Example of the Principle of Minimum Total Potential Energy

CEE 201L. Uncertainty, Design, and Optimization
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Spring, 2020

Consider the bending of a simply-supported beam, with simple supports at $x = 0$ and $x = L$ carrying a concentrated transverse point load $F$ at $x = x_o$, $(0 < x_o < L)$.

It is essential that any function approximating the transverse displacements of the beam, $v(x)$, adhere to the constraints of the supports (essential boundary conditions): $v(0) = 0$ and $v(L) = 0$. The assumed displacement function

$$v(x; a_1, a_2) = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L}$$

is admissible with respect to displacements at the supports. The semicolon notation indicates that $v$ is a function of $x$ and is parameterized by coefficients $a_1$ and $a_2$. Given this assumed equation for the transverse displacements, the coefficients $a_1$ and $a_2$ determine the displacements for any value of $x$. The curvature of deformation (assuming small deformation), is

$$v''(x; a_1, a_2) = -\frac{\pi^2}{L^2} a_1 \sin \frac{\pi x}{L} - \frac{4\pi^2}{L^2} a_2 \sin \frac{2\pi x}{L}.$$  

The internal bending strain energy is given by

$$U = \frac{1}{2} \int_0^L EI \left( v''(x; a_1, a_2) \right)^2 \, dx ,$$

the potential energy function of external forces is given by

$$V = -F \, v(x_o; a_1, a_2) ,$$

($F$ and $v(x_o; a_1, a_2)$ are collocated) and the total potential energy function is given by

$$\Pi = U + V .$$

The principle of minimum total potential energy states that $\Pi$ must be minimized with respect to the as-of-yet unknowns in the problem, $a_1$ and $a_2$.

$$\min_{a_1, a_2} \Pi(a_1, a_2) \Leftrightarrow \delta \Pi(a_1, a_2) = 0 \Leftrightarrow \frac{\partial U}{\partial a_i} \delta a_i + \frac{\partial V}{\partial a_i} \delta a_i = 0$$
\[
\frac{\partial U}{\partial a_i} = \frac{\partial}{\partial a_i} \frac{1}{2} \int_0^L EI \left(v''(x; a_1, a_2)\right)^2 \, dx \\
= \frac{1}{2} EI \int_0^L \frac{\partial}{\partial a_i} \left(v''(x; a_1, a_2)\right)^2 \, dx \\
= \frac{1}{2} EI \int_0^L 2 v''(x; a_1, a_2) \frac{\partial}{\partial a_i} v''(x; a_1, a_2) \, dx
\]

\[
\frac{\partial V}{\partial a_i} = -F \frac{\partial}{\partial a_i} v(x_o; a_1, a_2)
\]

These equations need to be evaluated separately for \(a_i = a_1\) and for \(a_i = a_2\).

For \(a_i = a_1\),

\[
0 = \delta \Pi_1 = EI \int_0^L \left(-\frac{\pi^2}{L^2} a_1 \sin \frac{\pi x}{L} - \frac{4\pi^2}{L^2} a_2 \sin \frac{2\pi x}{L}\right) \left(-\frac{\pi^2}{L^2} \sin \frac{\pi x}{L}\right) \, dx - F \sin \frac{\pi x_o}{L} \\
= EI \frac{\pi^4}{L^4} a_1 \int_0^L \sin^2 \frac{\pi x}{L} \, dx + EI \frac{4\pi^4}{L^4} a_2 \int_0^L \sin \frac{2\pi x}{L} \sin \frac{\pi x}{L} \, dx - F \sin \frac{\pi x_o}{L} \\
EI \frac{\pi^4}{L^4} a_1 \frac{L}{2} = F \sin \frac{\pi x_o}{L} \\
a_1 = \frac{1}{EI} \frac{2L^3}{\pi^4} F \sin \frac{\pi x_o}{L}
\]

