Classical Damping, Non-Classical Damping and Complex Modes

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1 Classical Damping

The equations of motion of an un-forced $N$ degree of freedom elastic structure with viscous damping are

$$M\ddot{r}(t) + C\dot{r}(t) + Kr(t) = 0,$$  
(1)

with initial conditions $r(0) = d_0$ and $\dot{r}(0) = v_0$. If the system is un-damped ($C = 0_{N \times N}$), the free response of the system will not decay with time, and a suitable trial solution to the differential equation (1) is $r(t) = \bar{r}\sin(\omega_n t)$, where $\bar{r}$ is a constant vector of dimension $N$. Differentiating $r(t)$ twice, $\ddot{r}(t) = -\omega_n^2 \bar{r}\sin(\omega_n t)$, and substituting the trial solution into equation (1) we obtain

$$-\omega_n^2 M\bar{r}\sin(\omega_n t) + K\bar{r}\sin(\omega_n t) = 0.$$  
(2)

For the assumed trial solution to be true for all time,

$$[K - \omega_n^2 M]\bar{r}_j = 0,$$  
(3)

which is a general eigen-value problem, in which eigen-values are squared natural frequencies, $\omega_n^2$, and the eigen-vectors are mode-shape vectors, $\bar{r}_j$. If the structure is modeled with $N$ degrees of freedom, then there will be $N$ natural frequencies and $N$ modal vectors. The modal matrix $\bar{R}$ is the column-wise concatenation of the $N$ mode-shape vectors, $\bar{R} = [\bar{r}_1 \bar{r}_2 \cdots \bar{r}_N]$. The modal matrix $\bar{R}$ diagonalizes both the mass and stiffness matrices. The Rayleigh quotient is the ratio of the diagonalized stiffness matrix to the diagonalized mass matrix.

$$\bar{R}^T K \bar{R} \bar{R}^T M \bar{R} = \begin{bmatrix} k_1^*/m_1^* & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ k_N^*/m_N^* & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \omega_{n1}^2 \\ \cdots \\ \omega_{nN}^2 \end{bmatrix} = \Omega^2.$$  
(4)

For mass-normalized modal vectors $\bar{R}^T M \bar{R} = I_N$ and $\bar{R}^T K \bar{R} = \Omega^2$.

A damping matrix that is diagonalizable by $\bar{R}$ is called a classical damping matrix.

$$\bar{R}^T C \bar{R} \bar{R}^T M \bar{R} = \begin{bmatrix} c_1^*/m_1^* & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ c_N^*/m_N^* & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} 2\zeta_1\omega_{n1} \\ \cdots \\ 2\zeta_{nN}\omega_{nN} \end{bmatrix}.$$  
(5)

where $\zeta_j$ is the damping ratio of the $i$-th mode, and $\omega_{ni}$ is the un-damped natural frequency of the $i$-th mode. Systems with classical damping are triple diagonalizeable. The modal vectors of triple diagonalizeable systems depend only on $M$ and $K$, and are independent of $C$, regardless of how heavily the system is damped. There are many ways to compute a classical damping matrix from mass and stiffness matrices.
A Rayleigh damping matrix is proportional to the mass and stiffness matrices [6],
\[ C = \alpha M + \beta K. \] (6)
where \( \alpha \) and \( \beta \) are related to damping ratios and frequencies by
\[ \zeta_k = \alpha \frac{1}{2\omega_k^2} + \beta \omega_k \] (7)
Mass proportional damping ratios decrease inversely with \( \omega \) and stiffness proportional damping ratios increase linearly with \( \omega \).

