Bilinear Hysteresis

CEE 541. Structural Dynamics
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Fall 2014

1 Non degrading bilinear hysteresis

Bilinear hysteresis is a mathematically convenient model for the behavior of yielding structures. The force-displacement relationship in bilinear hysteretic systems is composed of piecewise linear and continuous relationships. Initial loading and unloading follow lines with slope $k_1$. These lines can intersect the $x$ axis (zero force) at any number of residual displacements, e.g., $x_C$ in the figure below. Post yield behavior follows one of two lines, line A if $\dot{x} > 0$ and line B if $\dot{x} < 0$. These lines intersect the $x$ axis at points $x_A$ and $x_B$. This model is parameterized by the yield force $F_y$, the elastic stiffness $k_1$, and the post-yield stiffness, $k_2$. The yield displacement is found from the yield force and elastic stiffness, $x_y = F_y/k_1$. At points where $\dot{x}(t)$ passes through zero, $x(t)$ is at a relative maximum or minimum. These are called a turn-around points, and are marked at coordinates $(x_t, R_t)$. Equations for lines A, B, and C and their $x$ intercepts are at displacements can be found from the geometry of the modeled hysteretic behavior.

- line A: $R(x) = k_2x(t) + F_y(1 - k_2/k_1)$  
- line B: $R(x) = k_2x(t) - F_y(1 - k_2/k_1)$  
- line C: $R(x) = k_1x(t) + R_t - k_2x_t$
This model is called a “non-degrading” model since the stiffness of each branch of the model does not change as damage is accumulated.

The work dissipated per cycle $W_D$ is the area of a hysteresis loop with max and min displacements $x_{\text{max}}$ and $-x_{\text{max}}$. The work dissipated is the product of the lengths of line segments $IJ$ and $KL$. From geometry,

$$
\begin{align*}
IJ &= 2(x_{\text{max}} - x_y)\sqrt{1 + k_2^2} \\
JK &= 2\sqrt{x_y^2 + F_y^2} = 2F_y\sqrt{1 + 1/k_1^2} = 2x_y\sqrt{1 + k_1^2} \\
\theta &= \arctan k_1 - \arctan k_2 \\
\vec{KL} &= \vec{JK}\sin\theta \\
W_D &= \vec{JKL} \\
\bar{k} &= \frac{k_2}{k_1} + (x_y/x_{\text{max}})(k_1 - k_2)
\end{align*}
$$

The linear viscous damping ratio that is energy-equivalent to bilinear hysteretic models can be found from equating the area of the hysteresis loop with the energy absorbed per cycle by a linear viscous damper. In hysteretic systems, the energy dissipated per cycle depends upon the amplitude of motion, $x_{\text{max}}$ as well as the hysteretic parameters. This relation can be analyzed in terms of dimensionless quantities. The ductility ratio is the ratio of the peak displacement to the yield displacement, $x_{\text{max}}/x_y$. The post-yield stiffness ratio is the ratio $k_2/k_1$. The equivalent viscous damping ratio also depends upon a measure of stiffness. This can be taken to be the the elastic stiffness $k = k_1$ or the secant stiffness, $k = \bar{k}$.

$$
\zeta_{\text{eqv}} = \frac{W_D}{(2\pi x_{\text{max}}^2 k_1)} \left(\frac{\omega}{\omega_n}\right)
$$

For motions dominated by transients and in free response ($\omega/\omega_n \approx 1$), the equivalent viscous damping ratio can be plotted with respect to the ductility ratio.
Bilinear Hysteresis

% bilin.m ... equivalent viscous damping of bilinear hysteresis.

xy = 1; % yield displacement — this value doesn’t matter
Fy = 1000; % yield force — this value doesn’t really matter

A = xy*[1:0.2:6]; % ductility ratio ... independent variable

k1 = Fy/xy; % elastic stiffness
K2 = [0.0 0.1 0.2 0.5]*k1; % strain hardening stiffness

Omega1 = 1; % frequency ratio
wn1 = sqrt(k1); % natural frequency of elastic system
w = Omega1*wn1; % frequency of the motion

for kk = 1:length(K2)
    k2 = K2(kk);
    theta = atan(k1) - atan(k2);
    IJ = 2*(A'-xy)*sqrt(1+k2^2);
    KJ = 2*xy*sqrt(1+k1^2);
    Wd = IJ*KJ*sin(theta); % area of hysteresis loop
    Wn(:,kk) = Wd./(4*Fy*A)'; % ... normalized w.r.t. frictional damping
    k_ = k2 + (k1-k2)*xy./A; % equivalent dynamic stiffness
    Omega_eq = w./sqrt(k_); % equivalent frequency ratio
    Rp = k_*A; % peak restoring force, k_ * A
    zeq1(:,kk) = Wd./(2*pi*k1.*A.^2.*Omega1)'; % equivalent damping ratio, k1
    zeq_(:,kk) = Wd./(2*pi*k_.*A.^2.*Omega_eq)'; % equivalent damping ratio, k_
    K_(:,kk) = k_; %
end

Note that:

- The equivalent viscous damping ratio $\zeta_{eqv}$ is proportional to the energy dissipated per cycle $W_D$.
- The equivalent viscous damping ratio $\zeta_{eqv}$ is inversely proportional to $k$, $x_{max}^2$ and $(\omega/\omega_n)$.
- The equivalent viscous damping ratio $\zeta_{eqv}$ is maximized for ductility for $x_{max}/x_y$ between 2 and 3.
- The largest equivalent viscous damping ratio $\zeta_{eqv}$ is around 0.25 for elastic-plastic systems ($k_2 = 0$).
- The natural frequency of the hysteretic system $\sqrt{k/m}$ decreases with ductility, $x_{max}/x_y$.
- The secant stiffness, $\bar{k}$, decreases with ductility
- The net energy dissipation increases with ductility
- $\bar{k}/k_1 = 1 - W_D$
• The secant stiffness “$\bar{k}$” and frequency ratio, “$(\omega/\omega_n)$” for a hysteretic system are not precisely defined . . .

• So the manner in which 'equivalent viscous damping ratio' is ductility-dependent is somewhat subjective.

Project idea . . . Evaluate the equivalent viscous damping of bilinear hysteretic systems as a function of ductility and $k_2/k_1$ using the impulse response of the hysteretic system and the logarithmic decrement. Method for project . . . Simulate free response of a SDOF bilinear hysteretic oscillator. For each cycle:

• . . . call the average peak displacement, $A$.

• . . . set $\Omega = 1$ since transient-dominated response frequency is close to the natural frequency

• . . . set $k = (\text{average peak force}, R) / A$

• . . . compute $\zeta_{eqv}$ from $W_D/(2\pi k A^2 \Omega)$

• . . . plot $\zeta_{eqv}$ w.r.t. ductility for different values of $k_2/k_1$

2 Other hysteretic shapes

The bilinear hysteretic model can be extended to capture the behavior of degrading material and slipping interfaces.

Modified Johnston/Clough Degrading Hysteresis Model and the Flag Hysteresis Model.
http://www.eqnsols.com/Pages/HystereticModels.aspx
Figure 3. Equivalent linear viscous damping ratios for nondegrading bilinear hysteresis models.