The Three-Moment Equation for Continuous-Beam Analysis

CEE 201L. Uncertainty, Design, and Optimization
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Spring, 2009

Consider a continuous beam over several supports carrying arbitrary loads, $w(x)$.

Using the Moment-Area Theorem, we will analyze two adjoining spans of this beam to find the relationship between the internal moments at each of the supports and the loads applied to the beam. We will label the left, center, and right supports of this two-span segment L, C, and R. The left span has length L_L and flexural rigidity EI_L; the right span has length L_R and flexural rigidity EI_R (see figure (a)).

Applying the principle of superposition to this two-span segment, we can separate the moments caused by the applied loads from the internal moments at the supports. The two-span segment can be represented by two simply-supported spans (with zero moment at L, C, and R) carrying the external loads plus two simply-supported spans carrying the internal moments M_L, M_C, and M_R (figures (b), (c), and (d)). The applied loads are illustrated below the beam, so as not to confuse the loads with the moment diagram (shown above the beams). Note that we are being consistent with our sign convention: positive moments create positive curvature in the beam. The internal moments M_L, M_C, and M_R are drawn in the positive directions. The areas under the moment diagrams due to the applied loads on the simply-supported spans (figure (b)) are A_L and A_R; $\bar{x}_L$ represents the distance from the left support to the centroid of A_L, and $\bar{x}_R$ represents the distance from the right support to the centroid of A_R, as shown. The moment diagrams due to the unknown moments, M_L, M_C, and M_R are triangular, as shown in figures (c) and (d).

Examining the elastic curve of the continuous beam (figure (e)), we recognize that the rotation of the beam at the center support, θ_C, is continuous across support C. In other words, θ_C just to the left of point C is the same as θ_C just to the right of point C. This *continuity condition* may be expressed

$$\frac{\Delta_{L \tan C}}{L_L} = -\frac{\Delta_{R \tan C}}{L_R}, \tag{1}$$

where $\Delta_{L \tan C}$ is the distance from the tangent at C to point L, and $\Delta_{R \tan C}$ is the distance from the tangent at C to point R.

Using the second Moment-Area Theorem, and assuming that the flexural rigidity (EI) is constant within each span, we can find the terms $\Delta_{L \tan C}$, and $\Delta_{R \tan C}$ in terms of the unknown moments, M_L, M_C, and M_R and the known applied loads.

$$\Delta_{L \tan C} = \frac{1}{EI_L}\left[\bar{x}_L A_L + \frac{2}{3}L_L\frac{1}{2}M_C L_L + \frac{1}{3}L_L\frac{1}{2}M_L L_L\right], \tag{2}$$