For \(a_i = a_2\),

\[
0 = \delta \Pi_2 = EI \int_0^L \left(-\frac{\pi^2}{L^2} a_1 \sin \frac{\pi x}{L} - \frac{4\pi^2}{L^2} a_2 \sin \frac{2\pi x}{L}\right) \left(-\frac{4\pi^2}{L^2} \sin \frac{2\pi x}{L}\right) \, dx - F \sin \frac{2\pi x_o}{L} \\
= EI \frac{4\pi^4}{L^4} a_1 \int_0^L \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L} \, dx + EI \frac{16\pi^4}{L^4} a_2 \int_0^L \sin^2 \frac{2\pi x}{L} \, dx - F \sin \frac{2\pi x_o}{L} \\
EI \frac{16\pi^4}{L^4} a_2 \frac{L}{2} = F \sin \frac{2\pi x_o}{L} \\
= \frac{1}{EI} \frac{2L^3}{(2\pi)^4} F \sin \frac{2\pi x_o}{L}
\]

By extension, if the function for \(v(x)\) had two more terms, with \(\sin(3\pi x/L)\) and \(\sin(4\pi x/L)\), the coefficients would be

\[
a_3 = \frac{1}{EI} \frac{2L^3}{(3\pi)^4} F \sin \frac{3\pi x_o}{L} \\
a_4 = \frac{1}{EI} \frac{2L^3}{(4\pi)^4} F \sin \frac{4\pi x_o}{L}
\]

Note here that expressions for each coefficient \(a_i\) can be written separately, independent of the other coefficients, because of the orthogonality of sine functions. This series of sine functions is called a Fourier series. In general finding values for \(n\) coefficients involves solving a set of \(n\) linear equations with \(n\) unknowns, (a linear matrix equation).
It is important to note that the bending moments $M(x) = EIv''(x)$ corresponding to an assumed displacement function $v(x)$ satisfies equilibrium if and only if the assumption for $v(x)$ is correct. Of all the possible choices for the assumed function $v(x)$, the choice that results in the smallest total potential energy satisfies equilibrium. So given two possible assumptions for $v(x)$, the assumption that corresponds to the lower value of total potential energy $\Pi$ is the better assumption.

In this example of a simply supported beam, we know that the displacements at $x = 0$ and at $x = L$ must both be zero, and the assumed equations for $v(x)$ meet this requirement. Also, in a simply supported beam, the end moments are zero. Conveniently, for the assumed sine function basis for this solution, $v''(0) = 0$ and $v''(L) = 0$.

The figures on the next page show how increasing the number of terms in the sum of sines expansion for $v(x)$ results in:

- lower values of the total potential energy, and
- bending moments that are closer to satisfying equilibrium

The two figures correspond to the cases of $x_o = 0.7L$ and $x_o = 0.9L$.

The top half of each plot shows displacements for the four assumed functions for $v(x)$.

The bottom half of each plot shows the bending moments corresponding the $EIv''(x)$, and the solid black line shows the bending moments that satisfy equilibrium.

Values of the coefficients $a_i$ and the total potential energy $\Pi$ for the four assumed functions for $v(x)$ are shown. The absolute values of $a_i$ decrease with $i$, implying the Fourier series converges. The values of $\Pi$ become more negative, implying that more terms increases the accuracy of the solution.

We note here that the maximum deflection in both cases are insensitive to the number of terms in the sine expansion. Two terms seem to be sufficient to model the displacements in the $x_o = 0.7L$ case, and three terms seem sufficient for the $x_o = 0.9L$ case.

In order to model the bending moments accurately, more terms are required. Even with four terms, the difference between the moments corresponding to the assumed $v(x)$ and the true equilibrium moments is apparent, especially in the $x_o = 0.9L$ case. Indeed, a good sum-of-sines approximation for this triangular bending moment diagram would require very many terms.
v(x) / (FL^3/(\pi^4 EI))

point load at x_0/L = 0.70

a_1 = 1.6180, a_2 = -0.1189, a_3 = 0.0076, a_4 = 0.0046

\Pi_1 = -7.0205, \Pi_2 = -7.5753, \Pi_3 = -7.5803, \Pi_4 = -7.5987

\pi^2 EI v''(x) / (FL)

point load at x_0/L = 0.90

a_1 = 0.6180, a_2 = -0.0735, a_3 = 0.0200, a_4 = -0.0074

\Pi_1 = -1.1691, \Pi_2 = -1.4283, \Pi_3 = -1.5221, \Pi_4 = -1.5609
Example of the Principle of Minimum Total Potential Energy

