In a system with hysteresis, the system response lags the excitation [16, 21] — for any excitation [10]. Hysteretic behavior is commonly depicted as loops in graphs (2D plots) of periodic (oscillatory) output vs. periodic input. In rate-dependent hysteresis, the size and shape of the hysteresis loop changes with the rate or frequency of the input. If the loop collapses to a function (e.g., a curved line) for any input (e.g., quasi-static), then the system is not hysteretic [10]. Hysteresis implies a non-linear relationship between inputs and outputs: differential equation models for hysteresis must be nonlinear; convolution models for hysteresis must be nonhomogeneous. Linear visco-elastic materials are rate-dependent but are not hysteretic because forces and displacements are proportional in the limit of quasi-static loading. This document describes members of the Duhem [9, 15, 17, 20] class of nonlinear ordinary differential equations

$$\dot{z}(t) = f(z(t), u(t)) \ g(\dot{u}(t)) , \quad z(0) = z_0 .$$

(1)

Duhem models can be used to model the kinds of rate-independent hysteresis representative of material yielding and stick-slip friction.

Consider the following nonlinear ordinary differential equation

$$\dot{z}(t) = \dot{u}(t) - |\dot{u}(t)| \ z^\eta(t) , \quad z(0) = z_0 .$$

(2)

The variable $z(t)$ represents a level of force, normalized to be in the range $-1 < z(t) < 1$, so you can think of $z$ as a force level normalized by a yield force, or friction sliding force. The variable $u(t)$ represents a level of displacement, normalized by a yield displacement, so you can think of $u$ as a ductility ratio. In this way, equation (2) relates the rate of change of force to the rate of change of displacement (i.e., velocity). The exponent $\eta$ is restricted to be odd, for the time being.

Noting that $|\dot{u}| = \dot{u} \ \text{sgn}(\dot{u})$, equation (2) may be re-written as,

$$\dot{z} = (1 - z^\eta \ \text{sgn}(\dot{u})) \ \dot{u} , \quad z(0) = z_0$$

(3)

and the slope of the force-displacement relationship is

$$\frac{dz}{du} = \frac{\dot{z}}{\dot{u}} = 1 - z^\eta \text{sgn}(\dot{u})$$

(4)

From this expression it is easy to see that

- When the force is zero ($z = 0$), the dimensionless stiffness is 1 (the dimensional stiffness is the yield force divided by the yield displacement).
• As the force approaches the yield force \((z \to 1, \dot{u} > 0 \text{ or } z \to -1, \dot{u} < 0)\) \(dz/du\) approaches zero.

• When the velocity is positive, \(dz/du = 1 - z^n\), and when the velocity is negative, \(dz/du = 1 + z^n\).

• \(dz/du \geq 0\); \(\text{sgn}(\dot{z}) = \text{sgn}(\dot{u})\); and \((\dot{z})(\dot{u}) \geq 0\).

\[ \frac{dz}{du} = z \]

Figure 1. The vector field of \(dz/du\) for \(\dot{z} = (1 - z^n \text{sgn}(\dot{u})) \dot{u}\) depends on \(\text{sgn}(\dot{u})\). Larger values of \(n\) result in a sharper “knee.”

Extension 1

Replacing \(z^n\) with \(|z|^n \text{sgn}(z)\) in equation (3), and noting that
\[ \text{sgn}(a) \text{sgn}(b) = \text{sgn}(ab), \]
which allows the exponent \(n\) to be any positive value.

Extension 2

Adding parameters \(\beta\) and \(\gamma\) in equation (5), as follows,
\[ \dot{z} = (1 - |z|^n \text{sgn}(\dot{u}z)) \dot{u}, \]
allows for a wide range of hysteretic forms, as shown in figure 2.
If \(\beta + \gamma = 1\), then \(-1 < z < 1\). If \(\eta > 0\), \(\gamma > 0\) and \(-\gamma < \beta < \gamma\) the model respects the Second Law of Thermodynamics [1, 11, 14].

