{6} M_{j+1}L_j \right],$$

where span $j$ lies between support $j$ and support $j + 1$. The first term inside the brackets corresponds to a uniformly distributed load. The second term inside the brackets corresponds to a point load $P_k$ within span $j$, located a distance $x_k$ from the right end of span $j$, (support $j + 1$).
These ideas are implemented in the MATLAB function `three_moment.m`.

```matlab
function [M,R,V,xs,M_Diag,V_Diag,d_Diag,D_max] = three_moment(L,I,E,w,P,x)
% [M,R,V,xs,M_Diag,V_Diag,d_Diag,D_max] = three_moment(L,I,E,w,P,x)
% solve the three-moment equations for a continuous beam of N spans.
% INPUT:
% L is a vector of length N containing the lengths of each span.
% I is a vector of length N containing the moments of inertia of each span.
% E is a scalar constant for the modulus of elasticity.
% w is a vector of length N containing the uniform loads on each span.
% P is a vector of point loads magnitudes
% x is a vector of point load locations, from the start of the first span
% OUTPUT:
% M is a vector of length N+1 containing the moments at each support
% R is a vector of length N+1 containing the reactions at each support
% V is a matrix of size 2xN+1 containing end-shears of each span
% xs is a vector of the x-axis for the shear, moment, and displacement plots
% M_Diag is a vector of the moment diagram
% V_Diag is a vector of the shear diagram
% d_Diag is a vector of the displacement diagram
% D_max is a vector of the max absolute displacement of each span
% This program assumes that none of the supports are moment resisting,
% that there are no hinges in the beam, that all the spans are made of
% the same material, and that each spans is prismatic.
% H.P. Gavin, Civil and Environmental Engineering, Duke University, 3/24/09

N = length(L);                % number of spans
nP = length(x);               % number of concentrated point loads
span = zeros(1,nP);           % spans containing the point loads
sumL = cumsum(L);             % distances from point load to left support
xL = zeros(1,nP);             % distances from point load to left support
xR = zeros(1,nP);             % distances from point load to right support
for i=1:length(x)
    span(i) = min(find(x(i) < sumL));
end

for k=1:nP
    if span(k) == 1
        xL(k) = x(k);
    else
        xL(k) = x(k) - sumL(span(k)-1);
    end
    xR(k) = sumL(span(k)) - x(k);
end

F = zeros(N+1,N+1);           % initialize the flexibility matrix
for j=2:N
    F(j,j-1) = L(j-1) / I(j-1);
    F(j,j) = 2 * (L(j-1) / I(j-1) + L(j) / I(j));
end
```
\[ F(j,j+1) = \frac{L(j)}{I(j)}; \]

\[ F(1,1) = 1.0; \]
\[ F(N+1,N+1) = 1.0; \]

\[ d = \text{zeros}(N+1,1); \]

\[ \text{for } j=2:N \]
\[ \quad l = j-1; \quad \text{ \% } j-1 \text{ is the number of the left span} \]
\[ \quad r = j; \quad \text{ \% } j \text{ is the number of the right span} \]
\[ \quad \text{\texttt{d}(j) = -w(l)*L(l)^3 / (4*I(l)) - w(r)*L(r)^3 / (4*I(r));} \]
\[ \text{end} \]

\[ \text{\texttt{end}} \]

\[ \text{M = ( inv(F) * d )'}; \quad \% \text{compute the internal moments (7)} \]

\[ \text{R = zeros(1,N+1);} \quad \% \text{build the vector of reaction forces} \]

\[ \text{for } j=1:N \]
\[ \quad l = j-1; \quad \% j-1 \text{ is the number of the left span} \]
\[ \quad r = j; \quad \% j \text{ is the number of the right span} \]
\[ \quad \text{\texttt{if } j == 1} \]
\[ \quad \quad \text{R}(j) = w(r)*L(r)/2 - M(j)/L(r) + M(j+1)/L(r); \]
\[ \quad \text{\texttt{end}} \]
\[ \quad \text{\texttt{if } j == N+1} \]
\[ \quad \quad \text{R}(j) = w(l)*L(l)/2 - M(j)/L(l) + M(j-1)/L(l); \]
\[ \quad \text{\texttt{end}} \]
\[ \quad \text{\texttt{if } j > 1 && j < N+1} \]
\[ \quad \quad \text{R}(j) = w(l)*L(l)/2 + w(r)*L(r)/2 \ldots \]
\[ \quad \quad \quad \quad - M(j)/L(l) - M(j)/L(r) + M(j-1)/L(l) + M(j+1)/L(r); \]
\[ \quad \text{\texttt{end}} \]
\[ \text{\texttt{end}} \]