and

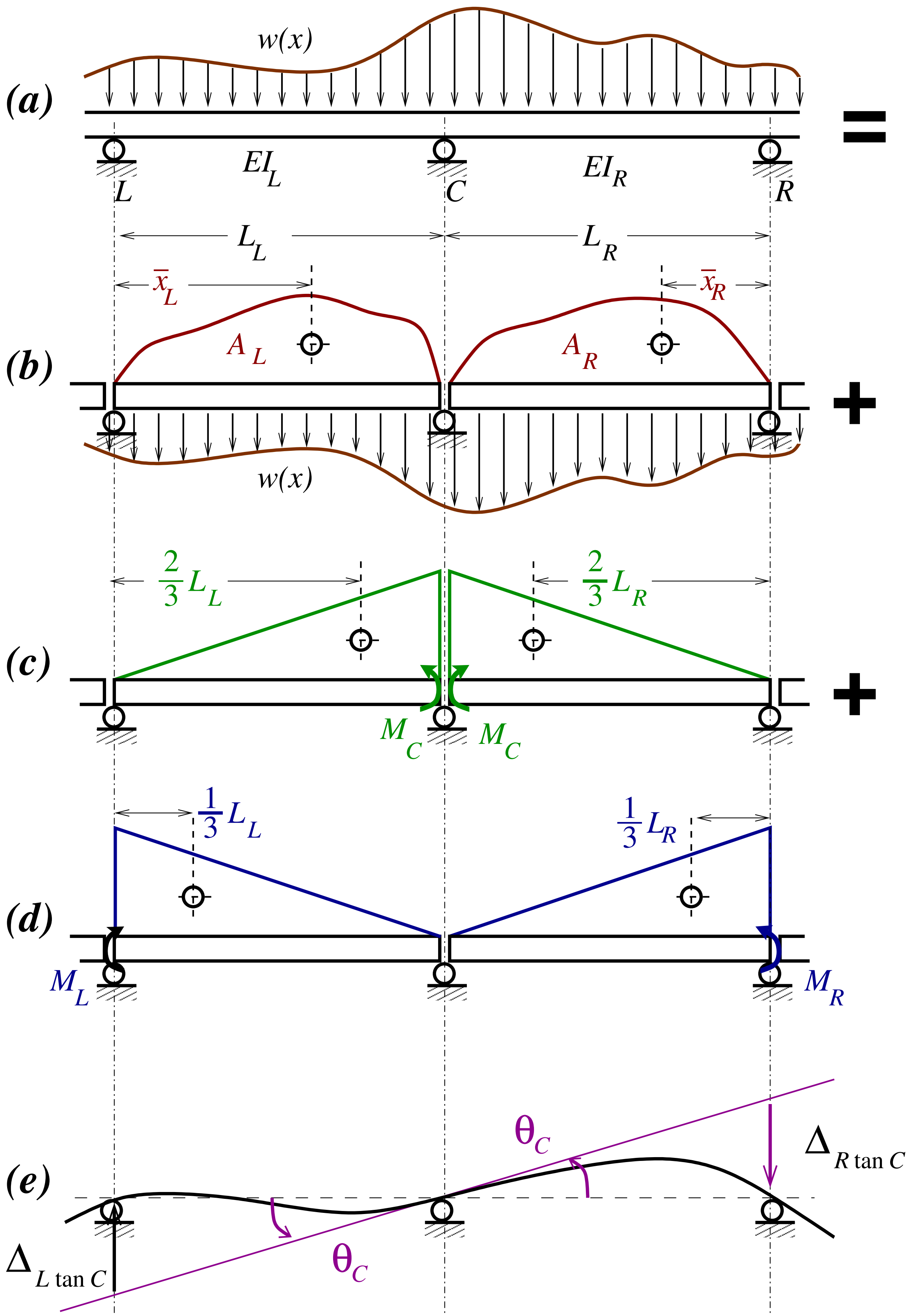
$$\Delta_{R \tan C} = \frac{1}{EI_R}\left[\bar{x}_R A_R + \frac{2}{3}L_R\frac{1}{2}M_C L_R + \frac{1}{3}L_R\frac{1}{2}M_R L_R\right]. \tag{3}$$

Substituting these expressions into equation (1) and re-arranging terms, leads to the three-moment equation. [1]

$$\frac{L_L}{EI_L}M_L + 2\left(\frac{L_L}{EI_L} + \frac{L_R}{EI_R}\right)M_C + \frac{L_R}{EI_R}M_R = -\frac{6\bar{x}_L A_L}{L_L EI_L} - \frac{6\bar{x}_R A_R}{L_R EI_R}. \tag{4}$$

Note that if $EI_L = EI_R = EI$, the three-moment equation is independent of EI.

[1] The three-moment equation was derived by Émile Clapeyron in 1857 using the differential equations of beam bending.



w(x)
(a)
=
EI_L
EI_R
L
C
R
L_L
L_R
x̄_L
x̄_R
A_L
A_R
(b)
+
w(x)
2/3 L_L
2/3 L_R
(c)
+
M_C
M_C
1/3 L_L
1/3 L_R
(d)
M_L
M_R
θ_C
Δ_R tan C
(e)
Δ_L tan C
θ_C

