Generalized Coordinates, Lagrange's Equations, and Constraints

CEE 541. Structural Dynamics
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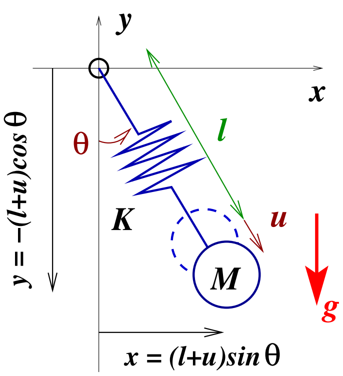
1 Cartesian Coordinates and Generalized Coordinates

The set of coordinates used to describe the motion of a dynamic system is not unique. For example, consider an elastic pendulum (a mass on the end of a spring). The position of the mass at any point in time may be expressed in Cartesian coordinates $(x(t), y(t))$ or in terms of the angle of the pendulum and the stretch of the spring $(\theta(t), u(t))$. Of course, these two coordinate systems are related. For Cartesian coordinates centered at the pivot point,



$$\begin{aligned} x(t) &= (l + u(t)) \sin\theta(t) & (1)\\ y(t) &= -(l + u(t)) \cos\theta(t) & (2) \end{aligned}$$

where l is the un-stretched length of the spring. Let's define

$$\boldsymbol{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} x(\theta(t), u(t)) \\ y(\theta(t), u(t)) \end{bmatrix} = \begin{bmatrix} (l + u(t)) \sin\theta(t) \\ -(l + u(t)) \cos\theta(t) \end{bmatrix} \quad (3)$$

and

$$\boldsymbol{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ u(t) \end{bmatrix} \quad (4)$$

so that $\boldsymbol{r}(t)$ is a function of $\boldsymbol{q}(t)$.

The potential energy, V, may be expressed in terms of $\boldsymbol{r}$ or in terms of $\boldsymbol{q}$.

In Cartesian coordinates, the velocities are

$$\dot{\boldsymbol{r}}(t) = \begin{bmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{u}(t) \sin\theta(t) + (l + u(t)) \cos\theta(t)\dot{\theta}(t) \\ -\dot{u}(t) \cos\theta(t) + (l + u(t)) \sin\theta(t)\dot{\theta}(t) \end{bmatrix} \quad (5)$$

So, *in general*, Cartesian velocities $\dot{\boldsymbol{r}}(t)$ can be a function of both the velocity *and* position of some other coordinates ($\dot{\boldsymbol{q}}(t)$ *and* $\boldsymbol{q}(t)$). Such coordinates $\boldsymbol{q}$ are called *generalized coordinates*. The kinetic energy, T, may be expressed in terms of either $\dot{\boldsymbol{r}}$ or, *more generally*, in terms of $\dot{\boldsymbol{q}}$ and $\boldsymbol{q}$.

Small changes (or *variations*) in the rectangular coordinates, $(\delta x, \delta y)$ *consistent with all displacement constraints*, can be found from variations in the generalized coordinates $(\delta\theta, \delta u)$.

$$\delta x = \frac{\partial x}{\partial \theta}\delta\theta + \frac{\partial x}{\partial u}\delta u = (l + u(t)) \cos\theta(t)\delta\theta + \sin\theta(t)\delta u \quad (6)$$

$$\delta y = \frac{\partial y}{\partial \theta}\delta\theta + \frac{\partial y}{\partial u}\delta u = (l + u(t))\sin\theta(t)\delta\theta - \cos\theta(t)\delta u \tag{7}$$

2 Principle of Virtual Displacements

Virtual displacements δr_i are any displacements consistent with the constraints of the system. The *principle of virtual displacements*[1] says that the work of real external forces through virtual external displacements equals the work of the real internal forces arising from the real external forces through virtual internal displacements consistent with the real external displacements. In system described by n coordinates r_i, with n internal inertial forces $m_i\ddot{r}_i(t)$, potential energy $V(\boldsymbol{r})$, and n external forces p_i *collocated* with coordinates r_i, the principle of virtual dispalcements says,

$$\sum_{i=1}^{n}\left(m_i\ddot{r}_i + \frac{\partial V}{\partial r_i}\right)\delta r_i = \sum_{i=1}^{n}(p_i(t))\,\delta r_i \tag{8}$$

3 Derivation of d'Alembert's principle from Minimum Total Potential Energy

The *total potential energy*, Π, is defined as the internal potential energy, V, minus the potential energy of external forces[2]. In terms of the $\boldsymbol{r}$ (Cartesian) coordinate system and n forces f_i collocated with the n displacement coordinates, r_i, the total potential energy is given by equation (**??**).

$$\Pi(\boldsymbol{r}) = V(\boldsymbol{r}) - \sum_{i=1}^{n} f_i r_i\ . \tag{9}$$

The *principle of minimum total potential energy* states that the total potential energy is minimized in a condition of equilibrium. Minimizing the total potential energy is the same as setting the variation of the total potential energy to zero. For any arbitrary set of "variational" displacements $\delta r_i(t)$, *consistent with displacement constraints*,

$$\min_{\boldsymbol{r}} \Pi(\boldsymbol{r}) \Rightarrow \delta\Pi(\boldsymbol{r}) = 0 \Leftrightarrow \sum_{i=1}^{n}\frac{\partial \Pi}{\partial r_i}\ \delta r_i = 0 \Leftrightarrow \sum_{i=1}^{n}\frac{\partial V}{\partial r_i}\ \delta r_i - \sum_{i=1}^{n} f_i\ \delta r_i = 0\ , \tag{10}$$

In a dynamic situation, the forces f_i can include inertial forces, damping forces, and external loads. For example,

$$f_i(t) = -m_i\ddot{r}_i(t) + p_i(t)\ , \tag{11}$$

where p_i is an *external* force collocated with the displacement coordinate r_i. Substituting equation (**??**) into equation (**??**), and rearranging slightly, results in the d'Alembert equations, the same as equation (**??**) found from the principle of virtual displacements,

$$\sum_{i=1}^{n}\left(m_i\ddot{r}_i(t) + \frac{\partial V}{\partial r_i} - p_i(t)\right)\delta r_i = 0\ . \tag{12}$$

[1] d'Alembert, J-B le R, 1743

[2] In solid mechanics, internal strain energy is conventionally assigned the variable U, whereas the potential energy of external forces is conventionally assigned the variable V. In Lagrange's equations potential energy is assigned the variable V and kinetic energy is denoted by T.

In equations (**??**) and (**??**) the virtual displacements (i.e., the variations) δr_i must be arbitrary and independent of one another; these equations must hold for each coordinate r_i individually.

$$m_i \ddot{r}_i(t) + \frac{\partial V}{\partial r_i} - p_i(t) = 0 \ . \tag{13}$$

4 Derivation of Lagrange's equations from d'Alembert's principle

For many problems equation (**??**) is enough to determine equations of motion. However, in coordinate systems where the kinetic energy depends on the position and velocity of some generalized coordinates, $\boldsymbol{q}(t)$ and $\dot{\boldsymbol{q}}(t)$, expressions for inertial forces become more complicated.

The first goal, then, is to relate the work of inertial forces ($\sum_i m_i \ddot{r}_i \ \delta r_i$) to the kinetic energy in terms of a set of generalized coordinates. To do this requires a change of coordinates from variations in n Cartesian coordinates $\delta\boldsymbol{r}$ to variations in m generalized coordinates $\delta\boldsymbol{q}$,

$$\delta r_i = \sum_{j=1}^{m} \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{m} \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\delta q_j \quad \text{and} \quad \frac{d}{dt}\delta r_i = \delta \dot{r}_i = \sum_{j=1}^{m} \frac{\partial \dot{r}_i}{\partial q_j}\delta q_j \ . \tag{14}$$

Now, consider the kinetic energy of a constant mass m. The kinetic energy in terms of velocities in Cartesian coordinates is given by $T(\dot{\boldsymbol{r}}) = \frac{1}{2}m(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2)$, and

$$\frac{\partial T}{\partial \dot{r}_i} \ \delta r_i = m_i \dot{r}_i \ \delta r_i \ . \tag{15}$$

Applying the product and chain rules of calculus to equation (**??**),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i} \ \delta r_i\right) = m_i \ddot{r}_i \ \delta r_i + m_i \dot{r}_i \ \frac{d}{dt}\delta r_i \tag{16}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i} \ \sum_{j=1}^{m} \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\delta q_j\right) = m_i \ddot{r}_i \ \delta r_i + \frac{\partial T}{\partial \dot{r}_i} \ \delta \dot{r}_i \tag{17}$$

$$m_i \ddot{r}_i \ \delta r_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i} \ \sum_{j=1}^{m} \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\delta q_j\right) - \frac{\partial T}{\partial \dot{r}_i} \ \sum_{j=1}^{m} \frac{\partial \dot{r}_i}{\partial q_j}\delta q_j \ . \tag{18}$$

Equation (**??**) is a step to getting the work of inertial forces ($\sum_i m_i \ddot{r}_i \ \delta r_i$) in terms of a set of generalized coordinates q_j. Next, changing the derivatives of the potential energy from r_i to q_j,

$$\frac{\partial V}{\partial r_i}\delta r_i = \sum_{j=1}^{m} \frac{\partial V}{\partial q_j}\frac{\partial q_j}{\partial r_i} \ \frac{\partial r_i}{\partial q_j}\delta q_j \ . \tag{19}$$

