1 Strain Energy in Elastic Solids

Consider an elastic object in equilibrium subjected to static forces and displacements.

\[ W = \int_0^R F(R') \, dR' + \int_S \int_0^r f(r') \, dr' \, dS \]  

where

- the forces \( F \) and \( f \) depend on displacements \( R \) and \( r \)
- \( R' \) and \( r' \) are dummy variables of integration
1.2 Internal Strain Energy

Strain energy is a kind of potential energy arising from stress and deformation of elastic solids. In nonlinear elastic solids, the strain energy of stresses increasing from 0 to $\sigma$ and working through strains from 0 to $\epsilon$ is

$$ U = \int_V \int_0^\epsilon \sigma \cdot d\epsilon' \, dV \tag{2} $$

where

- $V$ is the volume of the solid
- $\sigma = \{ \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz} \}$
- $\epsilon = \{ \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz} \}$
- $\epsilon'$ is a dummy variable of integration

1.3 The Principle of Real Work

In an elastic solid, the work of external forces, $W$, is stored entirely as elastic strain energy, $U$, within the solid.

$$ U = W \tag{3} $$

In linear elastic solids:

- Displacements and rotations increase linearly with forces and moments
  $$ F = kD \ldots \text{and} \ldots M = \kappa \Theta $$

- The work of an external force $F$ acting through a displacement $D$ on the solid is
  $$ W = \frac{1}{2} FD = \frac{1}{2} kD^2 = U $$

- The work of an external moment $M$ acting through a rotation $\Theta$ on the solid is
  $$ W = \frac{1}{2} M\Theta = \frac{1}{2} \kappa \Theta^2 = U $$
1.4 Strain energy in slender structural elements

In slender structural elements (bars, beams, or shafts) the internal forces, moments, shears, and torques can vary along the length of each element; so do the displacements and rotations.

The strain energy of spatially-varying internal forces $F(x)$ acting through spatially-varying internal displacements $D(x)$ within linear elastic solids is

$$U = \frac{1}{2} \int F(x) \cdot \frac{dD(x)}{dx} \, dx = \frac{1}{2} \int F(x) D'(x) \, dx$$  \hfill (4)

The strain energy of spatially-varying internal moments $M(x)$ acting through spatially-varying internal rotations $\Theta(x)$ within linear elastic solids is

$$U = \frac{1}{2} \int M(x) \cdot \frac{d\Theta(x)}{dx} \, dx = \frac{1}{2} \int M(x) \Theta'(x) \, dx$$  \hfill (5)

In slender structural elements, the relation between internal forces and moments $F$ and $M$, and internal displacements and rotations $v$ and $\phi$, depend on the kind of loading.

- Axial $N_x(x) = E(x)A(x)u'(x)$
- Bending $M_z(x) = E(x)I(x)v''(x)$
- Shear $V_y(x) = G(x)A_4(x)v'_y(x)$
- Torsion $T_x(x) = G(x)J_1(x)\phi'(x)$
- Geometric $N_x(x) = E(x)A(x) (u'(x) + (1/2)(v'(x))^2)$
Inserting these expressions into the general expressions for internal strain energy above,

<table>
<thead>
<tr>
<th></th>
<th>“force” deformation</th>
<th>strain energy (U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>$N_x(x)$ $u'(x)$</td>
<td>$\frac{1}{2} \int N_x(x) u'(x) dx$</td>
</tr>
<tr>
<td>Bending</td>
<td>$M_z(x)$ $v''(x)$</td>
<td>$\frac{1}{2} \int M_z(x) v''(x) dx$</td>
</tr>
<tr>
<td>Shear</td>
<td>$V_y(x)$ $v_s'(x)$</td>
<td>$\frac{1}{2} \int V_y(x) v_s'(x) dx$</td>
</tr>
<tr>
<td>Torsion</td>
<td>$T_x(x)$ $\phi'(x)$</td>
<td>$\frac{1}{2} \int T_x(x) \phi'(x) dx$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$N_x(x)$ $v'(x)$</td>
<td>$\frac{1}{2} \int N_x(x)(v'(x))^2 dx$</td>
</tr>
</tbody>
</table>

$E(x)$ is Young’s modulus
$G(x)$ is the shear modulus

$A(x)$ is the cross sectional area of a bar
$I(x)$ is the bending moment of inertia of a beam
$A(x)/\alpha$ is the effective shear area of a beam
$J(x)$ is the torsional moment of inertia of a shaft

$N_x(x)$ is the axial force within a bar
$M_z(x)$ is the bending moment within a beam
$V_y(x)$ is the shear force within a beam
$T_x(x)$ is the torque within a shaft

