

THERMOPLASTICITY OF SATURATED SOILS AND SHALES: CONSTITUTIVE EQUATIONS

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ABSTRACT: Plastic behavior of soils and shales due to heating and loading under constant elevated temperature is discussed in terms of a thermoplastic version of the critical state model. Rules for dependence of the yield surface on temperature in the elastic states and at yielding are proposed. The elastic domain is assumed to shrink during heating (thermal softening) and to expand during cooling, when the stress state is elastic. In a plastic state thermal softening occurs simultaneously with the plastic strain hardening. At a constant stress state, thermal softening may entirely be compensated by plastic strain hardening leading to thermal consolidation. Loading and unloading criteria are given to determine whether the soil response is thermoelastic or thermoplastic. As opposed to isothermal plasticity, stress rate excursions inside the current yield surface are admissible plastic processes, when temperature grows, even if strain hardening occurs. Also, outside stress rate excursions at the softening side may generate plastic strain, when cooling occurs. Thermally induced plastic strain rate non-associativity is discussed as well. Direct and inverse incremental strain-stress-temperature relationships are formulated. An analysis of the experimental results of thermomechanical testing of saturated clays is given in a companion paper.

INTRODUCTION

In this paper basic assumptions and constitutive equations are discussed for thermomechanical behavior of saturated soils and shales. A generalization of the critical state model (Schofield and Wroth 1968) to include thermal effects is developed. The work has been carried out for saturated clays and for the range of temperatures 15° C–115° C, corresponding to conditions of nuclear waste disposal in continental clays (Hueckel and Peano 1987). However, some features of the response to heat of sand, shale, and rock appear to be similar (Agar et al. 1986; Heuze 1983; Zimmerman et al. 1985). We therefore believe that the present work may also apply in other geotechnologies in which soil is subjected to heat in a similar range of temperatures. These are: bentonite buffering of radioactive waste material in hard-rock waste repositories, backfilling of underground high-voltage electric cables, enhanced oil recovery, highway pavements, and others. Also, a better understanding of sensitivity of soils to temperature changes in a laboratory environment may be beneficial in the analysis of the usual geotechnical tests.

The analysis in this paper is based on results obtained from tests on several different clay soils. These results are discussed in terms of the model presented in a companion paper (Hueckel and Baldi 1990).

Focus in this work is on thermomechanical behavior of the soil skeleton in drained conditions. Other aspects of heating of soils, such as thermal expansion of water in clay, including adsorbed water (Baldi et al. 1988), failure of saturated clay specimens in triaxial undrained conditions due to

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All authors of journal papers are asked to prepare their papers in SI units. To provide preliminary assistance to authors, the ASCE Committee on Metrication recommends the following conversion factors and guides:

To convert	To	Multiply by
acre-foot (acre-ft)	cubic meter (m ³)	1.23 × 10 ³
acre (acre)	hectare (ha)	0.405
pound mass (lbm)	kilogram (kg)	0.454
mile (mi)	kilometer (km)	1.61
pound force per square inch (psi)	kilopascal (kPa)	6.89
U.S. gallon (gal)	liter (L)	3.79
inch (in.)	millimeter (mm)	25.4
kilogram force (kgf)	newton (N)	9.81
pound force (lbf)	newton (N)	4.45

heating (Hueckel and Pellegrini 1989), coupling of hydraulic and mechanical fields in clay mass in thermal conditions (Baldi et al. 1987; Hueckel et al. 1987), coupling of clay solid and water response to heating through the adsorbed water, and mixture theory aspects of clay response to heat (Hueckel 1989), are dealt with separately.

