1 Time Domain Analysis

Dynamic test results on structural solids support damping mechanisms in which the energy dissipated per cycle is frequency-independent. Coulomb friction,

\[ f_D(r, \dot{r}) = F_f \, \text{sgn}[\dot{r}(t)] \quad (1) \]

is one such mechanism. Another is a mechanism which provides forces that are proportional to displacement, \( r(t) \), and are in phase with the velocity, \( \dot{r}(t) \),

\[ f_D(r, \dot{r}) = (\xi k) \, |r(t)| \, (\dot{r}(t)/|\dot{r}(t)|) \quad (2) \]

A damping force which is proportional to both displacement and velocity, and is in phase with velocity,

\[ f_D(r, \dot{r}) = (\xi k) \, |r(t)| \, \dot{r}(t) \quad (3) \]

is also frequency-independent. To examine the damping behavior resulting from these expressions, consider sinusoidal motion, \( r(t) = \bar{r} \cos(\omega t) \), and \( \dot{r}(t) = -\omega \bar{r} \sin(\omega t) \). Figures 1 and 2 show the force-displacement and force-velocity relationships for equations (2) and (3). Here we can see that the damping force is indeed proportional to displacement, in-phase with velocity, and independent of frequency, \( \omega \).

These damping expression are non-linear in terms displacements and velocities. Oscillators damped with the mechanisms of equation (2),

\[ m\ddot{r}(t) + (\xi k)|r(t)|\dot{r}(t)/|\dot{r}(t)| + kr(t) = f(t) \quad , \quad r(0) = 0 \quad , \quad \dot{r}(0) = 0 \quad (4) \]

are homogeneous. A doubling of the external forcing in equation (4) always results in a doubling of the response. Oscillators damped with the mechanisms of equations (1) and (3), on the other hand, are not homogeneous. They exhibit amplitude-dependent behavior.
Figure 1. Force-displacement and force-velocity relationship for complex-stiffness damping (equation (2), sometimes called “butterfly damping”) for $\omega = \pi$ (blue) and $\omega = 2\pi$ (green) and $\xi k = 0.1$.

Figure 2. Force-displacement and force-velocity relationship for complex-stiffness damping (equation (3)) for $\omega = \pi$ (blue) and $\omega = 2\pi$ (green) and $\xi k = 0.1$. 

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2 Equivalent Linear Viscous Damping

The equivalent linear viscous damping is conventionally defined in terms of equal energy dissipation per cycle of motion, \( W_D \). For any dissipative structural component deforming dynamically with period \( T \),

\[
W_D = \int_0^T f_D(t)\dot{r}(t) \, dt
\]  

(5)

and is geometrically equal to the area within the hysteresis loop of one closed cycle. It is not hard to see from figure 1 that for \( r(t) = \bar{r}\cos \omega t \),

\[
W_D = (4) \left( \frac{1}{2}\bar{r} \cdot \xi kr \right) = 2\xi k\bar{r}^2 ,
\]

(6)

and is independent of the frequency of oscillation, \( \omega \). The equivalent linear viscous damping rate is proportional to \( W_D \), and the equivalent viscous damping rate is amplitude-independent, but inversely proportional to the frequency of motion,

\[
c_{eqv}(\omega) = \frac{W_D}{\pi \omega \bar{r}^2} = \frac{2\xi k\bar{r}^2}{\pi \omega \bar{r}^2} = \frac{2\xi k}{\pi \omega} .
\]

(7)

The equivalent viscous damping ratio depends on the frequency ratio, \( \Omega = \omega/\omega_n \),

\[
\zeta_{eqv}(\Omega) = \frac{W_D}{2k\pi \Omega \bar{r}^2} = \frac{2\xi k\bar{r}^2}{2k\pi \Omega \bar{r}^2} = \frac{\xi k}{\pi \Omega} .
\]

(8)
3 Frequency Domain Analysis and Transmissibility

For linear viscous damping, the response amplitude $R$ is related to the amplitude of sinusoidal base motion $Z$ by

$$(m\lambda^2 + c\lambda + k) R e^{\lambda t} = -m\lambda^2 Z e^{\lambda t}.$$  \hspace{1cm} (9)

The transmissibility ratio $Tr(\omega)$ is the ratio of the total response amplitude, $(R + Z)$, to the input amplitude, $Z$. $Tr(\omega) = (R + Z)/Z$. For linear viscous damping,

$$Tr_{lv}(\omega) = \frac{ci\omega + k}{-m\omega^2 + ci\omega + k} = \frac{2i\zeta\Omega + 1}{-\Omega^2 + 2i\zeta\Omega + 1},$$  \hspace{1cm} (10)

where the frequency ratio is $\Omega = \omega/\omega_n$, the natural frequency is $\omega_n = \sqrt{k/m}$, and the damping ratio is $\zeta = c/(2k)$.

For damping that is proportional to displacement and is in-phase with velocity (equation (2)), the response amplitude $R$ is related to the amplitude of sinusoidal base motion $Z$ by

$$(m\lambda^2 R + (\xi k) |R| \lambda R / |\lambda R| + kR) e^{\lambda t} = -m\lambda^2 Z e^{\lambda t}
(m\lambda^2 R + (\xi k) |R| \lambda R / (|\lambda||R|) + kR) e^{\lambda t} = -m\lambda^2 Z e^{\lambda t}
(m\lambda^2 R + (\xi k) R \lambda/|\lambda| + kR) e^{\lambda t}|_{\lambda=i\omega} = -m\lambda^2 Z e^{\lambda t}|_{\lambda=i\omega}
(-m\omega^2 + (\xi k) i\omega/|\omega| + k) R e^{i\omega t} = m\omega^2 Z e^{i\omega t}
(-m\omega^2 + (k + i\xi)) R e^{i\omega t} = m\omega^2 Z e^{i\omega t}

The term $k(1 + i\xi)$ illustrates why this form of damping is called “complex-stiffness” damping. The transmissibility ratio for complex-stiffness damping is

$$Tr_{cs}(\omega) = \frac{k(i\xi + 1)}{-m\omega^2 + k(i\xi + 1)} = \frac{i\xi + 1}{-\Omega^2 + i\zeta + 1},$$  \hspace{1cm} (11)

where $\Omega = \omega/\omega_n$ and $\omega_n^2 = k/m$. The magnitudes of the transmissibility ratios given by equations (10) and (11) are plotted in Figure 3. Note that in the low-frequency range, linear viscous damping and complex-stiffness damping give similar transmissibilities, whereas in the high frequency (short period) range, the transmissibility for complex stiffness damping is less sensitive to increases in damping than is linear viscous damping. Note also that since complex-stiffness damped oscillators (equation (4)) are not linear, the transmissibility function (11) may not be used in Fourier expansions of general periodic responses.
Figure 3. Transmissibility ratio for viscous damping (left) and complex-stiffness damping (right).

\( \xi = 0.1, 0.2, 0.5, 0.7, 1.0 \) and \( \zeta = 0.1, 0.2, 0.5, 0.7, 1.0 \)

For viscous damping, \( H(\Omega) = H^*(-\Omega) \).

For complex stiffness damping, \( H(\Omega) = H(-\Omega) \).

References


