The Theorems of Castigliano

A linear force-displacement relationship between a force, $F$, and a collocated displacement, $D$, in statically determinate systems can be determined using the principle of real work,

$$ W = U $$

$$ \frac{1}{2} F \cdot D = \frac{1}{2} \int_V \{\sigma\}^T \{\epsilon\} \, dV. $$

The force-displacement relationships for systems with multiple external forces or distributed loads, or statically indeterminate systems, involve relationships between multiple forces and displacements. The external work is

$$ W = \frac{1}{2} \sum_{i=1}^{n} F_i D_i $$

A set of $n$ force-displacement relationships cannot be found with the single principle of real work equation, $W = U$. Instead, a new method must be developed.

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Castigliano’s Theorem - Part I (Force Theorem)

The strain energy in any elastic solid subjected to \( n \) point forces \( F_i \) is a function of the \( n \) collocated displacements, \( D_i \).

\[
U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} \int_{0}^{D_i} F_i(D_i) \, dD_i
\]

\[
\Delta U \approx F_j \Delta D_j , \quad F_j \approx \frac{\Delta U}{\Delta D_j} , \quad \Rightarrow \quad F_j = \frac{\partial U(D)}{\partial D_j}
\] (3)

The force, \( F_j \), on an elastic solid is equal to the partial derivative of the strain energy, \( U(D_1, D_2, \cdots, D_n) \), with respect to the collocated displacement, \( D_j \).

Castigliano’s Theorem - Part II (Deflection Theorem)

The complementary strain energy in any elastic solid subjected to \( n \) point forces \( F_i \) is a function of the \( n \) forces and is the complement of the strain energy.

\[
U^*(F_1, F_2, \cdots, F_n) = \sum_{i=1}^{n} F_i D_i - U(D_1, D_2, \cdots, D_n) = \sum_{i=1}^{n} F_i D_i - \sum_{i=1}^{n} \int_{0}^{D_i} F_i(D_i) \, dD_i
\]

\[
\Delta U^* \approx D_j \Delta F_j , \quad D_j \approx \frac{\Delta U^*}{\Delta F_j} , \quad \Rightarrow \quad D_j = \frac{\partial U^*(F)}{\partial F_j}
\] (4)

The partial derivative of the complementary strain energy of an elastic system, \( U^*(F) \), with respect to a selected force acting on the system, \( F_j \), gives the displacement of that force along its direction, \( D_j \).

If the solid is linear elastic, then \( U^*(F) = U(D) \).
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For linear elastic prismatic solids in equilibrium,

\[ U^*(F) = U(D) = \frac{1}{2} \int l \frac{N^2}{EA} \, dl + \frac{1}{2} \int l \frac{M_z^2}{EI_z} \, dl + \frac{1}{2} \int l \frac{M_y^2}{EI_y} \, dl + \frac{1}{2} \int l \frac{V_z^2}{G(A/\alpha_z)} \, dl + \frac{1}{2} \int l \frac{V_y^2}{G(A/\alpha_y)} \, dl + \frac{1}{2} \int l \frac{T^2}{GJ} \, dl \]. \tag{5}

So,

\[ \frac{\partial U^*}{\partial F_j} = \frac{\partial U}{\partial F_j} = \int l \frac{\partial N}{\partial F_j} \frac{\partial N}{\partial F_j} \, dl + \int l \frac{\partial M_z}{\partial F_j} \frac{\partial M_z}{\partial F_j} \, dl + \int l \frac{\partial M_y}{\partial F_j} \frac{\partial M_y}{\partial F_j} \, dl + \int l \frac{\partial V_z}{\partial F_j} \frac{\partial V_z}{\partial F_j} \, dl + \int l \frac{\partial V_y}{\partial F_j} \frac{\partial V_y}{\partial F_j} \, dl + \int l \frac{\partial T}{\partial F_j} \frac{\partial T}{\partial F_j} \, dl \]. \tag{6}

Superposition

Superposition is an extremely powerful method for separating a system with multiple linear force-displacement relationships into multiple systems with single linear force-displacement relationships.

The principle of superposition states:

*Any response of a linear system to multiple inputs can be represented as the sum of the responses to the inputs taken individually.*

By “response” we can mean a displacement, a strain, a stress, an internal force, a rotation, an internal moment, etc.

By “input” we can mean an externally applied load, a temperature change, a support settlement, etc.