$u(x)$ is the axial displacement along the bar
$u'(x)$ is the axial displacement per unit length, $du(x)/dx$, the axial strain

$v(x)$ is the transverse bending displacement of the beam
$v'(x)$ is the slope of the displacement of the beam
$v''(x)$ is the rotation per unit length, the curvature, approximately $d^2v(x)/dx^2$

$v_s(x)$ is the transverse shear displacement of the beam
$v_s'(x)$ is the transverse shear displacement per unit length, $dv_s(x)/dx$

$\phi(x)$ is the torsional rotation (twist) of the shaft
$\phi'(x)$ is the torsional rotation per unit length, $d\phi(x)/dx$
2 Virtual Work in Elastic Solids — The Principle of Virtual Displacements

Now consider a second set of loads, \( \delta F, \delta f \), in equilibrium and applied subsequently to the loads \( F \) and \( f \). The loads \( \delta F \) and \( \delta f \) give rise to displacements \( \delta R \) and \( \delta r \) collocated with forces \( F \) and \( f \), and internal stresses \( \delta \sigma \) and strains \( \delta \epsilon \). The displacements \( \delta R \) and \( \delta r \) are consistent with the support conditions of the system. In other words, the displacements \( \delta R \) and \( \delta r \) are admissible with respect to kinematic constraints.

Call \( \delta F \) and \( \delta f \) a set of “virtual” forces in equilibrium.

Call \( \delta R \) and \( \delta r \) a set of “virtual” admissible displacements, collocated with forces \( F \) and \( f \).

Forces \( F \) and \( f \) are held constant as loads \( \delta F \) and \( \delta f \) are applied. Stresses \( \sigma \), in equilibrium with forces \( F \) and \( f \), are therefore also held constant as loads \( \delta F \) and \( \delta f \) are applied. Forces \( F \) and \( f \) do not increase with displacements \( \delta R \) and \( \delta r \). Strains \( \delta \epsilon \) increase as loads \( \delta F \) and \( \delta f \) are applied.

The principle of virtual displacements states that the virtual external work of real external forces \( (f \) and \( F') \) moving through collocated virtual displacements \( (\delta r \) and \( \delta R ) \) equals the internal virtual work of real stresses \( (\sigma ) \) in equilibrium with real forces \( (f \) and \( F) \) with the virtual strains \( (\delta \epsilon ) \) compatible with the virtual displacements \( (\delta r \) and \( \delta R ) \), integrated over the volume of the solid.

\[
\delta W_I = \delta W_E \\
\int_V \sigma \cdot \delta \epsilon \, dV = \int_S f \cdot \delta r \, dS + \sum F_i \cdot \delta R_i
\]  

(6)
3 The Principle of Virtual Displacements for Dynamic Loading

The principle of virtual displacements applies to both static and dynamic forces. Elastic forces $k(x)\dot{r}(t, x)$ are present in structural systems responding to static or dynamic loads. Forces arising from dynamic effects only include viscous damping forces $c(x)\dot{r}(t, x)$ and inertial forces $m(x)\ddot{r}(t, x)$. Elastic forces, viscous damping forces, and inertial forces can be developed within slender structural elements in response to axial, bending, shear, and torsional deformations.

<table>
<thead>
<tr>
<th></th>
<th>real force deformation</th>
<th>virtual deformation</th>
<th>internal virtual work ($\delta W_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial</td>
<td>$N_x(t, x)$</td>
<td>$\delta u'(t, x)$</td>
<td>$\int N_x(t, x) \delta u'(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int E A(x) \dot{u}'(t, x) , dx$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \eta A(x) \ddot{u}(t, x) , dx$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \rho A(x) \dot{u}(t, x) , dx$</td>
</tr>
<tr>
<td>Bending</td>
<td>$M_z(t, x)$</td>
<td>$\delta v''(t, x)$</td>
<td>$\int M_z(t, x) \delta v''(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int EI(x) \delta v''(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \eta I(x) \ddot{v}(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \rho A(x) \dot{v}(t, x) , dx$</td>
</tr>
<tr>
<td>Shear</td>
<td>$V_y(t, x)$</td>
<td>$\delta v_s'(t, x)$</td>
<td>$\int V_y(t, x) \delta v_s'(t, x) , dx$</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>$\int G A_s(x) \delta v_s'(t, x) , dx$</td>
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<td></td>
<td></td>
<td></td>
<td>$\int \eta A_s(x) \ddot{v}_s(t, x) , dx$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \rho A(x) \dot{v}_s(t, x) , dx$</td>
</tr>
<tr>
<td>Torsion</td>
<td>$T_x(t, x)$</td>
<td>$\delta \phi'(t, x)$</td>
<td>$\int T_x(t, x) \delta \phi'(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int G J(x) \delta \phi'(t, x) , dx$</td>
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<tr>
<td></td>
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<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\int \rho J(x) \dot{\phi}(t, x) , dx$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$N_x(t, x)$</td>
<td>$\delta u'(t, x)$</td>
<td>$\int N_x(t, x) \delta u'(t, x) , dx$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\int N_x(t, x) \dot{u}'(t, x) , dx$</td>
</tr>
</tbody>
</table>

