In materials or elements with hysteresis, the response to a cycle reciprocating forcing depends on the forcing history — for any reciprocating forcing of a sufficiently large amplitude [10]. Hysteretic behavior is commonly depicted as loops in graphs (2D plots) of periodic (oscillatory) output vs. periodic input. In rate-dependent hysteresis, the size and shape of the hysteresis loop changes with the rate or frequency of the input. If the loop collapses to a function (e.g., a curved line) for any input (e.g., quasi-static), then the system is not hysteretic [10]. Hysteresis implies a non-linear relationship between inputs and outputs: differential equation models for hysteresis must be nonlinear; convolution models for hysteresis must be nonhomogeneous. Linear visco-elastic materials are rate-dependent but are not hysteretic because forces and displacements are proportional in the limit of quasi-static loading. This document describes dynamic hysteresis models that are members of the Duhem [9, 15, 17, 20] class of nonlinear ordinary differential equations

\[
\dot{z}(t) = f(z(t), u(t)) \ g(\dot{u}(t)) , \quad z(0) = z_o .
\]  

Duhem models can be used to model the kinds of rate-independent hysteresis representative of material yielding and stick-slip friction.

Consider the following nonlinear ordinary differential equation

\[
\dot{z}(t) = \dot{u}(t) - |\dot{u}(t)| z^\eta(t) , \quad z(0) = z_o .
\]  

The variable \(z(t)\) represents a level of force, normalized to be in the range \(-1 < z(t) < 1\), so you can think of \(z\) as a force level normalized by a yield force, or friction sliding force. The variable \(u(t)\) represents a level of displacement, normalized by a yield displacement, so you can think of \(u\) as a ductility ratio. In this way, equation (2) relates the rate of change of force to the rate of change of displacement (i.e., velocity). The exponent \(\eta\) is restricted to be odd, for the time being.

Noting that \(|\dot{u}| = \dot{u} \text{sgn}(\dot{u})\), equation (2) may be re-written as,

\[
\dot{z} = (1 - z^\eta \text{sgn}(\dot{u})) \ \dot{u} , \quad z(0) = z_o
\]  

(3)

and the slope of the force-displacement relationship is

\[
\frac{dz}{du} = \frac{\dot{z}}{\dot{u}} = 1 - z^\eta \text{sgn}(\dot{u})
\]  

(4)

From this expression it is easy to see that:
When the force is zero \((z = 0)\), the dimensionless stiffness is 1 (the dimensional stiffness is the yield force divided by the yield displacement).

As the force approaches the yield force \((z \to 1, \dot{u} > 0 \text{ or } z \to -1, \dot{u} < 0)\) \(dz/du\) approaches zero.

When the velocity is positive, \(dz/du = 1 - z^n\), and when the velocity is negative, \(dz/du = 1 + z^n\).

\[dz/du \geq 0 ; \text{sgn} (\dot{z}) = \text{sgn}(\dot{u}) ; \text{and } (\dot{z})(\dot{u}) \geq 0 .\]

\[z = f/f\]

\[\dot{z} = 1\]

\[dz/du = 0\]

\[\dot{u} < 0\]

\[dz/du = 2\]

\[\dot{u} > 0\]

\[dz/du = 0\]

\[z = 1\]

\[z = -1\]

\[dz/du = 2\]

\[u = d/d\]

\[dz/du = 1\]

\[dz/du = 1 - z^n\]

\[dz/du = z^n\]

\[u = d/d\]

\[dz/du = 0\]

\[z = 1\]

\[z = -1\]

\[dz/du = 0\]

\[\dot{z} = (1 - z^n \text{sgn}(z)) \dot{u}, \quad (5)\]

which allows the exponent \(\eta\) to be any positive value.

\[\dot{z} = (1 - |z|^n \text{sgn}(\dot{uz})) \dot{u}, \quad (5)\]

Extension 1

Replacing \(z^n\) with \(|z|^n\text{sgn}(z)\) in equation (3), and noting that \(\text{sgn}(a)\text{sgn}(b) = \text{sgn}(ab)\),

\[\dot{z} = (1 - |z|^n \text{sgn}(\dot{uz})) \dot{u}, \quad (5)\]

Extension 2

Adding parameters \(\beta\) and \(\gamma\) in equation (5), as follows,

\[\dot{z} = (1 - |z|^n (\beta \text{ sgn}(\dot{uz}) + \gamma)) \dot{u}, \quad (6)\]

allows for a wide range of hysteretic forms, as shown in figure 2.