% example of an unconstrained minimum total potential energy problem ...
% simply supported beam of length L with a point load at x=x0
% assume v(x) = a1 sin(pi x/L) + a2 sin(2 pi x/L)
L = 10; % length of the beam, m
EI = 1e0; % bending rigidity, kN m^2
F = 1; % load value, (F>0 : up-ward), kN
x0 = 9.0; % location of the point load, m

% coefficients ...

a1 = 1/EI * 2*L^3/(3*pi^4); % a normalization factor
a2 = 1/EI * 2*L^3/(3*pi^2); % a normalization factor
a3 = 1/EI * 2*L^3/(4*pi); % a normalization factor
a4 = 1/EI; % a normalization factor

x = [0:dx:L];
dx = L/100; % x-axis increment for plotting

% four levels of approximation for v(x) ...
v1 = a1 * sin(pi*x/L);
v2 = a1 * sin(pi*x/L) + 2 * sin(2*pi*x/L);
v3 = a1 * sin(pi*x/L) + 2 * sin(2*pi*x/L) + a2 * sin(3*pi*x/L);
v4 = a1 * sin(pi*x/L) + 2 * sin(2*pi*x/L) + a2 * sin(3*pi*x/L) + a3 * sin(4*pi*x/L);

% four levels of approximation for v''(x) ...
d2v1 = -a1 * (4*pi*x/L)^2 * sin(pi*x/L);
d2v2 = -a1 * (4*pi*x/L)^2 * sin(pi*x/L) - a2 * (2*pi*x/L)^2 * sin(2*pi*x/L);
d2v3 = -a1 * (4*pi*x/L)^2 * sin(pi*x/L) - a2 * (2*pi*x/L)^2 * sin(2*pi*x/L) - a3 * (3*pi*x/L)^2 * sin(3*pi*x/L);
d2v4 = -a1 * (4*pi*x/L)^2 * sin(pi*x/L) - a2 * (2*pi*x/L)^2 * sin(2*pi*x/L) - a3 * (3*pi*x/L)^2 * sin(3*pi*x/L) - a4 * (4*pi*x/L)^2 * sin(4*pi*x/L);

% compute the total potential energy PI = U + V for each of the approximations ...

PI1 = (1/2) * EI * trapz((diff(diff(v1)/dx)^2)*dx - F * v1(i0));
PI2 = (1/2) * EI * trapz((diff(diff(v2)/dx)^2)*dx - F * v2(i0));
PI3 = (1/2) * EI * trapz((diff(diff(v3)/dx)^2)*dx - F * v3(i0));
PI4 = (1/2) * EI * trapz((diff(diff(v4)/dx)^2)*dx - F * v4(i0));

% plotting ...

FL3EI = F*L^3/3*pi^4/EI; % a normalization factor
FLEI = F*L^2/2/EI; % a normalization factor

hold on
plot((0 x0/L), [0 -F*(x0/L)*(L-x0)] * pi^2 / (F*L), '-k', 'LineWidth', 1);
plot((1 x0/L), [0 -F*(x0/L)*(L-x0)] * pi^2 / (F*L), '-k', 'LineWidth', 1);
plot((0 x/L), [0 -F*(x0/L)*(L-x0)] * pi^2 / (F*L), '-k', 'LineWidth', 1);
plot((0 x/L), [0 -F*(x0/L)*(L-x0)] * pi^2 / (F*L), '-k', 'LineWidth', 1);
hold off
xlabel('(x/L)')
ylabel('bending moments | displacements ')
text(0.37,0.45,'v(x) / (FL^3/(pi^4 EI)) ', 'FontSize', 14);
text(0.44,0.2,\"\pi^2 EI v''''(x) / (FL) \", 'FontSize', 14);