As far as clay is concerned, it may be expected that thermomechanical behavior of its skeleton will be affected by the presence and performance of the adsorbed water, in analogy to its mechanical behavior (Rosegnant 1959; Murajama and Shibata 1966; Olson and Mesri 1970; Olson 1974; Sridharan and Jayadeva 1982). Microscopic studies performed by Pusch (1987) and Pusch and Güven (1988) on Na-bentonite autoclaved at 150° C and 200° C have shown irreversible microstructural changes resulting in a denser grouping of the stacks of flakes. Larger voids formed between the stacks than in the reference nonheated material, and between 150° C and 200° C, permanent collapse of a large fraction of the interlamellar voids occurred. Derjaguin et al. (1986) observed that the hydration water in small silica voids loses its special structure above 70° C. Hydration water then partially leaves the interlamellar space, as believed by Winterkorn and Bayer (1935), triggering collapse of the microstructure of the skeleton if the skeleton is subjected to stress.

However, experimental data are, at this time, insufficient to enable one to develop a micromechanical, quantitative model of the thermo-deformational behavior of clays, or soils in general. In part, this is because of the controversial nature of water in fine soil interstices (e.g., Nihnam 1981). Still, a macroscopic model based on phenomenological experiments may be developed. Such a model is presented here and is a combination of a suitably adapted thermoplasticity and the critical state model. A slightly simpler, earlier version of the model, combined with the water-flow continuity condition and Darcy's law, yielded a coupled system of governing equations, as discussed by Hueckel et al. (1987). These equations were implemented into a finite element code as a material subroutine. Cylindrical heat-source effects on mechanical stability of clay mass were subsequently analyzed numerically (Baldi et al. 1987). Supporting experiments on clays were performed on an appropriately arranged triaxial apparatus at ISMES, Bergamo, Italy, and are discussed in the companion paper (Hueckel and Baldi 1990).

The first conceptual model for strain response of clay to heating was given by Campanella and Mitchell (1968); see also Mitchell (1976). Thermal consolidation has been studied by Schiffmann (1971), Deriski and Kowalski (1979), and Bear and Corapcioglu (1981). However, no irreversible strains were considered explicitly in these works. Experimental studies on thermomechanical behavior of soils were conducted by Campanella and Mitchell (1968), Yong et al. (1969), Plum and Esrig (1969), Demars and Charles (1982), Paswell (1967), Morin and Silva (1984), Houston et al. (1985), Houston and Lin (1987), Agar et al. (1986), and others. Numerical analyses of thermal consolidation around spherical and cylindrical sources were performed by Palcauskas and Domenico (1982), Booker and Savvidou (1985), McTigue (1986), and others for an elastic medium, and for plastic clay by Hueckel et al. (1987).

CONSTITUTIVE EQUATIONS

A core of the mechanical part of the model is taken from the "critical state

soil mechanics" concept (Schofield and Wroth 1968). The critical state model leads to an incremental elasto-plastic model, consisting of an elasticity law, plastic flow rule, hardening law, and yield condition. These elements are generalized here to depend explicitly on temperature, consistent with a thermoplasticity theory (see, e.g., Prager 1958 or Naghdi 1960) suitably modified for soils. A particular form of this dependence for saturated clays is discussed in light of experimental data in the companion paper (Hueckel and Baldi 1990).

Elasticity

Preliminary data suggest that elastic domain varies with temperature. So, the current yield surface is assumed to be a function not only of stress and plastic volumetric strain, but also of temperature difference ΔT (referred to an environment temperature T_0), Fig. 1

$$f = f(p', q, \epsilon_p^e, \Delta T), \quad \Delta T = T - T_0 \dots \dots \dots (1)$$

where the effective stress invariants p' and q are defined for six component effective stress σ'_{ij} ; $i, j = 1, 2, 3$, as follows

$$p' = \frac{1}{3} \sigma'_{ii}; \quad q = \sqrt{3} \sqrt{\frac{1}{2} s_{ij} s_{ij}}; \quad s_{ij} = \sigma'_{ij} - \frac{1}{3} \sigma'_{kk} \delta_{ij} \dots \dots \dots (2)$$

In the case of triaxial tests, where $\sigma'_1, \sigma'_3 = \sigma'_2$ are principal effective stress components, the effective stress invariants become

$$p' = \frac{\sigma'_1 + 2\sigma'_3}{3} \dots \dots \dots (3a)$$

$$q = \sigma'_1 - \sigma'_3 \dots \dots \dots (3b)$$

In what follows, the triaxial formulation will be adopted whenever possible. A multidimensional matrix formulation has been discussed in a somewhat different context by Hueckel et al. (1987).