Finally, expressing the work of non-conservative forces p_i in terms of the new coordinates q_j,

$$\sum_{i=1}^{n} p_i(t) \ \delta r_i = \sum_{i=1}^{n} p_i(t) \sum_{j=1}^{m} \frac{\partial r_i}{\partial q_j} \ \delta q_j = \sum_{j=1}^{m} \ \sum_{i=1}^{n} p_i(t) \frac{\partial r_i}{\partial q_j} \ \delta q_j = \sum_{j=1}^{m} Q_j(t) \ \delta q_j \ , \tag{20}$$

where $Q_j(t)$ are *generalized forces*, collocated with the generalized coordinates, $q_j(t)$. Substituting equations (**??**), (**??**) and (**??**) into d'Alembert's equation (**??**), rearranging the order of the summations, factoring out the common δq_j, canceling the ∂r_i and $\partial \dot{r}_i$ terms, and eliminating the sum over i, leaves the equation in terms of q_j and $\dot{q}_j$,

$$\begin{aligned}
\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\sum_{j=1}^{m}\frac{\partial \dot{r}_i}{\partial \dot{q}_j}\delta q_j\right) - \frac{\partial T}{\partial \dot{r}_i}\sum_{j=1}^{m}\frac{\partial \dot{r}_i}{\partial q_j}\delta q_j + \sum_{j=1}^{m}\frac{\partial V}{\partial q_j}\frac{\partial q_j}{\partial r_i}\frac{\partial r_i}{\partial q_j}\delta q_j - p_i(t)\sum_{j=1}^{m}\frac{\partial r_i}{\partial q_j}\,\delta q_j\right] &= 0 \\
\sum_{j=1}^{m}\ \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_i}\ \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial \dot{r}_i}\ \frac{\partial \dot{r}_i}{\partial q_j} + \frac{\partial V}{\partial q_j}\frac{\partial q_j}{\partial r_i}\ \frac{\partial r_i}{\partial q_j} - p_i\frac{\partial r_i}{\partial q_j}\right]\ \delta q_j &= 0 \\
\sum_{j=1}^{m}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - Q_j\right]\ \delta q_j &= 0
\end{aligned}$$

The variations δq_j must be arbitrary and independent of one another; this equation must hold for each generalized coordinate q_j individually. Removing the summation over j and canceling out the common δq_j factor results in Lagrange's equations,[3]

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - Q_j = 0} \tag{21}$$

are which are applicable in *any* coordinate system, Cartesian or not.

5 The Lagrangian

Lagrange's equations may be expressed more compactly in terms of the Lagrangian of the energies,

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t) \tag{22}$$

Since the potential energy V depends only on the positions, $\boldsymbol{q}$, and not on the velocities, $\dot{\boldsymbol{q}}$, Lagrange's equations may be written,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - Q_j = 0} \tag{23}$$

[3] Lagrange, J.L., *Mecanique analytique,* Mm Ve Courcier, 1811.

6 Derivation of Hamilton's principle from d'Alembert's principle

The variation of the potentential energy $V(\boldsymbol{r})$ may be expressed in terms of variations of the coordinates r_i

$$\delta V = \sum_{i=1}^{n} \frac{\partial V}{\partial r_i}\ \delta r_i = \sum_{i=1}^{n} f_i\ \delta r_i\ . \tag{24}$$

where f_i are potential forces collocated with coordiantes r_i. In Cartesian coordinates, the variation of the kinetic energy $T(\dot{\boldsymbol{r}})$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i^2 \tag{25}$$

may be expressed in terms of variations of the coordinate velocities, $\dot{r}_i$

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{r}_i}\ \delta \dot{r}_i = \sum_{i=1}^{n} m \dot{r}_i\ \delta \dot{r}_i\ . \tag{26}$$

For a system of n particle masses m_i acted on by n *internal* forces f_i of the potential V, d'Alembert's principle (**??** or **??**), is

$$\sum_{i=1}^{n} m_i \ddot{r}_i\ \delta r_i + \sum_{i=1}^{n} f_i\ \delta r_i = 0 \tag{27}$$

Integrating d'Alembert's equation over a finite time period,

$$\begin{aligned}
\int_{t_1}^{t_2} \sum_{i=1}^{n} m_i \ddot{r}_i\ \delta r_i\ dt + \int_{t_1}^{t_2} \sum_{i=1}^{n} f_i\ \delta r_i\ dt &= 0 \\
\sum_{i=1}^{n} \int_{t_1}^{t_2} m_i \frac{d}{dt}\dot{r}_i\ \delta r_i\ dt + \int_{t_1}^{t_2} \delta V\ dt &= 0 \\
\sum_{i=1}^{n} \left[m_i \dot{r}_i\ \delta r_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m_i \dot{r}_i\ \frac{d}{dt}\delta r_i\ dt \right] + \int_{t_1}^{t_2} \delta V\ dt &= 0 \\
- \int_{t_1}^{t_2} \sum_{i=1}^{n} m_i \dot{r}_i\ \delta \dot{r}_i\ dt + \int_{t_1}^{t_2} \delta V\ dt &= 0 \\
- \int_{t_1}^{t_2} \delta T\ dt + \int_{t_1}^{t_2} \delta V\ dt &= 0 \\
\delta \int_{t_1}^{t_2} (T - V)\ dt &= 0
\end{aligned} \tag{28}$$

In this derivation we consider a variation of the coordinate motions $\delta r(t)$ from t_1 to t_2. That is, $\delta r(t_1) = 0$ and $\delta r(t_2) = 0$, which eliminates the first term in the third line. The fourth line involves a transposition of the variation and the derivitive ($d(\delta r)/dt = \delta \dot{r}$). The last line is a statement of Hamilton's principle, which is presented formally in the next section. Note that kinetic energy and potential energy are scalar valued quantities, invariant to changes in coordiante systems. So, while Hamilton's principle is derived here in the context of Cartesian coordinates, it applies to generalized coordinates, as well.

7 Derivation of Lagrange's equations from Hamilton's principle

Define a *Lagrangian* of kinetic and potential energies

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t) \tag{29}$$

and define an *action potential functional*

$$\mathcal{S}[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ dt \tag{30}$$

with end points $\boldsymbol{q}_1 = \boldsymbol{q}(t_1)$ and $\boldsymbol{q}_2 = \boldsymbol{q}(t_2)$. Consider the true path of $\boldsymbol{q}(t)$ from t_1 to t_2 and a variation of the path, $\delta\boldsymbol{q}(t)$ such that $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$.

Hamilton's principle:[4]
The solution $\boldsymbol{q}(t)$ is an extremum of the action potential $\mathcal{S}[\boldsymbol{q}(t)]$
$\iff \delta\mathcal{S}[\boldsymbol{q}(t)] = 0$
$\iff \displaystyle\int_{t_1}^{t_2} \delta L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ dt = 0$
Substituting the Lagrangian into Hamilton's principle,

$$\int_{t_1}^{t_2} \left\{ \sum_i \left[\frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i + \frac{\partial T}{\partial q_i}\delta q_i - \frac{\partial V}{\partial q_i}\delta q_i \right] \right\}\ dt = 0$$

We wish to factor out the independent variations δq_i, however the first term contains the variation of the derivative, $\delta\dot{q}_i$. If the conditions for admissible variations in position δq fully specify the conditions for admissible variations in velocity $\delta\dot{q}$, the variation and the differentiation can be transposed,

$$\frac{d}{dt}\delta q_i = \delta\dot{q}_i, \tag{31}$$

and we can integrate the first term by parts,

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i}\delta\dot{q}_i = \left[\frac{\partial T}{\partial \dot{q}_i}\delta q_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i\ dt$$

Since $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$,

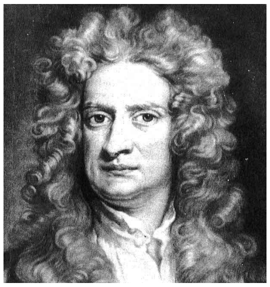
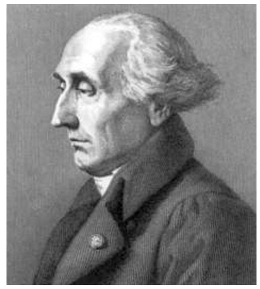
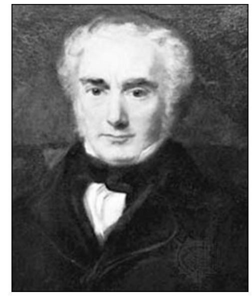
$$\int_{t_1}^{t_2} \left\{ \sum_i \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial T}{\partial q_i}\delta q_i - \frac{\partial V}{\partial q_i}\delta q_i \right] \right\}\ dt = 0$$

The variations δq_i must be arbitrary, so the term within the square brackets must be zero for all i.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0.}$$

[4] Hamilton, W.R., "On a General Method in Dynamics," *Phil. Trans. of the Royal Society* Part II (1834) pp. 247-308; Part I (1835) pp. 95-144

