Solving Partitioned Stiffness Matrix Equations

CEE 421L. Matrix Structural Analysis
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In stiffness matrix equations, there are two types of unknowns:

- displacement unknowns at un-restrained coordinates
- force (reaction) unknowns at restrained coordinates

Internal element forces (truss bar forces, for example) are also unknown, but we can find them directly from the nodal displacements. (See previous course notes.)

In a 2D truss, each node has 2 coordinates (horizontal X and vertical Y).
If a 2D truss has N nodes, then it has $2N$ coordinates, in all.
According to the common coordinate numbering convention, node number "j" has:

- coordinate number $2j-1$ in the X-direction, and
- coordinate number $2j$ in the Y-direction

Suppose the nodes of a truss are numbered so that coordinates $[1, \cdots, n]$ are un-restrained and the displacements of coordinates $[n+1, \cdots, 2N]$ are restrained by reaction forces. The external forces at the reaction coordinates are the set of the reaction forces, $\left[\begin{array}{ccc} r_{n+1}, & \cdots, & r_{2N} \end{array} \right]$ and any external forces applied directly to those reaction coordinates, $\left[\begin{array}{ccc} p_{n+1}, & \cdots, & p_{2N} \end{array} \right]$.

The vector of $2N$ forces $\mathbf{p}$ (external applied forces and reaction forces) and the vector of $2N$ displacements $\mathbf{d}$ (at all coordinates) are related by a matrix equation.

$$\left[\begin{array}{c} \\ \\ \\ \mathbf{K}_s \\ \\ \\ \\ \end{array} \right]_{2N\times 2N} \left[\begin{array}{c} d_1 \\ d_2 \\ \vdots \\ d_n \\ d_{n+1} \\ \vdots \\ d_{2N} \end{array} \right]_{2N\times 1} = \left[\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ r_{n+1} + p_{n+1} \\ \vdots \\ r_{2N} + p_{2N} \end{array} \right]_{2N\times 1} \tag{1}$$

This matrix equation has knowns and unknowns in both the vector of displacements and the vector of external loads and reactions. The vectors $\left[\begin{array}{ccc} d_{n+1}, & \cdots, & d_{2N} \end{array} \right]$ and $\left[\begin{array}{cccc} p_1, & p_2, & \cdots, & p_{2N} \end{array} \right]$

are known and the vectors $\left[\begin{array}{cccc} d_1, & d_2 & \cdots, & d_n \end{array}\right]$ and $\left[\begin{array}{ccc} r_{n+1}, & \cdots, & r_{2N} \end{array}\right]$ are unknown. There are still $2N$ equations with $2N$ unknowns, and we need a way to separate these knowns and unknowns from one another.

Let's call the set of coordinates with known external forces q. For example, $\mathsf{q} = [1, \cdots, n]$. Let's call the set of coordinates with known displacements r (i.e., reaction coordinates). For example, $\mathsf{r} = [n+1, \cdots, 2N]$.

We can partition the vectors and stiffness matrix in equation (1) as follows:

$$\left[\begin{array}{cc} \mathbf{K}_{\mathsf{qq}} & \mathbf{K}_{\mathsf{qr}} \\ \mathbf{K}_{\mathsf{rq}} & \mathbf{K}_{\mathsf{rr}} \end{array}\right] \left[\begin{array}{c} \mathbf{d}_{\mathsf{q}} \\ \mathbf{d}_{\mathsf{r}} \end{array}\right] = \left[\begin{array}{c} \mathbf{p}_{\mathsf{q}} \\ \mathbf{r}_{\mathbf{r}} + \mathbf{p}_{\mathsf{r}} \end{array}\right] \tag{2}$$

The submatrix $\mathbf{K}_{\mathsf{qq}}$ relates forces $\mathbf{p}_{\mathsf{q}}$ and displacements $\mathbf{d}_{\mathsf{q}}$ at the un-restrained coordinates. As long as the structure is adequately restrained and internally stable, the submatrix $\mathbf{K}_{\mathsf{qq}}$ is invertible. If the structure is insufficiently restrained or internally unstable, it can not resist loads through deformation; it would move without deformation or collapse when subjected to external loads. In such cases $\mathbf{K}_{\mathsf{qq}}$ is not invertible.

What are the dimensions of $\mathbf{p}_{\mathsf{q}}$, $\mathbf{d}_{\mathsf{r}}$, $\mathbf{K}_{\mathsf{qq}}$, and $\mathbf{K}_{\mathsf{qr}}$ in terms of n and N?

Equation (2) is just a generalization of a system of two equations with two unknowns. It's just that now each unknown is a vector instead of a scalar.

The first of these two equations is:

$$\mathbf{K}_{\mathsf{qq}}\mathbf{d}_{\mathsf{q}} + \mathbf{K}_{\mathsf{qr}}\mathbf{d}_{\mathsf{r}} = \mathbf{p}_{\mathsf{q}} \tag{3}$$

The only unknown in this equation is $\mathbf{d}_{\mathsf{q}}$, since the displacements at the reaction coordinates ($\mathbf{d}_{\mathsf{r}}$) are known. (Usually the displacements at reaction coordinates are set to zero, but they could be set to other prescribed values.) Solving equation (3) for $\mathbf{d}_{\mathsf{q}}$,

$$\mathbf{d}_{\mathsf{q}} = \mathbf{K}_{\mathsf{qq}}^{-1}(\mathbf{p}_{\mathsf{q}} - \mathbf{K}_{\mathsf{qr}}\mathbf{d}_{\mathsf{r}}) \tag{4}$$

Now that the displacements at all the coordinates are known, we can plug them into the second equation,

$$\mathbf{K}_{\mathsf{rq}}\mathbf{d}_{\mathsf{q}} + \mathbf{K}_{\mathsf{rr}}\mathbf{d}_{\mathsf{r}} = \mathbf{r}_{\mathsf{r}} + \mathbf{p}_{\mathsf{r}} \tag{5}$$

in order to compute the reaction forces $\mathbf{r}_{\mathsf{r}}$. The vector $\mathbf{p}_{\mathsf{r}}$ represents the external forces applied directly to the reaction coordinates. These kinds of forces arise in the analysis of beams and frames in which structural elements connecting to reaction coordinates are loaded with distributed loads. Of course, forces applied at reaction coordinates can not deform the structure and therefore do not affect the coordinate displacements. The vector of reactions forces, $\mathbf{r}_{\mathsf{r}}$, is computed from

$$\mathbf{r}_{\mathsf{r}} = \mathbf{K}_{\mathsf{rq}}\mathbf{d}_{\mathsf{q}} + \mathbf{K}_{\mathsf{rr}}\mathbf{d}_{\mathsf{r}} - \mathbf{p}_{\mathsf{r}} \tag{6}$$

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