To apply the three-moment equation numerically, the lengths, moments of inertia, and applied loads must be specified for each span. Two commonly applied loads are point loads and uniformly distributed loads. For point loads P_L and P_R acting a distance x_L and x_R from the left and right supports, respectively, the right hand side of the three-moment equation becomes

$$-\frac{6\bar{x}_L A_L}{L_L E I_L} - \frac{6\bar{x}_R A_R}{L_R E I_R} = -P_L \frac{x_L}{L_L E I_L}(L_L^2 - x_L^2) - P_R \frac{x_R}{L_R E I_R}(L_R^2 - x_R^2) \ . \tag{5}$$

For uniformly distributed loads w_L and w_R on the left and right spans,

$$-\frac{6\bar{x}_L A_L}{L_L E I_L} - \frac{6\bar{x}_R A_R}{L_R E I_R} = -\frac{w_L L_L^3}{4 E I_L} - \frac{w_R L_R^3}{4 E I_R}. \tag{6}$$

To find the internal moments at the $N+1$ supports in a continuous beam with N spans, the three-moment equation is applied to $N-1$ adjacent pairs of spans. For example, consider the application of the three-moment equation to a four-span beam. Spans a, b, c, and d carry uniformly distributed loads w_a, w_b, w_c, and w_d, and rest on supports 1, 2, 3, 4, and 5.

The three-moment equation is applied to spans $a-b$, spans $b-c$, and spans $c-d$.

$$\begin{aligned}
\frac{L_a}{EI_a} M_1 + 2\left(\frac{L_a}{EI_a} + \frac{L_b}{EI_b}\right) M_2 + \frac{L_b}{EI_b} M_3 &= -\frac{w_a L_a^3}{4EI_a} - \frac{w_b L_b^3}{4EI_b} \\
\frac{L_b}{EI_b} M_2 + 2\left(\frac{L_b}{EI_b} + \frac{L_c}{EI_c}\right) M_3 + \frac{L_c}{EI_c} M_4 &= -\frac{w_b L_b^3}{4EI_b} - \frac{w_c L_c^3}{4EI_c} \\
\frac{L_c}{EI_c} M_3 + 2\left(\frac{L_c}{EI_c} + \frac{L_d}{EI_d}\right) M_4 + \frac{L_d}{EI_d} M_5 &= -\frac{w_c L_c^3}{4EI_c} - \frac{w_d L_d^3}{4EI_d}
\end{aligned}$$

We know that $M_1 = 0$ and $M_5 = 0$ because they are at the ends of the span. Applying these end-moment conditions to the three three-moment equations and casting the equations into matrix form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{L_a}{EI_a} & 2\left(\frac{L_a}{EI_a} + \frac{L_b}{EI_b}\right) & \frac{L_b}{EI_b} & 0 & 0 \\ 0 & \frac{L_b}{EI_b} & 2\left(\frac{L_b}{EI_b} + \frac{L_c}{EI_c}\right) & \frac{L_c}{EI_c} & 0 \\ 0 & 0 & \frac{L_c}{EI_c} & 2\left(\frac{L_c}{EI_c} + \frac{L_d}{EI_d}\right) & \frac{L_d}{EI_d} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{w_a L_a^3}{4EI_a} - \frac{w_b L_b^3}{4EI_b} \\ -\frac{w_b L_b^3}{4EI_b} - \frac{w_c L_c^3}{4EI_c} \\ -\frac{w_c L_c^3}{4EI_c} - \frac{w_d L_d^3}{4EI_d} \\ 0 \end{bmatrix} . \tag{7}$$