8 A Little History

Sir Isaac **Newton**	Jean-Baptiste le Rond **d'Alembert**	Joseph-Louis **Lagrange**	Carl Fredrich **Gauss**	William Rowan **Hamilton**
1642-1727	1717-1783	1736-1813	1777-1855	1805-1865
Second Law 1687	d'Alembert's Principle 1743	Méchanique Analytique 1788	Gauss's Principle 1829	Hamilton's Principle 1834-1835

$$\boldsymbol{f}_i \ \mathsf{d}t = \mathsf{d}(m_i \boldsymbol{v}_i)$$

$$\sum(\boldsymbol{f}_i - m_i \boldsymbol{a}_i)\delta \boldsymbol{r}_i = 0$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

$$\begin{aligned} \mathcal{G} &= \tfrac{1}{2}(\ddot{\boldsymbol{q}} - \boldsymbol{a})^{\mathsf{T}} \boldsymbol{M} (\ddot{\boldsymbol{q}} - \boldsymbol{a}) \\ \delta\mathcal{G} &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{S} &= \textstyle\int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) dt \\ \delta\mathcal{S} &= 0 \end{aligned}$$

9 Example: an elastic pendulum

For an elastic pendulum (a mass swinging on the end of a spring), it is much easier to express the kinetic energy and the potential energy in terms of θ and u than x and y.

$$T(\theta, u, \dot{\theta}, \dot{u}) = \frac{1}{2}M((l+u)\dot{\theta})^2 + \frac{1}{2}M\dot{u}^2 \tag{32}$$

$$V(\theta, u) = Mg(l - (l+u)\cos\theta) + \frac{1}{2}Ku^2 \tag{33}$$

Lagrange's equations require the following derivatives (where $q_1 = \theta$ and $q_2 = u$):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{d}{dt}(M(l+u)^2\dot{\theta}) = M(2(l+u)\dot{u}\dot{\theta} + (l+u)^2\ddot{\theta}) \tag{34}$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial \theta} = 0 \tag{35}$$

$$\frac{\partial V}{\partial q_1} = \frac{\partial V}{\partial \theta} = Mg(l+u)\sin\theta \tag{36}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_2} = \frac{d}{dt}\frac{\partial T}{\partial \dot{u}} = \frac{d}{dt}M\dot{u} = M\ddot{u} \tag{37}$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial u} = M(l+u)\dot{\theta}^2 \tag{38}$$

$$\frac{\partial V}{\partial q_2} = \frac{\partial V}{\partial u} = Ku - Mg\cos\theta \tag{39}$$

Substituting these derivatives into Lagrange's equations for q_1 gives

$$M(2(l+u)\dot{u}\dot{\theta} + (l+u)^2\ddot{\theta}) + Mg(l+u)\sin\theta = 0 \ . \tag{40}$$

Substituting these derivatives into Lagrange's equations for q_2 gives

$$M\ddot{u} - M(l+u)\dot{\theta}^2 + Ku - Mg\cos\theta = 0 \ . \tag{41}$$

These last two equations represent a pair of coupled nonlinear ordinary differential equations describing the unconstrained motion of an elastic pendulum. Equation (**??**) represents moments (N.m) collocated with θ. Equation (**??**) represents forces (N) collocated with u. They may be written in matrix form as

$$\begin{bmatrix} M(l+u)^2 & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} -Mg(l+u)\sin\theta - 2M(l+u)\dot{u}\dot{\theta} \\ M(l+u)\dot{\theta}^2 - Ku + Mg\cos\theta \end{bmatrix} \tag{42}$$

or

$$\boldsymbol{M}(\boldsymbol{q}(t), t)\ \ddot{\boldsymbol{q}}(t) = \boldsymbol{Q}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) \tag{43}$$

As can be seen, the inertial terms (involving mass) are more complicated than just $M\ddot{r}$, and can involve position, velocity, and acceleration of the generalized coordinates. By carefully relating forces in Cartisian coordinates to those in generalized coordinates through free-body diagrams the same equations of motion may be derived, but doing so with Lagrange's equations is often more straight-forward once the kinetic and potential energies are derived.

10 Maple and Mathematica software to compute derivatives for Lagrange's equations

10.1 A forced spring-mass-damper oscillator

Consider a forced mass-spring-damper oscillator … basically the same as the elastic pendulum with just the $u(t)$ coordinate, (without the θ coordinate), but with some linear viscous damping in parallel with the spring, and external forcing $f(t)$, collocated with the displacement $u(t)$.

$$\begin{aligned} T(u,\dot{u}) &= \frac{1}{2}M\dot{u}(t)^2 \\ V(u) &= \frac{1}{2}Ku(t)^2 \\ p(u,\dot{u})\ \delta u &= -c\dot{u}(t)\ \delta u + f(t)\ \delta u \end{aligned}$$

where $(p\ \delta u)$ is the work of real non-conservative forces through a virtual displacement δu, in which the damping force is $c\dot{u}$ and the external driving force is $f(t)$.

10.1.1 Using Maple

The Maple software may be used to apply Lagrange's equations to these expressions for kinetic energy, T, potential energy V, and the work of non-conservative forces and external forcing, W. Here are the Maple commands:

```
> with(Physics):
> Setup(mathematicalnotation=true)

> v := diff(u(t),t);
                                                 d
                                            v := -- u(t)
                                                 dt
> T := (1/2) * M * v^2;
                                                       /d       \2
                                            T := 1/2 M |-- u(t)|
                                                       \dt      /

> V := (1/2) * K * u(t)^2;
                                                            2
                                            V := 1/2 K u(t)

> p := -C * diff(u(t),t)  + f(t) ;
                                                    /d       \
                                            p := -C |-- u(t)| + f(t)
                                                    \dt      /
```

The lines above setup Maple to invoke the desired functional notation, define the velocities `v` as the derivatives of time-dependent displacements `u(t)`, and represent the kinetic energy `T`, potential energy `V`, and the external and non-conservative forces `p`. The subsequent lines evaluate the derivatives and combine the derivatives into Lagrange's equations to give us the equations of motion.

```
> dTdv := diff(T, v);
                                                         /d      \
                                             dTdv := M |-- u(t)|
                                                         \dt     /
> ddt_dTdv := diff(dTdv,t);
                                                         / 2      \
                                                         |d       |
                                         ddt_dTdv := M |--- u(t)|
                                                         |  2     |
                                                         \dt      /
> dTdu := diff(T,u(t));
                                             dTdu := 0

> dVdu := diff(V,u(t));
                                             dVdu := K u(t)
> Q := p*diff(u(t),u(t));
                                                          /d      \
                                                Q := -C |-- u(t)| + f(t)
                                                          \dt     /

> EOM := ddt_dTdv - dTdu + dVdu = Q;

                         / 2      \
                         |d       |                 /d      \
              EOM := M |--- u(t)| + K u(t) = -C |-- u(t)| + f(t)
                         |  2     |                 \dt     /
                         \dt      /
> quit
```

…giving us the expected equation of motion (EOM) …

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t)$$

10.1.2 Using Mathematica

The same problem using Mathematica software is solved as follows:

```
v[t] = D[u[t], t]
u '[t]

T[t] = 1 / 2 * M * v[t]^2
(1/2) M u ' [t] 2

V[t] = 1 / 2 * K * u[t]^2
(1/2) K u[t] 2

Q[t] = - C D[u[t], t] + f[t]
f[t] - C u ' [t]

dTdv[t] = D[T[t], v[t]]
M u ' [t]

ddTdvdt[t] = D[dTdv[t], t]
M u" [t]

dVdu[t] = D[V[t], u[t]]
K u[t]

dTdu[t] = D[U[t], u[t]]
0

EOM = ddTdvdt[t] - dTdu[t] + dVdu[t] == Q[t]
K u[t] + M u"[t] == f[t] - C u'[t]
```

10.2 An elastic pendulum

For the elastic pendulum problem considered earlier, the first coordinate is $\theta(t)$, and is called `q(t)` in the Maple commands and `q[t]` in the Mathematica commands.. The second coordinate is $u(t)$ …called `u(t)` in Maple and `u[t]` in Mathematica.

10.2.1 Using Maple

```
> with(Physics):
> Setup(mathematicalnotation=true)

> v1 := diff(q(t),t);
                                                          d
                                                   v1 := -- q(t)
                                                          dt

> v2 := diff(u(t),t);
                                                          d
                                                   v2 := -- u(t)
                                                          dt
> T := (1/2) * M * (( l+u(t))*v1)^2  +  (1/2) * M * v2^2;

                                        2 /d      \2         /d      \2
                   T := 1/2 M (l + u(t))  |-- q(t)|  + 1/2 M |-- u(t)|
                                          \dt     /          \dt     /

> V := M * g * ( l - (l+u(t)) * cos(q(t)))  +  (1/2) * K * u(t)^2;

                                                                      2
                   V := M g (l - (l + u(t)) cos(q(t))) + 1/2 K u(t)

> EOMq := diff( diff(T,v1) , t) - diff(T,q(t)) + diff(V,q(t));

                                                                     / 2      \
                               /d      \ /d      \                2  |d       |
          EOMq := 2 M (l + u(t)) |-- q(t)| |-- u(t)| + M (l + u(t))  |--- q(t)|
                               \dt     / \dt     /                   |  2     |
                                                                     \dt      /
                   + M g (l + u(t)) sin(q(t))


> EOMu := diff( diff(T,v2) , t) - diff(T,u(t)) + diff(V,u(t));

                  / 2      \
                  |d       |                  /d      \2
         EOMu := M |--- u(t)| - M (l + u(t)) |-- q(t)|  - M g cos(q(t)) + K u(t)
                  |  2     |                  \dt     /
                  \dt      /
```

10.2.2 Using Mathematica

```
KE = 1/2 * M * ((l + u[t]) * D[q[t], t])^2 + 1/2 * M * D[u[t], t]^2
                              (1/2) M (l + u[t])^2 q'[t]^2 + (1/2) M u'[t]^2

PE = M * g * (l - (l + u[t]) * Cos[q[t]]) + 1/2 * K * u[t]^2
                              (1/2) K u[t]^2 + g M (l - Cos[q[t]] (l + u[t]))

Collect[EOMq = D[D[KE, D[q[t], t]], t] + D[PE - KE, q[t]], {q''[t], q'[t], q[t]}]
    g M Sin[q[t]] (l + u[t]) + 2 M (l + u[t]) q'[t] u'[t] + M (l + u[t])^2 q"[t]

Collect[EOMu = D[D[KE, D[u[t], t]], t] + D[PE - KE, u[t]], {u ''[t], u '[t], u[t]}]
                  -g M Cos[q[t]] - l M q'[t]^2 + u[t] (K - M q'[t]^2) + M u"[t]
```

The Maple and Mathematica expressions `EOMq` and `EOMu` are the same equations of motion as the previously-derived equations (**??**) and (**??**).