Extension 3

Adding a parameter \(A\),
\[ \dot{z} = (A - |z|^n (\beta \text{sgn}(\dot{u}z) + \gamma)) \dot{u}, \]
allows for scaling. If \(\beta + \gamma = A\) then \(-A < z < A\). This is the “Bouc-Wen” model for hysteresis [5, 6, 14, 26] and is an element of the Duhem class of hysteresis models [17, 25].
Isotropic bi-axial hysteretic behavior may be modeled in orthogonal directions $x$ and $y$ by coupling the hysteretic variables and velocities \cite{12, 23}.

\begin{equation}
\dot{z} = a(z, \dot{u}) \ z + b \ \dot{u} \tag{8}
\end{equation}

where $z = [z_x, z_y]$, $\dot{u} = [\dot{u}_x, \dot{u}_y]$, and

\begin{equation}
a(z, \dot{u}) = -\left[ \beta(|z_x\dot{u}_x| + |z_y\dot{u}_y|) + \gamma(z_x\dot{u}_x + z_y\dot{u}_y) \right] (z_x^2 + z_y^2)^{(\eta - 2)/2} \tag{9}
\end{equation}

**Extension 5**

Hysteretic behavior with a non-zero post yield stiffness may be simulated by combining equation (7) or (8) with

\begin{equation}
f(t) = f_y((1 - \kappa)z(t) + \kappa u(t)) \tag{10}
\end{equation}

where $f_y$ is a yield force level, $u$ is the displacement divided by the yield displacement (the ductility), and $\kappa$ is the ratio of the post yield stiffness to the pre-yield stiffness. When using (10), set $A = 1$ and $\beta + \gamma = 1$. 

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Figure 2. Dependence of hysteretic shape on $\beta$, $\gamma$, and $\eta$. 

Extension 4

Extension 5
Stick-Slip Friction

The Dahl friction model [8],

\[
\dot{z} = \left( 1 - z \, \text{sgn}(\dot{u}) \right)^\eta \ \dot{u}
\] (11)

is equivalent to the Bouc model for \( \eta = 1 \) (which is the usual choice for \( \eta \) in friction modeling). The “LuGre” friction model [7] is an extension of the Dahl model which captures the Stribeck (“stick-slip”) effect [2]. The LuGre model is an element of the Duhem class of hysteresis models [17, 25].

\[
\dot{z} = \left( 1 - z \, \text{sgn}(\dot{u})/g(\dot{u}) \right) \dot{u}
\] (12)

\[
k_o g(\dot{u}) = F_C + (F_S - F_C) \ e^{-\left(\dot{u}/v_S\right)^2}
\] (13)

\[
f(t) = k_o z(t) + c_1 \dot{z}(t) + c_2 \dot{u}(t)
\] (14)

In the LuGre model the velocity \( \dot{u} \) is not normalized by a pre-slip displacement. For high values of pre-slip stiffness, \( k_o \), the LuGre model can require very small time steps for numerical stability.

| Table 1. Representative parameter values for the LuGre friction model [7]. |
|-----------------|-----------------|--------|
| Parameter       | Definition       | Value  | Unit   |
| \( k_o \)       | pre-slip stiffness | \( 10^4 \) | N/m    |
| \( c_1 \)       | friction rate effect | \( \sqrt{10^4} \) | Ns/m   |
| \( c_2 \)       | viscous rate effect | 0.4   | Ns/m   |
| \( F_C \)       | Coulomb sliding friction force | 1   | N      |
| \( F_S \)       | Stribeck sticking friction force | 1.5  | N      |
| \( v_S \)       | Stribeck velocity      | 0.001 | m/s    |

Degrading Behavior

To model the accumulation of damage [4], strength \( f_y \) and other model parameters may be linked to a damage accumulation index \( D \), where \( \dot{D} \approx |\dot{u} - \dot{z}| \) and

\[
f_y(t) = \frac{f_y(0)}{1 + D(t)/a(f_y)},
\] (15)

where \( a(f_y) \) is a positive constant.

Further Extensions

Further generalizations to these equations can account for orthotropic behavior, degrading stiffness, and pinching hysteresis, as described in the references below.
References