The 5×5 matrix on the left hand side of equation (7) is called a *flexibility matrix* and is tri-diagonal and symmetric. This equation can be written symbolically as $\mathbf{F}\ \mathbf{m} = \mathbf{d}$. By examining the general form of this expression, we can write a matrix representation of the three-moment equation for arbitrarily many spans. If a numbering convention is adopted in which support j lies between span $j-1$ and span j, the three non-zero elements in row j of matrix $\mathbf{F}$ are given by

$$F_{j,j-1} = \frac{L_{j-1}}{EI_{j-1}}, \tag{8}$$

$$F_{j,j} = 2\left(\frac{L_{j-1}}{EI_{j-1}} + \frac{L_j}{EI_j}\right), \tag{9}$$

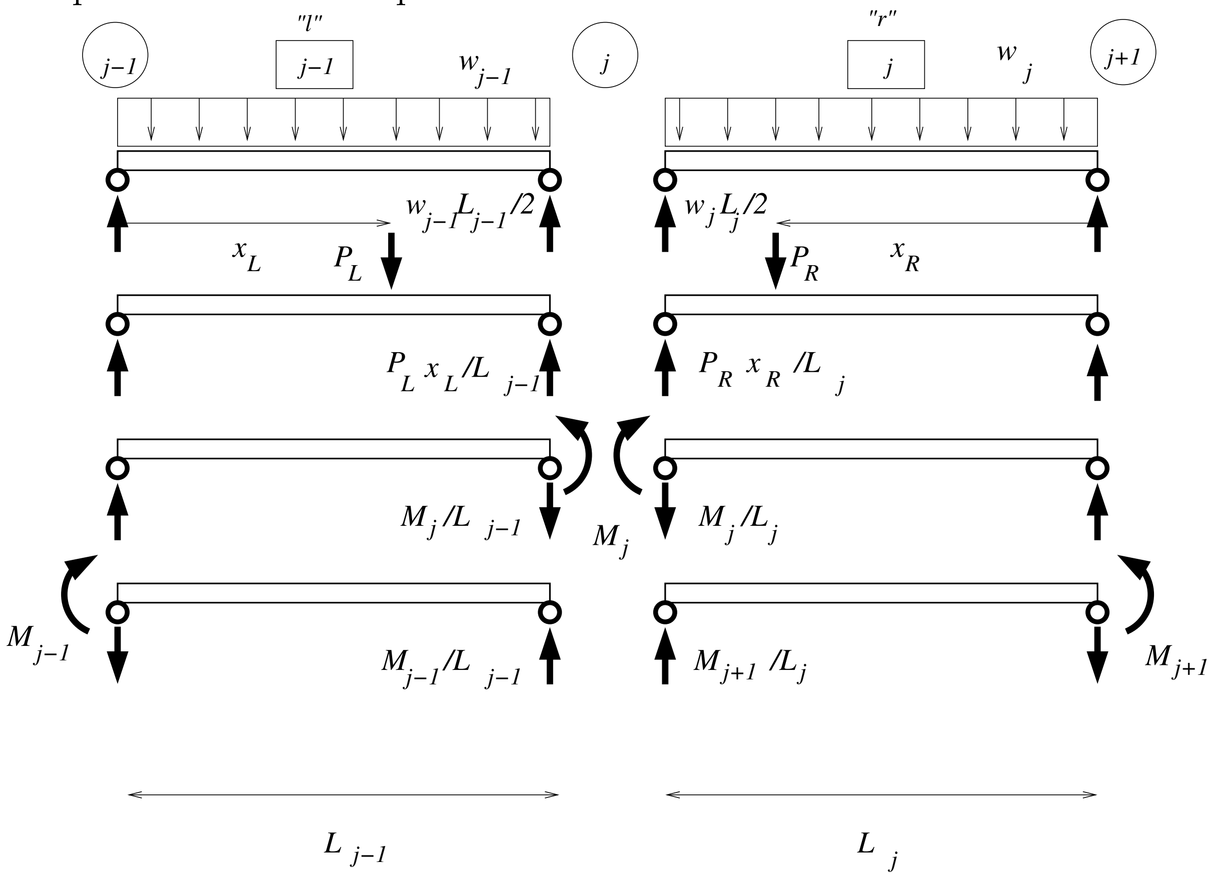
$$F_{j,j+1} = \frac{L_j}{EI_j}. \tag{10}$$

For the case of uniformly distributed loads, row j of vector $\mathbf{d}$ is

$$d_j = -\frac{w_{j-1} L_{j-1}^3}{4EI_{j-1}} - \frac{w_j L_j^3}{4EI_j} . \tag{11}$$

The moments at the supports are computed by solving the system of equations $\mathbf{F}\ \mathbf{m} = \mathbf{d}$ for the vector $\mathbf{m}$.