11 Matlab simulation of an elastic pendulum

To simulate the dynamic response of a system described by a set of ordinary differential equations, the system equations may be written in a state variable form, in which the highest-order derivatives of each equation ($\ddot{\theta}$ and $\ddot{u}$ in this example) are written in terms of the lower-order derivatives ($\dot{\theta}$, $\dot{u}$, θ, and u in this example). The set of lower-order derivatives is called the *state vector*. In this example, the equations of motion are re-expressed as

$$\ddot{\theta} = -(2\dot{u}\dot{\theta} + g\sin\theta)/(l+u) \tag{44}$$

$$\ddot{u} = (l+u)\dot{\theta}^2 - (K/M)u + g\cos\theta \tag{45}$$

In general, the state derivative $\dot{x} = [\dot{\theta}, \dot{u}, \ddot{\theta}, \ddot{u}]^{\mathsf{T}}$ is written as a function of the state, $x = [\theta, u, \dot{\theta}, \dot{u}]^{\mathsf{T}}$, $\dot{x}(t) = f(t, x)$. The equations of motion are written in this way in the following MATLAB simulation.

```
% elastic_pendulum_sim
% simulate the free response of an elastic pendulum

animate = 1;      % 1: animate the response
MakeMovie = 1;    % 1: make a movie, 0: don't.


% system constants are global variables ...

 global  l  g  K  M

 l =  1.0;                    % un-stretched length of the pendulum, m
 g =  9.8;                    % gravitational acceleration, m/s^2
 K = 25.6;                    % elastic spring constant, N/m
 M =  1.0;                    % pendulum mass, kg

 fprintf('spring-mass frequency = %f Hz\n', sqrt(K/M)/(2*pi))
 fprintf('pendulum frequency = %f Hz\n', sqrt(g/l)/(2*pi))

 t_final = 36;                           % simulation duration, s
 delta_t = 0.02;                         % time step increment, s
 points = floor(t_final/delta_t);        % number of data points
 t = [1:points]*delta_t;                 % time data, s

% initial conditions

 theta_o     = 0.0;           % initial rotation angle, rad
 u_o         = 1;             % initial spring stretch, m
 theta_dot_o = 0.5;           % initial rotation rate, rad/s
 u_dot_o     = 0.0;           % initial spring stretch rate, m/s

 x_o = [ theta_o ; u_o ; theta_dot_o ; u_dot_o ];      % initial state

 % solve the equations of motion
 [t,x,dxdt,TV] = ode4u ( 'elastic_pendulum_sys' , t , x_o );

 theta     = x(1,:);
 u         = x(2,:);
 theta_dot = x(3,:);
 u_dot     = x(4,:);

 T         = TV(1,:);          %  kinetic  energy
 V         = TV(2,:);          % potential energy

% convert from "q" coordinates to "r" coordinates ...

 x =  (l+u).*sin(theta);                                  % eq'n (1)
 y = -(l+u).*cos(theta);                                  % eq'n (2)
```

```
function [dxdt,TV] = elastic_pendulum_sys(t,x)
% [dxdt,TV] = elastic_pendulum_sys(t,x)
% system equations for an elastic pendulum
% compute the state derivative, dxdt, the kinetic energy, T, and the
% potential energy, V,  of an elastic pendulum.

% system constants are pre-defined global variables ...

 global  l g K M

% define the state vector

 theta     = x(1);              % rotation angle, rad
 u         = x(2);              % spring stretch, m
 theta_dot = x(3);              % rotation rate, rad/s
 u_dot     = x(4);              % spring stretch rate, m/s

% compute the acceleration of theta and u

 theta_ddot = -(2*u_dot*theta_dot + g*sin(theta)) / (l+u);          % eq'n (35)

 u_ddot     = (l+u)*theta_dot^2  -  (K/M)*u  +  g*cos(theta);       % eq'n (36)

% assemble the state derivative

 dxdt = [ theta_dot ; u_dot ; theta_ddot ; u_ddot ];

 TV = [ (1/2) * M * ((l+u).*theta_dot).^2 + (1/2) * M *  u_dot.^2 ; % K.E. (25)
         M*g*(l-(l+u).*cos(theta)) + (1/2) * K * u.^2 ];            % P.E. (26)


% ---------------------------------------------------------------- elastic_pendulum_sys
```

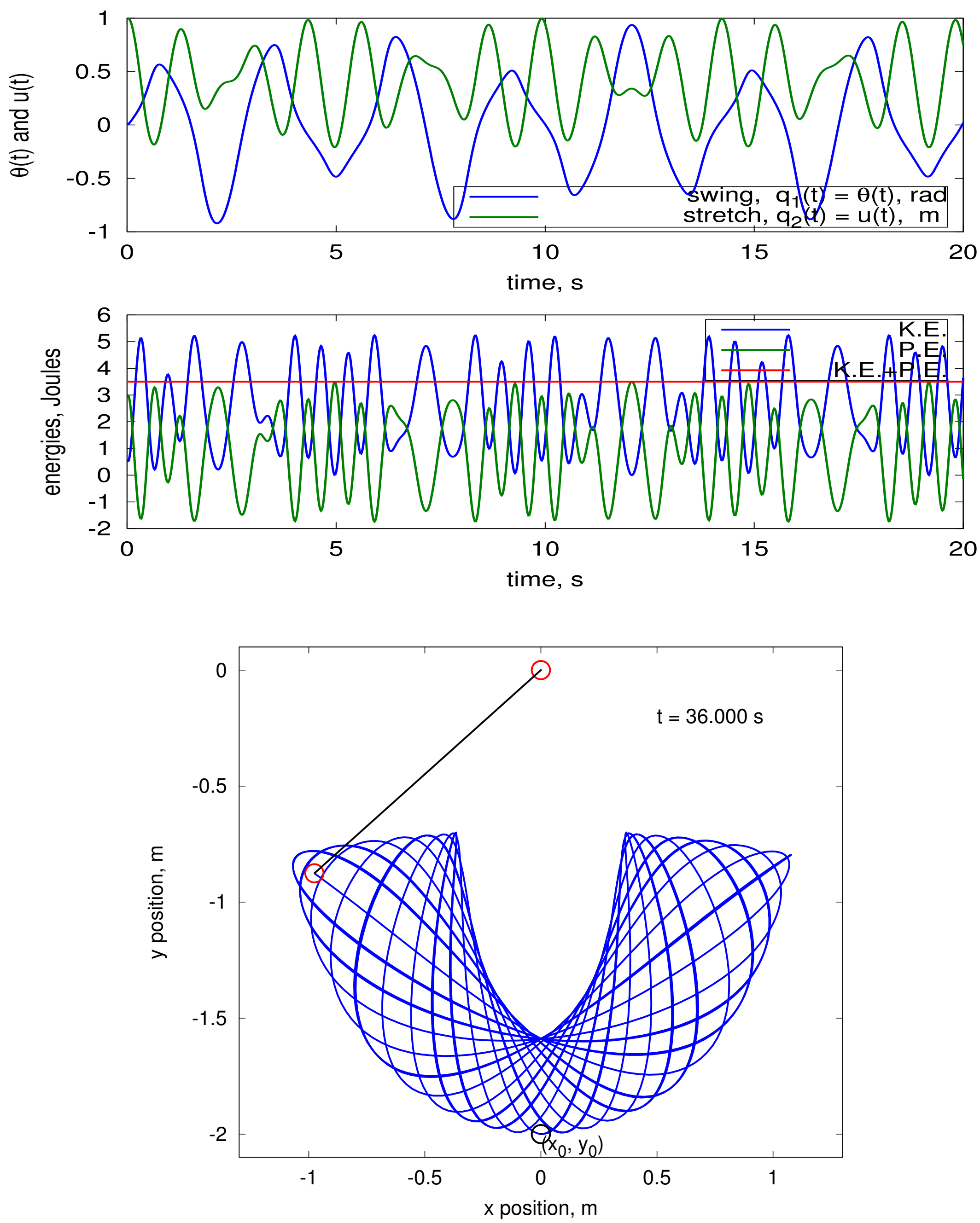


Figure 1. Free response of an elastic pendulum from an initial rotational velocity, $\dot{\theta}(0)$ of 0.5 rad/s for $l = 1.0$ m, $g = 9.8$ m/s^2, $K = 25.6$ N/m, and $M = 1$ kg. $T_k = 2\pi\sqrt{M/K} = 1.242$ s. $T_l = 2\pi\sqrt{l/g} = 2.007$ s. Even though $T_k/T_l \approx (\sqrt{5} - 1)/2$, the *least rational* number, the record repeats every 24 seconds. Note that in the absence of internal damping and external forcing, the sum of kinetic energy and potential energy is constant. Why is it that the internal potential energy becomes *negative* as compared to it's initial value? What is the initial configuration of the system $(\theta(0), u(0))$ in this example? About what values do $\theta(t)$ and $u(t)$ oscillate for $t > 0$? (spyrograph!) An animation of this motion is here: http://www.duke.edu/~hpgavin/cee541/elastic_pendulum_sim.mp4

12 Constraints

Suppose that in a dynamical system described by m generalized dynamic coordinates,

$$\boldsymbol{q}(t) = \left[\; q_1(t), \quad q_2(t), \quad \cdots , \quad q_m(t) \; \right]$$

the coordinate trajectories *must* satisfy additional (constraining) relationships $\mathbf{g}(\boldsymbol{q}) = 0$. For example, suppose the elastic pendulum must move along a particular line, $g(\theta(t), u(t)) = 0$, or that the pendulum can not move past a particular line, $g(\theta(t), u(t)) \leq 0$. If these constraints to the motion of the system can be written pureley in terms of the *positions* of the coordinates, then the system (including the differential equations and the constraints) is called a *holonomic system*.