\[ \text{\texttt{for } k=1:nP} \quad \% \text{loop over all concentrated point loads} \]
\[ \quad \text{\texttt{if span}(k) == l} \quad \% \text{the point load is in the left span} \]
\[ \quad \quad \text{R}(j) = \text{R}(j) + P(k)*xL(k)/L(l); \]
\[ \quad \text{\texttt{end}} \]
\[ \quad \text{\texttt{if span}(k) == r} \quad \% \text{the point load is in the right span} \]
\[ \quad \quad \text{R}(j) = \text{R}(j) + P(k)*xR(k)/L(r); \]
\[ \text{\texttt{end}} \]
\[ \text{\texttt{end}} \]

\[ \text{slope = zeros(1,N);} \quad \% \text{compute the slopes (15)} \]
\[ \text{\texttt{for } j=1:N} \]
\[ r = j; \] % \textit{j} \textit{i}s the \textit{span} to the \textit{right} of reaction \textit{j} \\
\[ \text{\textbf{slop}}e(j) = w(r) \times L(r)^3 / 24 + M(j+1) \times L(r) / 6 + M(j) \times L(r) / 3; \] \\
\textbf{for} \textbf{k}=1:nP \% loop over all concentrated point loads \\
\textbf{if} \text{span(k)} == r \% the point load \textit{i}s in the \textit{right} \textit{span} \\
\textbf{slop}e(j) = \textbf{slop}e(j) + P(k) \times xR(k) / L(r) \times (L(r)^2 - xR(k)^2) / 6; \\
\textbf{end} \\
\textbf{end} \\
\text{\textbf{slop}e(j) = -slop}e(j) / (E \times I(r)); \\
\textbf{end} \\
\textbf{if} ( \text{abs} ( \text{sum}(R) - \text{sum}(w \times L) - \text{sum}(P) ) < 1e-9 ) \\
\textbf{disp}'yes!' \% equilibrium check \ldots should be close to zero \\
\textbf{end} \\
\textbf{\%} \textit{shear, moment, slope, and deflection data and plots} \\
\textbf{for} \textbf{j}=1:N \% \textit{x-axis} data for \textit{shear}, \textit{moment}, \textit{slope}, \textit{and} \textit{deflection diagrams} \\
\textbf{x}(::,j) = [0:L(j)/157:L(j)]'; \\
\textbf{end} \\
\textbf{for} \textbf{j}=1:N \% \textit{j} \textit{i}s the \textit{span} \textit{number} \\
\textbf{V}_{\text{Diag}}(:,j) = V_{\text{Diag}}(:,j) + w(j) \times \textbf{x}(::,j); \\
\textbf{for} \textbf{k}=1:nP \% loop over all concentrated point loads \\
\textbf{if} \text{span(k)} == j \% the point load \textit{i}s in \textit{span} to the \textit{right} \\
\textbf{i}1 = \textbf{find}(\textbf{x}(::,j)<xL(k)); \\
\textbf{i}2 = \textbf{find}(\textbf{x}(::,j)>xL(k)); \\
\textbf{V}_{\text{Diag}}(\textbf{i}1,j) = \textbf{V}_{\text{Diag}}(\textbf{i}1,j) - P(k) \times xR(k) / L(j); \\
\textbf{V}_{\text{Diag}}(\textbf{i}2,j) = \textbf{V}_{\text{Diag}}(\textbf{i}2,j) + P(k) \times (1 - xR(k) / L(j)); \\
\textbf{end} \\
\textbf{end} \\
\textbf{M}_{\text{Diag}}(:,j) = M(j) + \textbf{cumtrapz}(-\textbf{V}_{\text{Diag}}(:,j)) \times \textbf{x}(2,j); \\
\textbf{s}_{\text{Diag}}(:,j) = \textbf{cumtrapz}(\textbf{M}_{\text{Diag}}(:,j)) \times \textbf{x}(2,j) / (E \times I(j)) + \textbf{slop}e(j); \\
\textbf{d}_{\text{Diag}}(:,j) = \textbf{cumtrapz}(\textbf{s}_{\text{Diag}}(:,j)) \times \textbf{x}(2,j); \\
\textbf{end} \\
\textbf{\%} \textit{display key results to the screen} \\
\textbf{fprintf}('------------------------------------------------------------------
'); \\
\textbf{fprintf}(' Moment Shear Deflection
'); \\
\textbf{fprintf}(' Maximum %12.5e %12.5e %12.5e
', ... \\
\textbf{max}(\textbf{max}(\textbf{M}_{\text{Diag}})), \textbf{max}(\textbf{max}(-\textbf{V}_{\text{Diag}})), \textbf{max}(\textbf{max}(\textbf{d}_{\text{Diag}})) ); \\
\textbf{fprintf}(' Minimum %12.5e %12.5e %12.5e
', ... \\
\textbf{min}(\textbf{min}(\textbf{M}_{\text{Diag}})), \textbf{min}(\textbf{min}(-\textbf{V}_{\text{Diag}})), \textbf{min}(\textbf{min}(\textbf{d}_{\text{Diag}})) ); \\
\textbf{fprintf}('------------------------------------------------------------------
'); \\
\textbf{for} \textbf{j}=1:N \% \textit{j} \textit{i}s the \textit{span} \textit{number} \\
\textbf{V}(1,j) = -\textbf{V}_{\text{Diag}}(1,j); \% shear force at left \textit{end} \textit{of} \textit{span} \\
\textbf{V}(2,j) = -\textbf{V}_{\text{Diag}}(158,j); \% shear force at right \textit{end} \textit{of} \textit{span} \\
\textbf{end} \\
\textbf{for} \textbf{j}=2:N \% \textit{x-axis} data for \textit{shear} and \textit{moment diagram} plots \\
\textbf{x}(::,j) = \textbf{x}(::,j) + \textbf{sum}(\textbf{L}(j-1));
end