If \(\beta + \gamma = 1\), then \(-1 < z < 1\). If \(\eta > 0, \gamma > 0\) and \(-\gamma < \beta < \gamma\) the model respects the Second Law of Thermodynamics [1, 11, 14].
Adding a parameter $A$, 
$$\dot{z} = \left( A - |z|^\eta (\beta \text{sgn}(\dot{u}z) + \gamma) \right) \dot{u},$$  
(7)
allows for scaling. If $\beta + \gamma = A$ then $-A < z < A$. This is the “Bouc-Wen” model for
hysteresis [5, 6, 14, 26] and is an element of the Duhem class of hysteresis models [17, 25].

 Extension 4

Isotropic bi-axial hysteretic behavior may be modeled in orthogonal directions $x$ and $y$
by coupling the hysteretic variables and velocities [12, 23].
$$\dot{z} = a(z, \dot{u}) \ z + \ b \ \dot{u}$$
(8)
where $z = [z_x, z_y]$, $\dot{u} = [\dot{u}_x, \dot{u}_y]$, and
$$a(z, \dot{u}) = - [\beta(|z_x \dot{u}_x| + |z_y \dot{u}_y|) + \gamma(z_x \dot{u}_x + z_y \dot{u}_y)] (z_x^2 + z_y^2)^{\eta - 2}/2$$
(9)

 Extension 5

Hysteretic behavior with a non-zero post yield stiffness may be simulated by combining
equation (7) or (8) with
$$f(t) = f_y((1 - \kappa)z(t) + \kappa u(t)),$$  
(10)
where $f_y$ is a yield force level, $u$ is the displacement divided by the yield displacement (the ductility), and $\kappa$ is the ratio of the post yield stiffness to the pre-yield stiffness. When using (10), set $A = 1$ and $\beta + \gamma = 1$.

Stick-Slip Friction

The Dahl friction model [8],

$$\dot{z} = (1 - z \text{sgn}(\dot{u}))^\eta \dot{u}$$

is equivalent to the Bouc model for $\eta = 1$ (which is the usual choice for $\eta$ in friction modeling). The “LuGre” friction model [7] is an extension of the Dahl model which captures the Stribeck (“stick-slip”) effect [2]. The LuGre model is an element of the Duhem class of hysteresis models [17, 25].

$$\dot{z} = (1 - z \text{sgn}(\dot{u})/g(\dot{u})) \dot{u}$$

$$k_0 g(\dot{u}) = F_C + (F_S - F_C) e^{-(\dot{u}/v_S)^2}$$

In the LuGre model the velocity $\dot{u}$ is not normalized by a pre-slip displacement. For high values of pre-slip stiffness, $k_0$, the LuGre model can require very small time steps for numerical stability.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_o$</td>
<td>pre-slip stiffness</td>
<td>$10^4$</td>
<td>N/m</td>
</tr>
<tr>
<td>$c_1$</td>
<td>friction rate effect</td>
<td>$\sqrt{10^4}$</td>
<td>Ns/m</td>
</tr>
<tr>
<td>$c_2$</td>
<td>viscous rate effect</td>
<td>0.4</td>
<td>Ns/m</td>
</tr>
<tr>
<td>$F_C$</td>
<td>Coulomb sliding friction force</td>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>$F_S$</td>
<td>Stribeck sticking friction force</td>
<td>1.5</td>
<td>N</td>
</tr>
<tr>
<td>$v_S$</td>
<td>Stribeck velocity</td>
<td>0.001</td>
<td>m/s</td>
</tr>
</tbody>
</table>

Degrading Behavior

To model the accumulation of damage [4], strength $f_y$ and other model parameters may be linked to a damage accumulation index $D$, where $D \approx |\dot{u} - \dot{z}|$ and

$$f_y(t) = \frac{f_y(0)}{1 + D(t)/a_{(f_y)}}$$

where $a_{(f_y)}$ is a positive constant.

Further Extensions

Further generalizations to these equations can account for orthotropic behavior, degrading stiffness, and pinching hysteresis, as described in the references below.
References