There are a number of intriguing terms connected to constrained dynamical systems.

- A system with constraints that depend only on the *position* of the coordinates,

  $$\boldsymbol{g}(\boldsymbol{q}(t), t) = \mathbf{0}$$

  is *holonomic*.

- A system with constraints that depend on the *position and velocity* of the coordinates,

  $$\boldsymbol{g}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) = \mathbf{0}$$

  (in which $\boldsymbol{g}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$ can not be integrated into a holonomic form) is *nonholonomic*.

- A system with constraints that are independent of time,

  $$\boldsymbol{g}(\boldsymbol{q}) = \mathbf{0} \qquad \text{or} \qquad \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathbf{0}$$

  is *scleronomic*.

- A system with constraints that are explicitely dependent on time, as in the constraints listed under the first two definitions is *rheonomic*.

- A system with constraints that are linear in the velocities,

  $$\boldsymbol{g}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) = \boldsymbol{f}(\boldsymbol{q}(t)) \cdot \dot{\boldsymbol{q}}(t)$$

  is *pfafian*.

Any unconstrained system must be forced to adhere to a prescribed constraint. The required constraint forces $\boldsymbol{Q}^{\mathrm{C}}$ are collocated with the generalized coordinates $\boldsymbol{q}$, and the virtual work of the constraint forces acting through virtual displacements is zero[5]

$$\boldsymbol{Q}^{\mathrm{C}} \cdot \delta \boldsymbol{q} = 0.$$

Geometrically, the constraint forces are normal to the displacement variations.

[5]Lanczos, C., *The Variational Principles of Mechanics, 4th ed.*, Dover 1986

12.1 Holonomic systems

Consider a constraint for the elastic pendulum in which the pendulum must move along a curve $g(\theta(t), u(t)) = 0$. For example, consider the constraint

$$u(t) = l - l\theta^2(t) \qquad \Longleftrightarrow \qquad g(\theta(t), u(t)) = 1 - \theta^2(t) - u(t)/l \ .$$

Clearly the elastic pendulum will not follow the path $g = 0$ all by itself; it needs to be forced, somehow, to follow the prescribed trajectory. In a holonomic system (in which the constraints are on the coordinate positions) it can be helpful to think of a frictionless guide that enforces the dynamics to evolve along a partiular line, $g(\boldsymbol{q}) = 0$. The guide exerts constraint forces Q^{C} in a direction transverse to the guide, but not along the guide.

For relatively simple systems such as this, incorporating a constraint can be as simple as solving the constraint equation for one of the coordinates, for example, $u(t) = l - l\theta^2(t)$ and substituting that expression into the Lagrangian or the unconstrained equations of motion. With the substitution of $u(t) = l - l\theta^2(t)$ into the expressions for kinetic energy and potential energy, Lagranges equations can be written in terms of the remaining coordinate, θ. This is a perfectly acceptable means of incorporating holonomic constraints into an analysis. However, in general, a set of c constraint equations $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$ need not be re-arranged to express the position of c coordinates in terms of the remaining $(m - c)$ coordinates. Furthermore, reducing the dimension of the system by using the constraint equation to eliminate variables does not give us the forces required to enforce the constraints, and therefore misses an important aspect of the behavior of the system. So a more general approach can be followed in order to derive the equations of motion for constrained systems.

Recall that an admissible variation, $\delta\boldsymbol{q}$ *must* adhere to *all* constraints. For example, the solution to the constrained elastic pendulum, perterbued by $[\delta\theta, \delta u]$, must lie along the curve $1 - \theta^2 - u/l = 0$. In order for the variation $\delta\boldsymbol{q}$ to be admissible, the perturbed solution *must also* lie on the line of the constraint. A constraint evaluated at a perturbed solution $\boldsymbol{q} + \delta\boldsymbol{q}$, in general, can be approximated via a truncated Taylor series

$$\boldsymbol{g}(\boldsymbol{q} + \delta\boldsymbol{q}, t) = \boldsymbol{g}(\boldsymbol{q}, t) + \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right] \delta\boldsymbol{q} + \text{h.o.t.} \ .$$

The constraints at the perturbed solution are satisfied for infinitessimal pertubations as long as the variation $\delta\boldsymbol{q}$ is orthogonal to the gradient of $\boldsymbol{g}$ with respect to $\boldsymbol{q}$,

$$\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right] \ \delta\boldsymbol{q} = \boldsymbol{0} \ . \tag{46}$$

Because admissible variations $\delta\boldsymbol{q}$ are normal to $[\partial\boldsymbol{g}/\partial\boldsymbol{q}]$ and the constraint force $\boldsymbol{Q}^{\mathrm{C}}$ is normal to $\delta\boldsymbol{q}$, the constraint force must lie within the subspace spanned by $[\partial\boldsymbol{g}/\partial\boldsymbol{q}]$,

$$\boldsymbol{Q}^{\mathrm{C}} = \boldsymbol{\lambda}^{\mathsf{T}} \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right] \ .$$

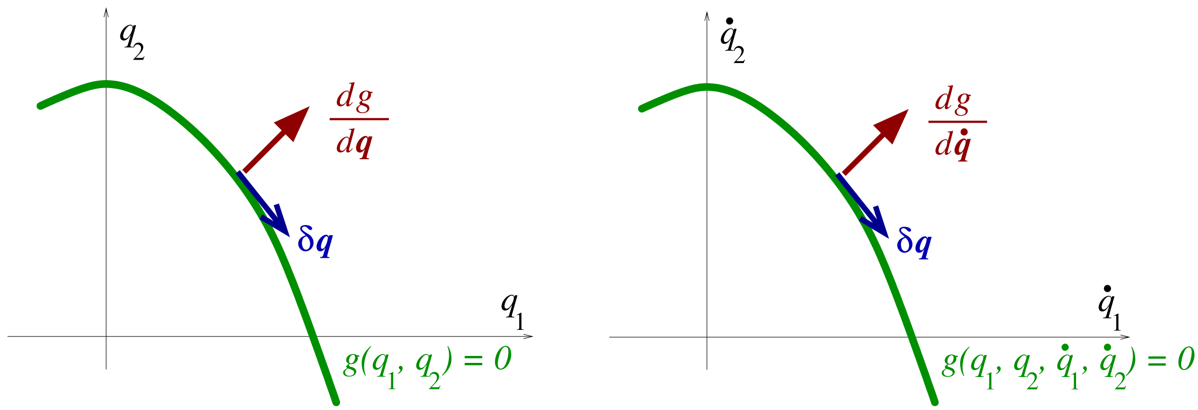


Figure 2. Admissibile variations $\delta\boldsymbol{q}$ for holonomic (left) and nonholonomic (right) systems. In holonomic systems $\delta\boldsymbol{q}$ adheres to the constraint $g(\boldsymbol{q}) = 0$ since $\delta\boldsymbol{q}$ is normal to the gradient of g with respect to $\boldsymbol{q}$. In nonholonomic systems $\delta\boldsymbol{q}$ need only be normal to $\partial g/\partial\dot{\boldsymbol{q}}$.

The constraint forces Q_j^{C} may now be added into Lagrange's equations as the forces required to adhere the motion to the constraints $g_i(\boldsymbol{q}) = 0$.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - \sum_i \lambda_i \frac{\partial g_i}{\partial q_j} - Q_j = 0} \tag{47}$$

in which

$$\sum_i \lambda_i \frac{\partial g_i}{\partial q_j}$$

is the generalized force on coordinate q_j applied through the system of constraints,

$$\boldsymbol{g}(\boldsymbol{q}, t) = \mathbf{0} \ . \tag{48}$$

These forces are precisely the actions that enforce the constraints. Equations (**??**) and (**??**) uniquely describe the dynamics of the system. In numerical simulations these two systems of equations are solved simultaneously for the accelerations, $\ddot{q}(t)$, and the Lagrange multipliers, $\boldsymbol{\lambda}(t)$, from which the constraint forces, $\boldsymbol{Q}^{\mathrm{C}}(t)$, can be found.