% Plotting

xs = xs(:);
M_Diag = M_Diag(:);
V_Diag = -V_Diag(:);
s_Diag = s_Diag(:);
d_Diag = d_Diag(:);
z = zeros(1,length(xs));

D_max = max(abs(d_Diag));

figure(1)
cf
subplot(411)
plot (xs, z, '-k', xs, V_Diag, '-b', 'LineWidth', 2)
ylabel('Internal Shear')

subplot(412)
plot (xs, z, '-k', xs, M_Diag, '-b', 'LineWidth', 2)
ylabel('Internal Moment')

subplot(413)
plot (xs, z, '-k', xs, s_Diag, '-b', 'LineWidth', 2)
ylabel('Slope')

subplot(414)
plot (xs, z, '-k', xs, d_Diag, '-b', 'LineWidth', 2)
ylabel('Deflection')

% -------------------------------------------------------- three_moment.m
Example

\begin{verbatim}
L = [100 150 150 50]; % lengths of each span
I = [ 500 1000 2000 100 ]; % bending moment of inertia of each span
E = 1000; % elastic modulus
w = [ 0.10 0.20 0.10 0.30 ]; % uniformly distributed load, each span
P = [ 10 20 40 20 5 ]; % concentrated interior point loads
x = [ 110 130 300 330 420 ]; % location of point loads from x=0

[M,R,V] = three_moment(L,I,E,w,P,x)

yes!
\end{verbatim}

---

<table>
<thead>
<tr>
<th>Moment</th>
<th>Shear</th>
<th>Deflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>1.30355e+03</td>
<td>4.89758e+01</td>
</tr>
<tr>
<td>Minimum</td>
<td>-1.10017e+03</td>
<td>-2.60242e+01</td>
</tr>
</tbody>
</table>

\begin{verbatim}
M = 0.0000e+00 -3.0056e+02 -1.1002e+03 -2.7880e+02 0.0000e+00
R = 1.9944 43.0082 73.9732 42.1003 3.9239
V = 1.9944 35.0026 48.9758 16.0761
\end{verbatim}