Once the internal moments are found, the reactions at the supports can be computed from static equilibrium.



$$R_j = \frac{1}{2}w_{j-1}L_{j-1} + \frac{1}{2}w_j L_j + P_L\frac{x_L}{L_{j-1}} + P_R\frac{x_R}{L_j} - \frac{M_j}{L_{j-1}} - \frac{M_j}{L_j} + \frac{M_{j-1}}{L_{j-1}} + \frac{M_{j+1}}{L_j} \quad (12)$$

where the first two terms on the right hand side correspond to a uniformly distributed load and the next two terms correspond to interior point loads.

Having computed the reactions and internal moments, we can find the shear and moment diagrams from equilibrium equations. For example, consider span j between support j and support $j+1$. The internal shear force at support j in span j is

$$V_{j,j} = \frac{M_j - M_{j+1}}{L_j} - \frac{1}{2}w_j L_j - P_k\frac{x_k}{L_j}, \quad (13)$$

and the internal shear force at support $j+1$ in span j is

$$V_{j+1,j} = \frac{M_j - M_{j+1}}{L_j} + \frac{1}{2}w_j L_j + P_k\left(1 - \frac{x_k}{L_j}\right). \quad (14)$$

The term $w_j L_j/2$ in these two equations corresponds to a uniformly distributed load, P_k is a point load within span j, and x_k is the distance from the right end of span j to the point load P_k.

Beam rotations at the supports may be computed from equations (1), (2), and (3). The slope of the beam at support j is $\tan\theta_j$. From the second Moment-Area Theorem,

$$\tan\theta_j = -\frac{1}{EI_j}\left[\frac{1}{24}w_j L_j^3 + \frac{1}{6}P_k\frac{x_k}{L_j}(L_j^2 - x_k^2) + \frac{1}{3}M_j L_j + \frac{1}{6}M_{j+1}L_j\right], \quad (15)$$

where span j lies between support j and support $j+1$. The first term inside the brackets corresponds to a uniformly distributed load. The second term inside the brackets corresponds to a point load P_k within span j, located a distance x_k from the right end of span j, (support $j+1$).