A system with m coordinates and c holonomic constraints can be reduced to a system of $m - c$ equations of motion by substituting constraint equations into the equations of motion and eliminating coordinates. Alternatively, the Lagrange multipliers can be treated as dynamic variables, and the constraint equations can be enforced in the acceleration form, by differentiating them twice with respect to time. In either case, the initial conditions must be admissible, and numerical integration errors that would lead to dynamic responses that violate $\boldsymbol{g}(\boldsymbol{q}, t) = \mathbf{0}$ must be numerically corrected.

Let's apply this to the elastic pendulum, constrained to move along a path

$$u(t) = l - l\theta^2(t) \qquad \Longleftrightarrow \qquad g(\theta(t), u(t)) = 1 - \theta^2(t) - u(t)/l \ . \tag{49}$$

The derivitives of T and V are as they were. The new derivitives required to model the actions of the constraints are

$$\frac{\partial g}{\partial q_1} = \frac{\partial g}{\partial \theta} = -2\theta \tag{50}$$

$$\frac{\partial g}{\partial q_2} = \frac{\partial g}{\partial u} = -1/l \tag{51}$$

Note that these derivites are related to, but are not the same as, the gradient of $g(\boldsymbol{q})$. Equation (**??**) now includes a new term for the constraint *moment* in the θ direction, $+\lambda(2\theta)$, and the new term for the constraint *force* in the u direction in equation (**??**) is $+\lambda(1/l)$. (This problem has one constraint, and therefore one Lagrange multiplier, but two coordinates, and therefore two generalized constraint forces.) The Lagrange multiplier in this problem has units of *moment* (N.m). There are no other non-conservative forces Q_j applied to this system. The problem now involves three equations (two equations of motion and the constraint) and three unknowns θ, u (or their derivitives) and λ. In principle, a solution can be found. In solving the equations of motion numerically, as a system of first-order o.d.e's, we solve for the highest order derivitives in each equation in terms of the lower-order derivitives. In this case the highest order derivitives are $\ddot{\theta}$ and $\ddot{u}$. In the case of constrained dynamics, we also need to solve for the Lagrange multiplier(s), λ. The constraint equation(s) give(s) the additional equation(s) to do so. But in this problem the constraint equation is in terms of positions θ and u. However, by differentiating the constraint we can put this in terms of accelerations $\ddot{\theta}$ and $\ddot{u}$. So doing, with some rearrangement,

$$2l\theta\ddot{\theta} + \ddot{u} = -2l\dot{\theta}^2 \ . \tag{52}$$

Now we have three equations and three unknowns for $\ddot{\theta}$, $\ddot{u}$, and λ.

$$M(l+u)^2\ddot{\theta} + 2\theta\lambda = -Mg(l+u)\sin\theta - 2M(l+u)\dot{u}\dot{\theta} \tag{53}$$

$$M\ddot{u} + \frac{1}{l}\lambda = M(l+u)\dot{\theta}^2 - Ku + Mg\cos\theta \tag{54}$$

$$2l\theta\ddot{\theta} + \ddot{u} = -2l\dot{\theta}^2 \tag{55}$$

The constraint *moment* in the θ direction is $\lambda(2\theta)$ and the constraint *force* in the u direction is λ/l. The constraint forces acting along $g(\boldsymbol{q}) = 0$ are λ/l in the u direction and $2\theta\lambda/(l+u)$ perpendicular to the u direction. The three equations may be written in matrix form ...

$$\begin{bmatrix} M(l+u)^2 & 0 & 2\theta \\ 0 & M & 1/l \\ 2\theta & 1/l & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} -Mg(l+u)\sin\theta - 2M(l+u)\dot{u}\dot{\theta} \\ M(l+u)\dot{\theta}^2 - Ku + Mg\cos\theta \\ -2\dot{\theta}^2 \end{bmatrix} \tag{56}$$

Note that the initial condition of the system must also adhere to the constraints,

$$l\theta_0^2 + u_0 = l \tag{57}$$

$$2l\theta_0\dot{\theta}_0 + \dot{u}_0 = 0 \tag{58}$$

These equations can be integrated numerically as was shown in the previous MATLAB example, except for the fact that in the presence of constraints, the accelerations and Lagrange multiplier need to be evaluated as a solution of three equations with three unknowns.

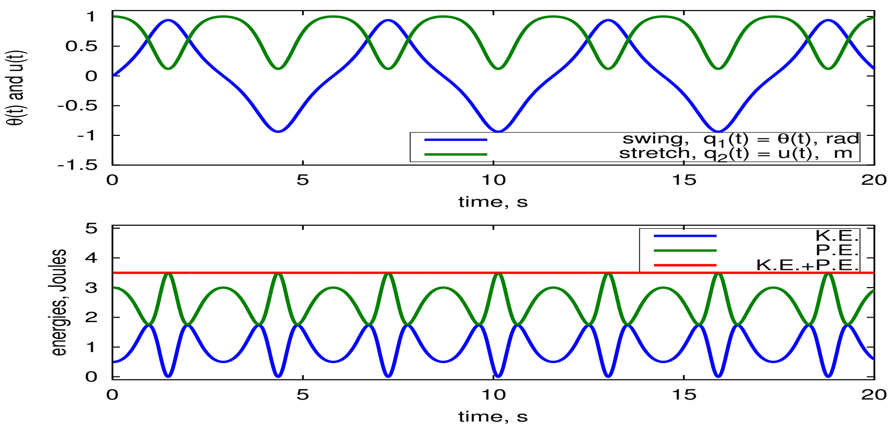


Figure 3. Free response of an undamped elastic pendulum from an initial condition $u_o = l$ and $\dot{\theta}_o = 1$ constrained to move along the arc $u = l - l\theta^2$. The motion is periodic and the total energy is conserved exactly.

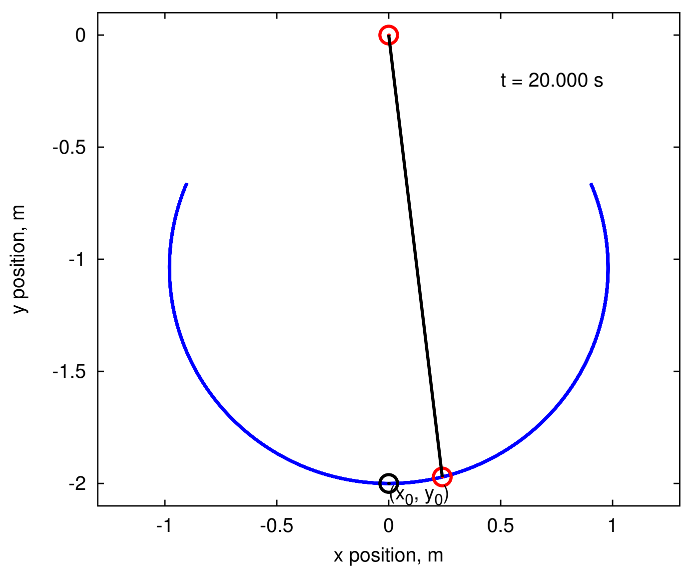


Figure 4. The constrained motion of the pendulum is seen to satisfy the equation $u = l - l\theta^2$ for $l = 1$ m. An animation of this motion and the dynamic constraint forces is here: http://www.duke.edu/~hpgavin/cee541/elastic_pendulum_H_sim.mp4

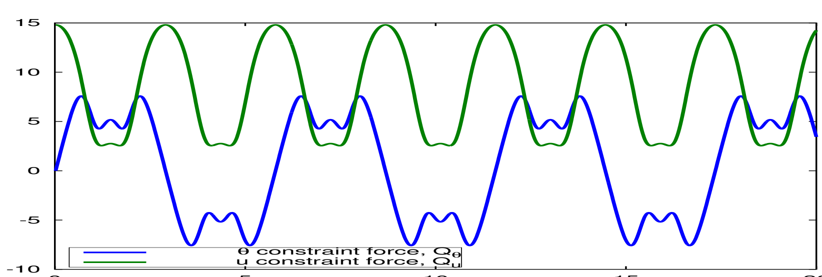


Figure 5. The constraint *moment* in the θ direction (blue) and *force* in the u direction (green) required to enforce the constraint $u(t) = l - l\theta^2(t)$.

12.2 Nonholonomic systems

A nonholonomic system has internal constraint forces $Q_j^{\rm C}$ which adhere the responses to a non-integrable relationship involving the positions and velocities of the coordinates,

$$\boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{0} \ . \tag{59}$$

The constraint forces $Q_j^{\rm C}$ are normal to the constraint surface $\boldsymbol{g}(q, \dot{q}, t) = \boldsymbol{0}$. As always, any admissible variations must satisfy the constraints

$$\boldsymbol{g}(\boldsymbol{q} + \delta\boldsymbol{q}, \dot{\boldsymbol{q}} + \delta\dot{\boldsymbol{q}}, t) = \boldsymbol{g}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right] \delta\boldsymbol{q} + \left[\frac{\partial \boldsymbol{g}}{\partial \dot{\boldsymbol{q}}}\right] \delta\dot{\boldsymbol{q}} + \text{h.o.t.} = \boldsymbol{0}$$

So, for infinitessimal variations,

$$\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \ , \ \frac{\partial \boldsymbol{g}}{\partial \dot{\boldsymbol{q}}}\right] \ \cdot \ \left[\begin{array}{c} \delta\boldsymbol{q} \\ \delta\dot{\boldsymbol{q}} \end{array}\right] = \boldsymbol{0} \ .$$

It may be shown[6] that this condition is equivalent to

$$\left[\frac{\partial \boldsymbol{g}}{\partial \dot{\boldsymbol{q}}}\right] \ \cdot \ \delta\boldsymbol{q} = \boldsymbol{0} \tag{60}$$

which is the constraint on the displacement variations for nonholonomic systems.

