ENERGY METHODS AND CASTIGLIANO’S THEOREMS
CEE 421L. Matrix Structural Analysis
Fall, 2012

- Strain Energy: \[ U = \frac{1}{2} \int_V \{\sigma\}^T \{\varepsilon\} \, dV \]
- External Work: \[ W = \int F \, dD \]; Complementary Work: \[ W^* = \int D \, dF \]
- Superposition: \[ N = N_o + \sum n_i F_i; \quad M = M_o + \sum m_i F_i; \quad V = V_o + \sum v_i F_i; \text{ etc.} \]
- Castigliano’s First Theorem: \[ F_i = \frac{\partial U}{\partial D_i} \]
- Castigliano’s Second Theorem: \[ D_i = \frac{\partial U^*}{\partial F_i} \]
- Linear Elastic Systems: \[ U = U^* \]

Mechanical Loads

Axial \[ U = \frac{1}{2} \int \frac{N^2}{EA} \, dl = \frac{1}{2} \sum \frac{N_i^2 L}{EA} \quad n_i = \frac{\partial N}{\partial F_i} \quad D_i = \int \frac{N n_i}{EA} \, dl = \sum \frac{N n_i L}{EA} \]
Bending \[ U = \frac{1}{2} \int \frac{M^2}{EI} \, dl \quad m_i = \frac{\partial M}{\partial F_i} \quad D_i = \int \frac{M m_i}{EI} \, dl \]
Shear \[ U = \frac{1}{2} \int \frac{V^2}{G(A/\alpha)} \, dl \quad v_i = \frac{\partial V}{\partial F_i} \quad D_i = \int \frac{V v_i}{G(A/\alpha)} \, dl \]
Torsion \[ U = \frac{1}{2} \int \frac{T^2}{GJ} \, dl \quad t_i = \frac{\partial T}{\partial F_i} \quad D_i = \int \frac{T t_i}{GJ} \, dl \]

Temperature Loads

Axial \[ U = \sum N \alpha \Delta TL \quad \frac{\partial U}{\partial F_i} = \sum \frac{\partial N}{\partial F_i} \alpha \Delta TL \]
Bending \[ U = \int M \alpha \left[ \frac{\Delta T_b - \Delta T_t}{h} \right] \, dl \quad \frac{\partial U}{\partial F_i} = \int \frac{\partial M}{\partial F_i} \alpha \left[ \frac{\Delta T_b - \Delta T_t}{h} \right] \, dl \]

Statically Indeterminate Structures and Superposition

1. Remove \( I \) redundant forces, \( R_i, \quad i = 1, \ldots, I \), where \( I \) is the degree of indeterminacy.
2. Solve for the internal forces, \( M_o, N_o, V_o \), in the resulting statically determinate structure (without the redundant forces), due to the real applied loads.
3. Now, remove all of the real applied loads, and apply \( I \) unit loads to the structure, collocated with the redundant forces, one at a time.
4. Solve for \( I \) sets of internal forces, \( m_i, n_i, v_i \), in each of the \( I \) different statically determinate systems.
5. Apply superposition for bending moments, axial forces, and shear forces.
   \[ M = M_o + \sum_{i=1}^I R_i m_i \quad N = N_o + \sum_{i=1}^I R_i n_i \quad V = V_o + \sum_{i=1}^I R_i v_i \]
6. Write \( I \) statements of Castigliano’s Second Theorem, one for each virtual system, enforce compatibility with respect to support settlement and relative positions, and solve for the redundant forces, \( R_i \).