These ideas are implemented in the MATLAB function three_moment.m.

```
function [M,R,V,xs,M_Diag,V_Diag,d_Diag,D_max] = three_moment(L,I,E,w,P,x)
% [M,R,V,xs,M_Diag,V_Diag,d_Diag,D_max] = three_moment(L,I,E,w,P,x)
% solve the three-moment equations for a continuous beam of N spans.
%
% INPUT:
%   L is a vector of length N containing the lengths of each span.
%   I is a vector of length N containing the moments of inertia of each span.
%   E is a scalar constant for the modulus of elasticity.
%   w is a vector of length N containing the uniform loads on each span.
%   P is a vector of point loads magnitudes
%   x is a vector of point load locations, from the start of the first span
%
% OUTPUT:
%   M      is a vector of length N+1 containing the moments at each support
%   R      is a vector of length N+1 containing the reactions at each support
%   V      is a matrix of size 2xN+1 containing end-shears of each span
%   xs     is a vector of the x-axis for the shear, moment, and displ plots
%   M_Diag is a vector of the moment diagram
%   V_Diag is a vector of the shear diagram
%   d_Diag is a vector of the displacement diagram
%   D_max  is a vector of the max absolute displacement of each span
%
% This program assumes that none of the supports are moment resisting,
% that there are no hinges in the beam, that all the spans are made of
% the same material, and that each spans is prismatic.

% H.P. Gavin, Civil and Environmental Engineering, Duke University, 3/24/09

 N = length(L);                         % number  of spans

 nP = length(x);                        % number of concentrated point loads
 span = zeros(1,nP);                    % spans containing the point loads
 sumL = cumsum(L);
 xL   = zeros(1,nP);                    % distance from point load to left rctn
 xR   = zeros(1,nP);                    % distance from point load to right rctn
 for i=1:length(x)
        span(i) = min( find( x(i) < sumL ) );
 end

 for k=1:nP                             % loop over all concentrated point loads
        if span(k) == 1                 % the point load is in the first span
                xL(k) = x(k);
        else
                xL(k) = x(k) - sumL(span(k)-1);
        end
        xR(k) = sumL(span(k)) - x(k);
 end

 F = zeros(N+1,N+1);                    % initialize the flexibility matrix

 for j=2:N                              % create the flexibility matrix (8)-(10)

        F(j,j-1) = L(j-1) / I(j-1);

        F(j,j)   = 2 * (  L(j-1) / I(j-1)  +  L(j) / I(j)  );
```

```

        F(j,j+1) = L(j) / I(j);
end

F(1,1)     = 1.0;
F(N+1,N+1) = 1.0;

d = zeros(N+1,1);
for j=2:N                                % create the right-hand-side vector

        l = j-1;                         % j-1  is the number of the left  span
        r = j;                           % j    is the number of the right span

        d(j) = -w(l)*L(l)^3 / (4*I(l)) - w(r)*L(r)^3 / (4*I(r));

        for k=1:nP                       % loop over all concentrated point loads
                if span(k) == l          % the point load is in the left span
                        d(j) = d(j) - P(k)*xL(k)/(L(l)*I(l))*(L(l)^2-xL(k)^2);
                end
                if span(k) == r          % the point load is in the right span
                        d(j) = d(j) - P(k)*xR(k)/(L(r)*I(r))*(L(r)^2-xR(k)^2);
                end
        end
end

M = ( inv(F) * d )';                     % compute the internal moments  (7)

R = zeros(1,N+1);                        % build the vector of reaction forces
for j=1:N+1                              % j is the reaction number

        l = j-1;                         % j-1  is the number of the left  span
        r = j;                           % j    is the number of the right span

        if j == 1
                R(j) = w(r)*L(r)/2 - M(j)/L(r) + M(j+1)/L(r);
        end
        if j == N+1
                R(j) = w(l)*L(l)/2 - M(j)/L(l) + M(j-1)/L(l);
        end