Since admissible variations $\delta\boldsymbol{q}$ are normal to $[\partial\boldsymbol{g}/\partial\dot{\boldsymbol{q}}]$, the constraint forces must therefore lie within $[\partial\boldsymbol{g}/\partial\dot{\boldsymbol{q}}]$,

$$\boldsymbol{Q}^{\rm C} = \boldsymbol{\lambda}^{\sf T} \left[\frac{\partial \boldsymbol{g}(\boldsymbol{q}, \dot{q})}{\partial \dot{\boldsymbol{q}}}\right] \ .$$

Including these constraint forces into Lagrange's equations gives the nonholonomic form of Lagrange's equations,[7] [8]

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - \sum_i \lambda_i \frac{\partial g_i}{\partial \dot{q}_j} - Q_j = 0 \ ,} \tag{61}$$

in which

$$\sum_i \lambda_i \frac{\partial g_i}{\partial \dot{q}_j}$$

is the generalized force on coordinate q_j applied through the system of constraints $\boldsymbol{g} = 0$. Equations (**??**) and (**??**) uniquely prescribe the constraint forces (Lagrange multipliers $\boldsymbol{\lambda}$) and the dynamics of a system constrained by a function of velocity and displacement.

---

[6]P.S. Harvey, Jr., *Rolling Isolation Systems: Modeling, Analysis, and Assessment,* Ph.D. dissertation, Duke Univ, 2013

[7]M.R. Flannery, "The enigma of nonholonomic constraints," *Am. J. Physics* 73(3) 265-272 (2005)

[8]M.R. Flannery, "d'Alembert-Lagrange analytical dynamics for nonholonomic systems," *J. Mathematical Physics* 52 032705 (2011)

As an example, suppose the dynamics of the elastic pendulum are controlled by some internal forces so that the direction of motion $\arctan(dy/dx)$ is actively steered to an angle ϕ that is proportional to the stretch in the spring, $\phi = -bu/l$. The velocity transvese to the steered angle must be zero, giving the constraint,

$$\frac{\dot u}{l\dot\theta} = \tan(\theta + bu/l) \qquad \Longleftrightarrow \qquad g(\theta, u, \dot\theta, \dot u) = \dot u \cos(\theta + bu/l) - l\dot\theta \sin(\theta + bu/l) \tag{62}$$

The additional derivitives required for the nonholonomic form of Lagranges equations are

$$\frac{\partial g}{\partial \dot q_1} = \frac{\partial g}{\partial \dot\theta} \;=\; -l\ \sin(\theta + bu/l) \tag{63}$$

$$\frac{\partial g}{\partial \dot q_2} = \frac{\partial g}{\partial \dot u} \;=\; \cos(\theta + bu/l) \tag{64}$$

Equation (**??**) now includes a new term for the constraint *moment* in the θ direction, $-\lambda(ls)$, and the new term for the constraint *force* in the u direction in equation (**??**) is $+\lambda(c)$, where $s = \sin(\theta + bu/l)$ and $c = \cos(\theta + bu/l)$. Note that in this case, the constraint forces are dependent upon the position of the pendulum (θ and u), but not on the velocities. The constraint equation along with the equations of motion uniquely define the solution $\theta(t)$, $u(t)$, and $\lambda(t)$. Since, for numerical simulation purposes, we are interested in solving for the accelerations, $\ddot\theta(t)$ and $\ddot u(t)$, we can differentiate the constraint equation to transform it into a form that is linear in the accelerations,

$$\dot g = -b\dot u^2 s/l + \ddot u c - l\dot\theta^2 c - l\ddot\theta s - \dot\theta\dot u(s + bc) \tag{65}$$

Our three equations are now,

$$\begin{aligned} M(l+u)^2\ddot\theta - (ls)\lambda &= -Mg(l+u)\sin\theta - 2M(l+u)\dot u\dot\theta & (66)\\ M\ddot u + (c)\lambda &= M(l+u)\dot\theta^2 - Ku + Mg\cos\theta & (67)\\ -(ls)\ddot\theta + (c)\ddot u &= l\dot\theta^2 c + \dot\theta\dot u(s+bc) + b\dot u^2 s/l & (68) \end{aligned}$$

with an initial condition that also satisfies the constraint, $\dot u_0 = l\dot\theta_0 \tan(\theta + bu/l)$. The three equations may be written in matrix form . . .

$$\begin{bmatrix} M(l+u)^2 & 0 & -ls \\ 0 & M & c \\ -ls & c & 0 \end{bmatrix} \begin{bmatrix} \ddot\theta \\ \ddot u \\ \lambda \end{bmatrix} = \begin{bmatrix} -Mg(l+u)\sin\theta - 2M(l+u)\dot u\dot\theta \\ M(l+u)\dot\theta^2 - Ku + Mg\cos\theta \\ l\dot\theta^2 c + \dot\theta\dot u(s+bc) + b\dot u^2 s/l \end{bmatrix} \tag{69}$$

Note that the upper-left 2×2 blocks in the matrices of equations (**??**) and (**??**) are the same as the corresponding matrix $\boldsymbol{M}$ in the unconstrained system (**??**). The same is true for the first two elements of the right-hand-side vectors of equations (**??**) and (**??**) and the right-hand-side vector $\boldsymbol{Q}$ of the unconstrained system (**??**). Furthermore, if the differentiated constraint equations (**??**) and (**??**) are written as

$$\hat{\boldsymbol{g}} = \boldsymbol{A}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\ \ddot{\boldsymbol{q}}(t) - \boldsymbol{b}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)$$

then both equations (**??**) and (**??**) may be written

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{A}^\mathsf{T} \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{b} \end{bmatrix} . \tag{70}$$

This expression can be obtained directly from Gauss's principle of least constraint, which provides an appealing interpretation of constrained dynamical systems.

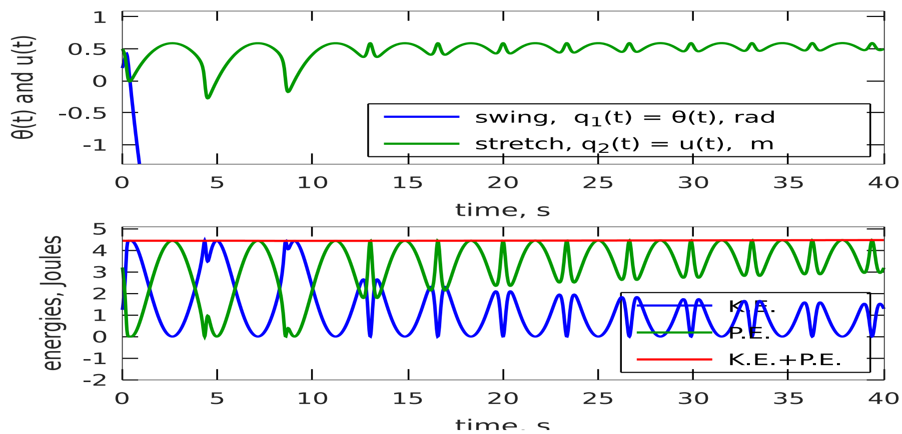


Figure 6. Free response of an undamped elastic pendulum from an initial condition $u_o = l/2$, $\theta_o = 0.2$ rad, $\dot{\theta}_o = 1$ rad/s, with trajectory constrained by $l\dot{\theta}/\dot{u} = \tan(\theta + bu/l)$, with $b = 5$. The motion is not periodic but total energy is conserved exactly and the constraint is satisfied at all times.

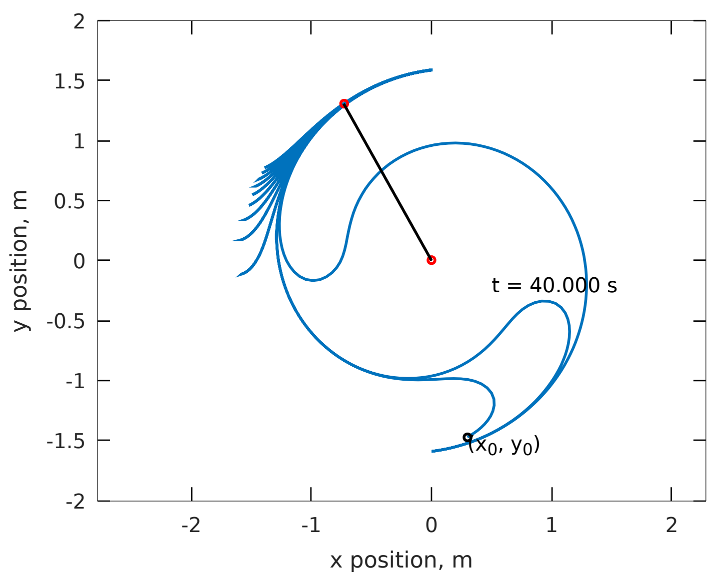


Figure 7. The constrained motion of the pendulum does not follow a fixed relation between u and θ — the constraint is nonholonomic. An animation of this motion and the dynamic constraint forces is here: http://www.duke.edu/~hpgavin/cee541/elastic_pendulum_NH_sim.mp4 (ying & yang)

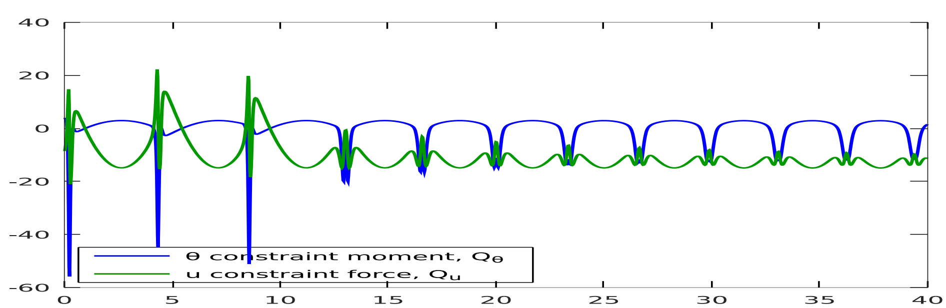


Figure 8. The constraint *moment* in the θ direction (blue) and the *force* u direction (green) directions required to enforce the constraint $l\dot{\theta}/\dot{u} = \tan(\theta + bu/l)$.