In this table:

- The internal virtual work of viscous effects is derived assuming linear viscous stress - strain-rate relations: $\sigma = \eta_\dot{\epsilon}$ and $\tau = \eta_\dot{\gamma}$. As will be seen later in the course, the damping properties of real structural materials are more complicated.

- Rotatory inertia effects are neglected in the virtual work of inertial forces in bending beams.
4 Generalized Coordinates

It is often helpful to assume the shape of a dynamic deformation in terms of a sum of products of kinematically-admissible spatially-dependent functions \( \psi_k(x) \) and associated time-dependent variables \( q_k(t) \).

\[
r(t, x) = \sum_k q_k(t) \psi_k(x)
\]

Within this assumption, kinematically admissible virtual displacements are variations in \( r(t, x) \) with respect to the set of coordinates \( q_k(t) \)

\[
\delta r(t, x) = \sum_j \frac{\partial r(t, x)}{\partial q_j(t)} \delta q_j(t) = \sum_j \psi_j(x) \delta q_j(t)
\]

and the following derivatives can be evaluated:

\[
r(t, x) = \sum_k q_k(t) \psi_k(x) \quad \dot{r}(t, x) = \sum_k \dot{q}_k(t) \psi_k(x) \quad \ddot{r}(t, x) = \sum_k \ddot{q}_k(t) \psi_k(x)
\]

\[
r'(t, x) = \sum_k q_k(t) \psi'_k(x) \quad \dot{r}'(t, x) = \sum_k \dot{q}_k(t) \psi'_k(x) \quad \ddot{r}'(t, x) = \sum_k \ddot{q}_k(t) \psi'_k(x)
\]

\[
r''(t, x) = \sum_k q_k(t) \psi''_k(x) \quad \dot{r}''(t, x) = \sum_k \dot{q}_k(t) \psi''_k(x) \quad \ddot{r}''(t, x) = \sum_k \ddot{q}_k(t) \psi''_k(x)
\]

External virtual work can be expressed in terms of generalized virtual displacements (that is, the variations in the generalized coordinates), \( \delta q_j(t) \).

\[
\delta W_E = \int_I f(x) \cdot \delta r(x) \, dx + \sum_i F_i \cdot \delta R_i
\]

\[
= \int_I f(x) \cdot \sum_j \psi_j(x) \delta q_j(t) \, dx + \sum_i F_i \cdot \sum_j \psi_j(x_i) \delta q_j(t)
\]

\[
= \sum_j \left[ \int_I f(x) \cdot \psi_j(x) \, dx \right] \delta q_j(t) + \sum_j \left[ \sum_i F_i \cdot \psi_j(x_i) \right] \delta q_j(t)
\]

Internal virtual work can also be expressed in terms of generalized virtual displacements, for example for elastic axial forces and virtual deformation in a bar,

\[
\delta W_I = \int_I EA(x) \, u'(t, x) \, \delta u'(t, x) \, dx
\]

\[
= \int_I EA(x) \sum_k q_k(t) \psi'_k(x) \sum_j \psi'_j(x) \delta q_j(t) \, dx
\]

\[
= \sum_k \sum_j \left[ \int_I EA(x) \psi'_j(x) \psi'_k(x) \, dx \right] q_k(t) \delta q_j(t) \tag{7}
\]

And for inertial axial forces and virtual displacements in a bar,

\[
\delta W_I = \int_I \rho A(x) \, \ddot{u}(t, x) \, \delta u(t, x) \, dx
\]

\[
= \int_I \rho A(x) \sum_k \ddot{q}_k(t) \psi_k(x) \sum_j \psi_j(x) \delta q_j(t) \, dx
\]

\[
= \sum_k \sum_j \left[ \int_I \rho A(x) \psi_j(x) \psi_k(x) \, dx \right] \ddot{q}_k(t) \delta q_j(t) \tag{8}
\]