Rayleigh damping can be extended. It can be shown that the damping matrix
\[ C = \alpha M + \beta K + \gamma MK^{-1}M + \delta KM^{-1}K \] (8)
is a classical damping matrix. An extended Rayleigh damping matrix, called Caughey damping [1, 2], can be computed from
\[ C = M \sum_{j=n_1}^{j=n_2} \alpha_j (M^{-1}K)^j \] (9)
where \( n_1 \) and \( n_2 \) can be positive or negative, as long as \( n_1 < n_2 \). The coefficients \( \alpha_j \) are related to the damping ratios, \( \zeta_k \), by
\[ \zeta_k = \frac{1}{2} \frac{1}{\omega_k \sum_{j=n_1}^{j=n_2} \alpha_j \omega_k^{2j}}. \] (10)
Alternatively, a classical damping matrix can be computed for a specified set of modal damping ratios \( \zeta_j \) from the mass matrix and all \( N \) modal vectors and natural frequencies.
\[ C = M \bar{R} \begin{bmatrix} 2\zeta_1 \omega_n^1/m_1^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2\zeta_N \omega_n^N/m_1^N \end{bmatrix} \bar{R}^T M. \] (11)
The displacements \( r(t) \) of triple-diagonalizeable systems can always be expressed as a linear combination of real-valued modal coordinates, \( q(t) \),
\[ r(t) = \bar{r}_1 q_1(t) + \bar{r}_2 q_2(t) + \cdots + \bar{r}_N q_N(t) = \bar{R}q(t). \] (12)
Substituting equation (12) into equation (1) and pre-multiplying by \( \bar{R}^T \) gives
\[ \bar{R}^T \bar{M} \bar{R} \bar{q}(t) + \bar{R}^T \bar{C} \bar{R} \dot{q}(t) + \bar{R}^T K \bar{R} q(t) = 0, \] (13)
or, for each mode, \( i, \ 1 \leq i \leq N, \)
\[ \ddot{q}_j(t) + 2\zeta_j \omega_n j \dot{q}_j(t) + \omega_n^2 q_j(t) = 0, \] (14)
which are the \( N \) uncoupled equations of motion in modal coordinates. The damped free response of each modal coordinate decays exponentially with time
\[ q_j(t) = e^{-\zeta_j \omega_n j t} (\bar{q}_{c,j} \cos \omega_d j t + \bar{q}_{s,j} \sin \omega_d j t), \] (15)
where \( \omega_d j \) is the \( j \)-th damped natural frequency, is related to the \( j \)-th un-damped natural frequency and damping ratio by \( \omega_d j = \omega_n j \sqrt{1 - \zeta_j^2} \), and the coefficients \( \bar{q}_{c,j}, \bar{q}_{s,j} \) depend on the initial conditions, the modal vectors, and the mass matrix.
2 Non-Classical Damping

In general, the damping is not classical, $R^TCR$ is not a diagonal matrix, and the natural frequencies, damping ratios, and modal vectors depend on the mass, stiffness, and damping matrices of the structural system. To determine the mode-shape vectors, natural frequencies, and damping frequencies, damping ratios, and modal vectors depend on the mass, stiffness, and damping matrices.

Solving equation (1) for $\ddot{q}(t)$ is necessary to write the 2nd order differential equation (1) as two sets of first order differential equations. Defining the velocity $v(t) = \dot{r}(t)$, so that $\ddot{r}(t) = \dot{v}(t)$, and solving equation (1) for $\ddot{r}(t)$,

$$\frac{d}{dt}v(t) = \ddot{r}(t) = -M^{-1}Kr(t) - M^{-1}C\dot{r}(t).$$

Re-writing these two sets of first order differential equations in matrix form,

$$\frac{d}{dt}\begin{bmatrix} r(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0_{N \times N} & I_N \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} r(t) \\ v(t) \end{bmatrix}.$$  \hspace{1cm} (17)

The $2N$-by-$2N$ matrix in the square brackets is called the dynamics matrix. Note that it is not symmetric.

For any damped system (classically or non-classically damped) we must assume that the free-vibration response decays with time,

$$r(t) = 2\bar{r}_te^{\sigma t} \cos(\omega_d t) - 2\bar{r}_1e^{\sigma t} \sin(\omega_d t).$$  \hspace{1cm} (18)

All of the terms in equation (18) are real valued, however, it will be convenient to express this equation in terms of complex values. We now introduce a complex mode shape vector $\bar{r} = \bar{r}_r + i\bar{r}_i$ and a complex modal coordinate.

$$q(t) = q_r(t) + iq_i(t) = e^{\sigma t}(\cos(\omega_d t) + i \sin(\omega_d t)),$$  \hspace{1cm} (19)

where $\bar{r}_r$ and $\bar{r}_i$ are the real and imaginary parts of $\bar{r}$ and $q_r(t)$ and $q_i(t)$ are the real and imaginary parts of $q(t)$. With these new definitions, the trial function may be written compactly as

$$r(t) = \bar{r}q(t) + \bar{r}^*q^*(t).$$

Note here that the subscripts “$r$” and “$i$” indicate real and imaginary and are not indices. Note also that

$$e^{\sigma t}(\cos(\omega_d t) + i \sin(\omega_d t)) = e^{\lambda t}$$  \hspace{1cm} (20)

where $\lambda = \sigma + i\omega_d$. So, the complex modal coordinate, $q(t)$, can be written $q(t) = e^{\lambda t}$. The real part of $\lambda$ equals $-\zeta\omega_n$, the imaginary part of $\lambda$ equals $\omega_d = \omega_n\sqrt{\zeta^2 - 1}$, and $\lambda\lambda^* = \omega_n^2$.