        if j > 1 && j < N+1
                R(j) = w(l)*L(l)/2 + w(r)*L(r)/2 ...
                      - M(j)/L(l) - M(j)/L(r) + M(j-1)/L(l) + M(j+1)/L(r);
        end

        for k=1:nP                       % loop over all concentrated point loads
                if span(k) == l          % the point load is in the left span
                        R(j) = R(j) + P(k)*xL(k)/L(l);
                end
                if span(k) == r          % the point load is in the right span
                        R(j) = R(j) + P(k)*xR(k)/L(r);
                end
        end

end

slope = zeros(1,N);
for j=1:N                                % compute the slopes  (15)
```

```
        r = j;                  % j is the span to the right of reaction j

        slope(j) =  w(r)*L(r)^3 / 24  +  M(j+1)*L(r) / 6  +  M(j)*L(r) / 3;

        for k=1:nP                         % loop over all concentrated point loads
                if span(k) == r          % the point load is in the right span
                        slope(j) = slope(j)+P(k)*xR(k)/L(r)*(L(r)^2-xR(k)^2)/6;
                end
        end

        slope(j) = -slope(j) / ( E*I(r) );

end

if (  abs ( sum(R) - sum (w .* L) - sum(P) ) < 1e-9  )
        disp (' yes! ')         % equilibrium check ... should be close to zero
end

% ---------- shear, moment, slope, and deflection data and plots ----------

for j=1:N       % x-axis data for shear, moment, slope, and deflection diagrams
        xs(:,j) = [ 0:L(j)/157:L(j) ]' ;
end

for j=1:N                              % j is the span number
        Vo = ( M(j) - M(j+1) ) / L(j) - w(j)*L(j)/2;             % shear at left
        V_Diag(:,j) = Vo + w(j)*xs(:,j);
        for k=1:nP                         % loop over all concentrated point loads
                if span(k) == j          % the point load is in span to the right
                        i1 = find(xs(:,j)<xL(k));
                        i2 = find(xs(:,j)>xL(k));
                        V_Diag(i1,j) = V_Diag(i1,j) - P(k)*xR(k)/L(j);
                        V_Diag(i2,j) = V_Diag(i2,j) + P(k)*(1-xR(k)/L(j));
                end
        end
        M_Diag(:,j) = M(j) + cumtrapz( -V_Diag(:,j) ) * xs(2,j);
        s_Diag(:,j) = cumtrapz( M_Diag(:,j) ) * xs(2,j) / (E*I(j)) + slope(j) ;
        d_Diag(:,j) = cumtrapz( s_Diag(:,j) ) * xs(2,j) ;
end

% ------------- display key results to the screen -------------
fprintf('------------------------------------------------------------------\n');
fprintf('                    Moment                Shear            Deflection\n');
fprintf('   Maximum      %12.5e         %12.5e      %12.5e\n', ...
           max(max(M_Diag)), max(max(-V_Diag)), max(max(d_Diag)) );
fprintf('   Minimum      %12.5e         %12.5e      %12.5e\n', ...
           min(min(M_Diag)), min(min(-V_Diag)), min(min(d_Diag)) );
fprintf('------------------------------------------------------------------\n');

for j=1:N                              % j is the span number
        V(1,j) = -V_Diag(1,j);             % shear force at left  end of span
        V(2,j) = -V_Diag(158,j);           % shear force at right end of span
end

for j=2:N                  % x-axis data for shear and moment diagram plots
        xs(:,j) = xs(:,j) + sumL(j-1);
```

```
 end

% Plotting

 xs      =  xs(:);
 M_Diag  =  M_Diag(:);
 V_Diag  = -V_Diag(:);
 s_Diag  =  s_Diag(:);
 d_Diag  =  d_Diag(:);
 z = zeros(1,length(xs));

 D_max  = max(abs(d_Diag));

 figure(1)
  clf
  subplot(411)
   plot ( xs, z, '-k', xs, V_Diag, '-b', 'LineWidth', 2 )
    ylabel('Internal Shear')

  subplot(412)
   plot ( xs, z, '-k', xs, M_Diag, '-b', 'LineWidth', 2 )
    ylabel('Internal Moment')

  subplot(413)
   plot ( xs, z, '-k', xs, s_Diag, '-b', 'LineWidth', 2 )
    ylabel('Slope')

  subplot(414)
   plot ( xs, z, '-k', xs, d_Diag, '-b', 'LineWidth', 2 )
    ylabel('Deflection')

% ------------------------------------------------------------ three_moment.m
```