13 Gauss's Principle of Least Constraint

Consider the equations of motion of the unconstrained system written as equation (**??**),

$$\boldsymbol{M}(\boldsymbol{q},t)\ \boldsymbol{a} = \boldsymbol{Q}(\boldsymbol{q},\dot{\boldsymbol{q}},t)\ .$$

The matrix $\boldsymbol{M}$ is assumed to be symmetric and positive definite. Define the accelerations of the unconstrained system as

$$\boldsymbol{a} \equiv \boldsymbol{M}^{-1}\boldsymbol{Q}$$

and write the differentiated constraints in terms of the constrained accelerations $\ddot{\boldsymbol{q}}$ as

$$\boldsymbol{A}(\boldsymbol{q},\dot{\boldsymbol{q}},t)\ \ddot{\boldsymbol{q}} = \boldsymbol{b}(\boldsymbol{q},\dot{\boldsymbol{q}},t).$$

The constrained system requires additional actions $\boldsymbol{Q}^{\mathrm{C}}$ to enforce the constraint, so the equations of motion of the constrained system are

$$\boldsymbol{M}(\boldsymbol{q},t)\ \ddot{\boldsymbol{q}} = \boldsymbol{Q}(\boldsymbol{q},\dot{\boldsymbol{q}},t) + \boldsymbol{Q}^{\mathrm{C}}(\boldsymbol{q},\dot{\boldsymbol{q}},t)\ .$$

Lastly, define a quadratic form of the accelerations,

$$\mathcal{G} = \frac{1}{2}\ (\ddot{\boldsymbol{q}} - \boldsymbol{a})^{\mathsf{T}}\ \boldsymbol{M}\ (\ddot{\boldsymbol{q}} - \boldsymbol{a})$$

Gauss's Principle of Least Constraint[9][10] states that the accelerations of the constrained system $\ddot{q}$ minimize $\mathcal{G}$ subject to the constraints $\boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b}$. The naturally-constrained accelerations $\ddot{\boldsymbol{q}}$ are as close as possible to the unconstrained accelerations $\boldsymbol{a}$ *in a least-squares sense* and *at each instant in time* while satisfying the constraint $\boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b}$.

To minimize a quadratic objective subject to a linear constraint the objective can be augmented via Lagrange multipliers $\boldsymbol{\lambda}$,

$$\begin{aligned} \mathcal{G}_A &= \frac{1}{2}\ (\ddot{\boldsymbol{q}} - \boldsymbol{a})^{\mathsf{T}}\ \boldsymbol{M}\ (\ddot{\boldsymbol{q}} - \boldsymbol{a}) + \boldsymbol{\lambda}^{\mathsf{T}}(\boldsymbol{A}\ \ddot{\boldsymbol{q}} - \boldsymbol{b}) \qquad (71)\\ &= \frac{1}{2}\ddot{\boldsymbol{q}}^{\mathsf{T}}\boldsymbol{M}\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{q}}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{a} + \frac{1}{2}\boldsymbol{a}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{a} + \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{A}\ddot{\boldsymbol{q}} - \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{b} \end{aligned}$$

minimized with respect to $\ddot{\boldsymbol{q}}$,

$$\left(\frac{\partial \mathcal{G}_A}{\partial \ddot{\boldsymbol{q}}}\right)^{\mathsf{T}} = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{M}\ddot{q} = \boldsymbol{Q} - \boldsymbol{A}^{\mathsf{T}}\boldsymbol{\lambda} \qquad (72)$$

and maximized with respect to $\boldsymbol{\lambda}$,

$$\left(\frac{\partial \mathcal{G}_A}{\partial \boldsymbol{\lambda}}\right)^{\mathsf{T}} = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{A}\ddot{\boldsymbol{q}} = \boldsymbol{b} \qquad (73)$$

These conditions are met in equation (**??**), as derived earlier for both holonomic and non-holonomic systems,

$$\left[\begin{array}{cc} \boldsymbol{M} & \boldsymbol{A}^{\mathsf{T}} \\ \boldsymbol{A} & \boldsymbol{0} \end{array}\right] \left[\begin{array}{c} \ddot{q} \\ \boldsymbol{\lambda} \end{array}\right] = \left[\begin{array}{c} \boldsymbol{Q} \\ \boldsymbol{b} \end{array}\right]\ .$$

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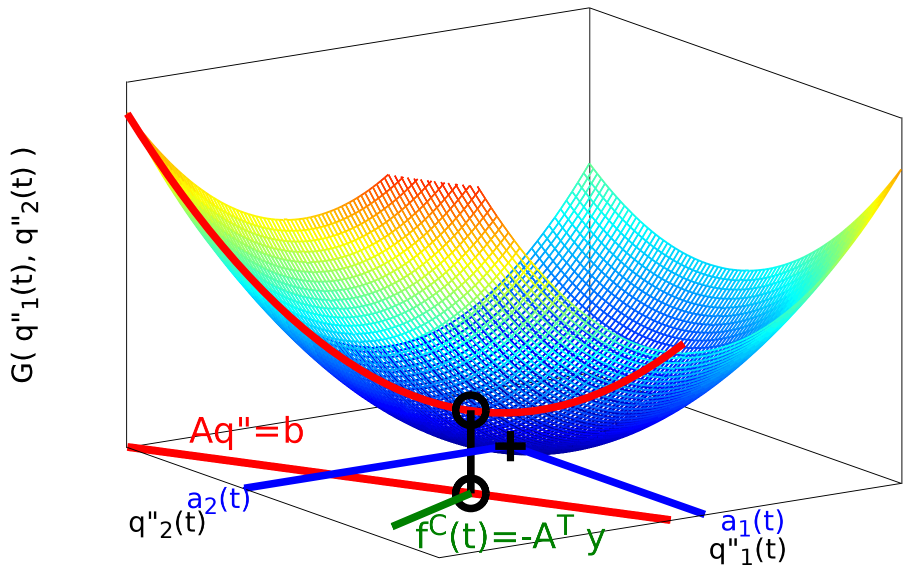
[9] Hofrath and Gauss, C.F., "Uber ein Neues Allgemeines Grundgesatz der Mechanik,", *J. Reine Angewandte Mathematik,* 4:232-235. (1829)

[10] Udwadia, F.E. and Kalaba, R.E., "A New Perspective on Constrained Motion," *Proc. Mathematical and Physical Sciences,* 439(1906):407-410. (1992).



The force of the constraints is now apparent, $\boldsymbol{Q}^{\mathrm{C}} = -\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\lambda}$. While equation (**??**) is sufficient for analyzing the behavior and simulating the response of constrained dynamical systems, it is also useful in building an understanding of their nature. Solving the first of the equations for $\ddot{q}$,

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}\boldsymbol{Q} - \boldsymbol{M}^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\lambda}$$

and inserting this expression into the constraint equation,

$$\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{Q} - \boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\lambda} = \boldsymbol{b}$$

the Lagrange multipliers may be found,

$$\boldsymbol{\lambda} = -(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathsf{T}})^{-1}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{Q}) \ ,$$

and substituting $\boldsymbol{a} = \boldsymbol{M}^{-1}\boldsymbol{Q}$, the constraint force, $\boldsymbol{Q}^{\mathrm{C}} = -\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\lambda}$, can be found as well,

$$\boldsymbol{Q}^{\mathrm{C}} = -\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathsf{T}})^{-1}(\boldsymbol{A}\boldsymbol{a} - \boldsymbol{b}) \ .$$

or

$$\boldsymbol{Q}^{\mathrm{C}} = \boldsymbol{K}(\boldsymbol{A}\boldsymbol{a} - \boldsymbol{b}) \ .$$

The difference $(\boldsymbol{A}\boldsymbol{a} - \boldsymbol{b})$ is the amount of the constraint violation associated with the unconstrained accelerations $\boldsymbol{a}$. If the unconstrained accelerations satisfy the constraint, then the constraint force is zero. The matrix $\boldsymbol{K} = -\boldsymbol{A}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^{\mathsf{T}})^{-1}$ is a kind of natural nonlinear feedback gain matrix that relates the constraint violation of the unconstrained accelerations $(\boldsymbol{A}\boldsymbol{a} - \boldsymbol{b})$ to the constriant force required to bring the constraint to zero, $\boldsymbol{Q}^{\mathrm{C}}$.

The constrained minimum of the *quadratic* objective $\mathcal{G}$ gives the correct equations of motion.