Re-writing and differentiating equation (18) to solve the first order differential equations (17),

$$r(t) = \bar{r}e^{\lambda t} + \bar{r}^*e^{\lambda^* t}$$  \hspace{1cm} (21)

$$v(t) = \lambda\bar{r}e^{\lambda t} + \lambda^*\bar{r}^*e^{\lambda^* t},$$  \hspace{1cm} (22)

or

$$\begin{bmatrix} r(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & \bar{r}^* \\ \lambda\bar{r} & \lambda^*\bar{r}^* \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ e^{\lambda^* t} \end{bmatrix},$$  \hspace{1cm} (23)

and

$$\frac{d}{dt}\begin{bmatrix} r(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & \bar{r}^* \\ \lambda\bar{r} & \lambda^*\bar{r}^* \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} \begin{bmatrix} e^{\lambda t} \\ e^{\lambda^* t} \end{bmatrix}.$$  \hspace{1cm} (24)
Substituting equations (23) and (24) into the differential equations (17),

\[
\begin{bmatrix}
\ddot{\vec{r}} & \ddot{\vec{r}}^* \\
\lambda \vec{r} & \lambda^* \vec{r}^*
\end{bmatrix}
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda^*
\end{bmatrix}
\begin{bmatrix}
e^{\lambda t} \\
e^{\lambda^* t}
\end{bmatrix}
= 
\begin{bmatrix}
0_{N \times N} & I_N \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\begin{bmatrix}
\vec{r} & \vec{r}^* \\
\lambda \vec{r} & \lambda^* \vec{r}^*
\end{bmatrix}
\begin{bmatrix}
e^{\lambda t} \\
e^{\lambda^* t}
\end{bmatrix},
\]

(25)

For this equation to be true for all time,

\[
\begin{bmatrix}
\ddot{\vec{r}} & \ddot{\vec{r}}^* \\
\lambda \vec{r} & \lambda^* \vec{r}^*
\end{bmatrix}
\begin{bmatrix}
\lambda & 0 \\
0 & \lambda^*
\end{bmatrix}
= 
\begin{bmatrix}
0_{N \times N} & I_N \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\begin{bmatrix}
\vec{r} & \vec{r}^* \\
\lambda \vec{r} & \lambda^* \vec{r}^*
\end{bmatrix},
\]

(26)

which represents a complex-conjugate pair of standard eigen-value problems:

\[
\begin{bmatrix}
0_{N \times N} & I_N \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\begin{bmatrix}
\vec{r} \\
\lambda \vec{r}
\end{bmatrix} = 
\begin{bmatrix}
\vec{r} \\
\lambda \vec{r}
\end{bmatrix} \lambda
\]

(27)

and

\[
\begin{bmatrix}
0_{N \times N} & I_N \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\begin{bmatrix}
\vec{r}^* \\
\lambda^* \vec{r}^*
\end{bmatrix} = 
\begin{bmatrix}
\vec{r}^* \\
\lambda^* \vec{r}^*
\end{bmatrix} \lambda^*.
\]

(28)

The solution to one of these two standard eigen-value problems implies the solution to the other.

A relationship between the modal vectors found by solving the general eigen-value problem (3) and the standard eigen-value problem (27) can be found by solving equation (27) for the un-damped case \(C = 0_{N \times N}\):

\[
\det \left( \begin{bmatrix}
-M^{-1}K & -\lambda I_N \\
I_N & -\lambda I_N
\end{bmatrix} \right) = \det \left( \lambda^2 I_N + M^{-1}K \right) = 0
\]

(29)

Comparing this characteristic equation to the general eigen-value problem, it can be seen that \(\lambda^2 = -\omega_n^2\), or that \(\lambda = \pm i\omega_n\). The eigen-vectors of this standard eigen-value problem for the un-damped system, \([\vec{r}^T \ i\omega_n \vec{r}^T]^T\), are directly related to the solution of the general eigen-value problem. Recall that eigen-vectors may be arbitrarily scaled, and it is not uncommon for numerical solutions to (27) to be scaled so that \(\vec{r}\) is imaginary and \(i\omega_n \vec{r}\) is real. For the un-damped case, the eigen-vectors can be more-intuitively scaled so that \(\vec{r}\) is purely real and \(i\omega_n \vec{r}\) is purely imaginary.

The real modes arising from systems with zero or classical damping have nodes, which are stationary points at which the structure has zero displacement. In contrast, for a complex modal vector, \(\vec{r} = \vec{r}_r + i\vec{r}_i\), there is not always a point on the structure at which the modal displacement is zero at all times within a periodic cycle.
3 Numerical Examples

The MATLAB programs Cmodes3run.m, Cmodes3analysis.m, and N.dof.anim.m, may be used to explore the modal characteristics of non-classically damped structures. These programs make plots of the real and imaginary parts of the displacement modal vector, $\bar{r}$, the modal phasors for each degree of freedom, the real and imaginary parts of the displacement modal coordinates, $q(t)$, and the displacement responses of the coordinates of a three-degree-of-freedom building model, for which,

$$
M = \begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 \end{bmatrix}, \quad C = \begin{bmatrix}
c_1 + c_2 & -c_2 & 0 \\
-c_2 & c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3 \end{bmatrix}, \quad K = \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3 \end{bmatrix}
$$

Values for the floor masses, $m_i$, inter-story viscous damping rates, $c_i$, inter-story stiffnesses, $k_i$, and displacement initial conditions, $r(0)$, are specified in Cmodes3run.m. Running Cmodes3run.m results in plots and an animation of the free response to the specified initial conditions.