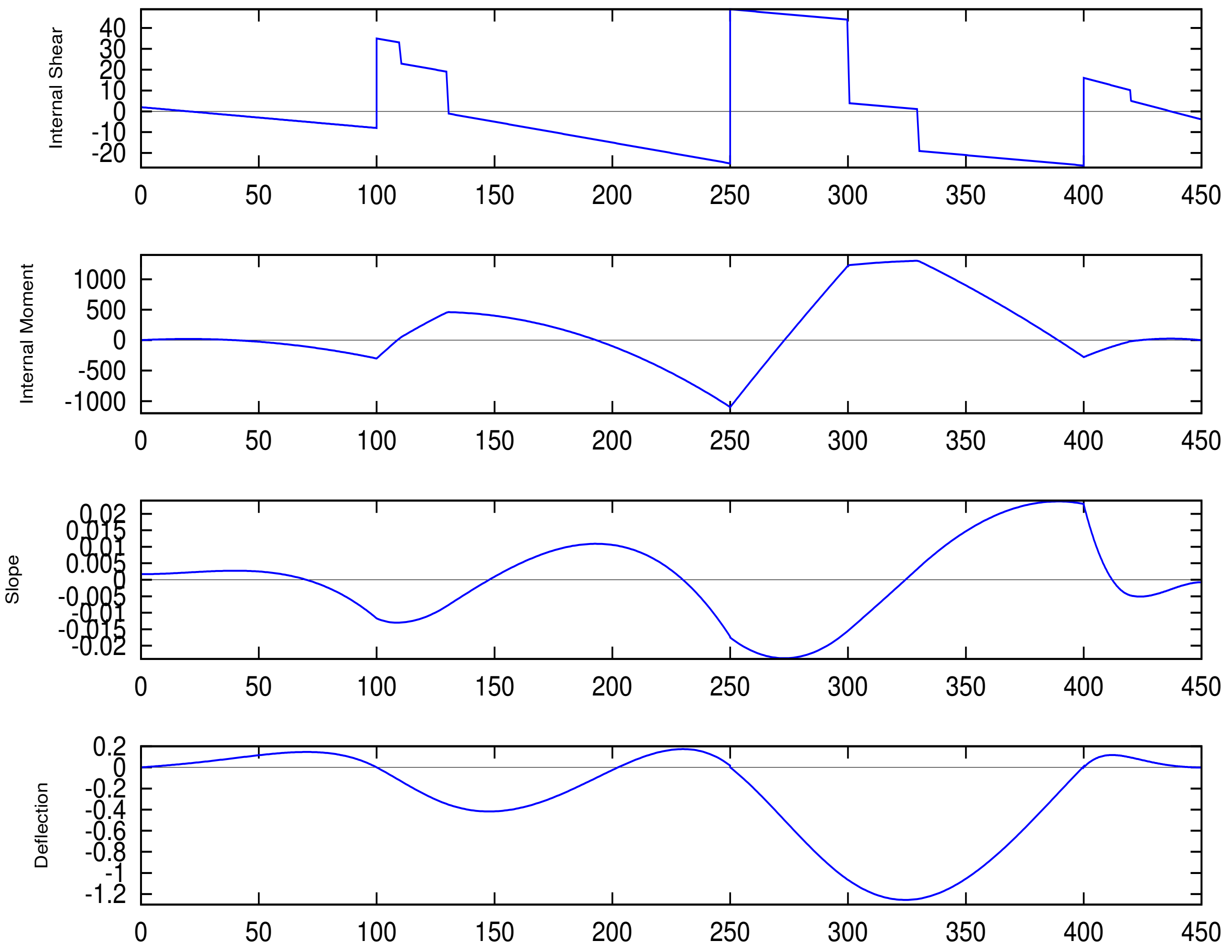
Example

```
L = [100   150   150    50];          % lengths of each span
I = [ 500 1000 2000 100 ];            % bending moment of inertia of each span
E =  1000;                            % elastic modulus
w = [ 0.10   0.20   0.10   0.30 ];    % uniformly distributed load, each span
P = [  10    20    40    20     5 ];  % concentrated interior point loads
x = [ 110   130   300   330   420 ];  % location of point loads from x=0

[M,R,V] = three_moment(L,I,E,w,P,x)
 yes!
------------------------------------------------------------------
                 Moment              Shear             Deflection
   Maximum       1.30355e+03         4.89758e+01       1.73453e-01
   Minimum      -1.10017e+03        -2.60242e+01      -1.25557e+00
------------------------------------------------------------------
M = 0.0000e+00  -3.0056e+02  -1.1002e+03  -2.7880e+02   0.0000e+00

R = 1.9944   43.0082   73.9732   42.1003    3.9239

V = 1.9944   35.0026   48.9758   16.0761
   -8.0056  -24.9974  -26.0242   -3.9239
```


Internal Shear
40
30
20
10
0
-10
-20
0 50 100 150 200 250 300 350 400 450
Internal Moment
1000
500
0
-500
-1000
0 50 100 150 200 250 300 350 400 450
Slope
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0 50 100 150 200 250 300 350 400 450
Deflection
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0 50 100 150 200 250 300 350 400 450