In the .m-function Cmodes3analysis.m, each complex mode vector $\bar{r}_j$ is scaled by a rotation $\theta_j$ in the complex plane (via multiplication by the complex scalar $e^{-i\theta_j}$) so that the real part of the displacement the mode shape, $\text{Re}(\bar{r})$, is maximized (and the imaginary part is minimized). For this rotation, $\tan \theta_j = \text{Im}(\bar{r}_j)/\text{Re}(\bar{r}_j)$, where $\bar{r}_j = \max |\bar{r}_j|$. The magnitude of each mode is then scaled so that the displacement parts of the modes are mass-normalized by dividing the real and imaginary parts of $\bar{r}_j$ and $i\omega_n\bar{r}_j$ by $\bar{r}_j^T M \bar{r}_j = I_N$.

When running Cmodes3run.m, you may try to:

1. Run a simulation with the as-provided default values for $m_i$, $c_i$, $k_i$, and $r_0$ ($m_i = 1$ tonne, $c_i = [0, 3, 0]$ N/mm/s, $k_i = 1000$ N/mm, $r_{oi} = [1, -2, 3]$ mm). Observe how the real part of mode $j$ has $j - 1$ zero-crossings; how the free response of each modal displacement $q_j(t)$ contains only a single frequency, the damped natural frequency, $\omega_n$: how all three modes are damped even if there is damping in one story only; and how the free response of a higher-frequency mode decays faster (in less time) than that of a lower-frequency mode, even if the higher-frequency mode has slightly less damping.

2. Confirm that if $C = 0$ the modes are purely real (with the normalization implemented as described above.)

3. Examine modal characteristics for systems with a Rayleigh damping matrix. For example by setting $k_i = 1000$ N/mm and $c_i = 2.0$ N/mm/s, $C$ is stiffness-proportional ($C = 0.002K$). Is $\bar{R}$ real or complex in this case?

4. Determine values of $c_i$ that will give approximately 5 percent damping in all three modes, for $m_i = 1$ tonne and $k_i = 1000$ N/mm. This will involve some trial-and-error iteration on the three values of $c_i$. (hint: $c_1 > c_2 > c_3$; $11 < c_1 < 13$ N/mm/s; and $2 < c_2 < 4$ kN/mm/s) Are the resulting modes real or complex? Is there anything unusual or surprising about any of the values of $c_i$ required to meet this goal? Does this finding imply a fallacy in the concept of “damped real normal modes” with arbitrary modal damping ratios?

5. Set the initial displacement, $r_o = r(0)$, proportional to each of the three mode shape vectors, and observe that the free response consists almost entirely of that mode. In Cmodes3run, if you set $r_{oi} = j$, where $j \in [1, 2, 3]$, $r_o$ will be set to $\bar{r}_j$. Next select some other set of initial displacements and observe that the free response contains all three modes.
6. The phasor matrix, $\Phi$, of a complex modal matrix, $\bar{R}$, is given by $\Phi_{ij} = \arctan(\bar{R}_{ij}/\bar{R}_{r ij})$ ($-\pi/2 < \Phi_{ij} < +\pi/2$). How does multiplying a modal vector by $\sqrt{-1}$ affect the associated column of $\Phi$? For a complex-valued mode, are values in the associated column of $\Phi$ equal to one another? Why, or why not? The “complexity” of modal vector $\bar{r}_j$ can be characterized by $C_j = \max_i |\Phi_{ij} - \Phi(i-1)j|$ Using the phasor plots generated by Cmodes3run.m with $m_i = 1$ tonne and $k_i = 1000$ N/mm, find values of $c_1, c_2, c_3$ that give a mode with a complexity greater than about 30 degrees.

7. Explore the effects of changing the values of mass, damping, and stiffness. When changing a value of $m_i, c_i, k_i,$ and $r_{oi}$, try to predict the effect of the change on the natural frequencies, damping ratios, mode-shapes, modal responses, and floor responses; then use Cmodes3run.m to check yourself.

(a) What happens if you increase a value of $c_i$ so that the damping of one of the modes approaches 100 percent?
(b) What happens if a single value of $c_i$ is negative?
(c) What happens if a value of $c_i$ is so negative that one of the modal damping ratios becomes slightly negative ($\approx -0.50\%$)?
(d) What happens if one of the stiffness coefficients is much much larger than the other coefficients?
(e) What happens if one of the stiffness coefficients is slightly negative?
(f) What happens if one of the mass coefficients is very negative?

References