The elasticity law is generalized to thermal conditions by introducing a reversible thermal isotropic strain and by allowing for thermal changes of bulk modulus. The later assumption leads to a coupled thermoelasticity (No-

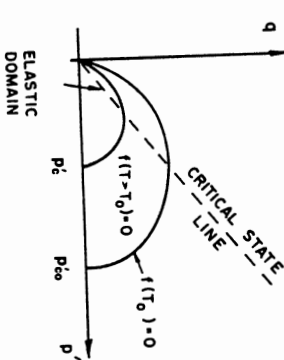


FIG. 1. Yield Surface Shrinking due to Temperature at Elastic State

wacki 1962). On the other hand, at least in clays, the thermal reversal expansion coefficient was found to depend on the effective stress value (Baldi et al. 1988). By definition, thermoelastic strain and energy should be recoverable for any closed stress-temperature cycle. Thus, the thermoelastic law should be derivable from a potential. Let us define an isotropic part, Ω' , of a thermoelastic complementary energy potential $\Omega = \Omega(\sigma'_{ij}, \Delta T)$, in such a way that it yields a logarithmic stress-strain law and a nonlinear stress-dependent thermal expansion coefficient. Thus

$$\Omega' = (\alpha_0^* + \alpha_2 \Delta T) \Delta T (p' - p'_g) + \left(\frac{K_i}{1 + e_0} + \alpha_1 \Delta T + \alpha_3 \Delta T^2 \right) p'_g \left[\frac{p'}{p'_g} \left(\ln \frac{p'}{p'_g} - 1 \right) + 1 \right] \dots \dots \dots (4)$$

where K_i is the isothermal bulk modulus; e_0 = initial void ratio; and p'_g = the isotropic component of geostatic stress, at which elastic strain is null. In laboratory conditions, p'_g is understood as stress at the moment of the end of saturation. Coefficients α_0^* , α_1 , α_2 , and α_3 are constant. Volumetric thermoelastic strain is then defined as

$$\epsilon_v^e = \frac{\partial \Omega}{\partial p'}, \quad \epsilon_v^e = \epsilon_v^i + 2\epsilon_v^e \dots \dots \dots (5)$$

Setting a priori ΔT equal to zero, Ω' reduces to the isothermal isotropic elastic complementary energy potential from which the usual elastic logarithmic law for volumetric strain in soil is recovered through Eq. 5. The thermo-elastic volumetric strain becomes, through Eq. 5

$$\epsilon_v^e = \left[\frac{K_i}{1 + e_0} + (\alpha_1 + \alpha_3 \Delta T) \Delta T \right] \ln \frac{p'}{p'_g} + (\alpha_0^* + \alpha_2 \Delta T) \Delta T \dots \dots \dots (6a)$$

or, regrouping terms differently

$$\epsilon_v^e = \frac{K_i}{1 + e_0} \ln \frac{p'}{p'_g} + \alpha(\Delta T, p') \Delta T \dots \dots \dots (6b)$$

where

$$\alpha(\Delta T, p') = \alpha_0^* + \alpha_2 \Delta T + (\alpha_1 + \alpha_3 \Delta T) \Delta T \ln \frac{p'}{p'_g} \dots \dots \dots (6c)$$

The elastic shear strain is assumed as purely mechanical, and the shear modulus as independent of temperature, i.e.

$$\epsilon_{ij}^e = \frac{1}{2G} (s_{ij} - s_{ij}^0) \dots \dots \dots (7a)$$

or

$$\epsilon_{ij}^e = \frac{1}{3G} (q - q_g) \dots \dots \dots (7b)$$

where G = the shear modulus; q_g = deviatoric stress component at the geostatic state and

$$\epsilon_q = \sqrt{\frac{2}{3}} \epsilon_{ij} \epsilon_{ij}, \quad \epsilon_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij} \dots \dots \dots (8a)$$

In triaxial conditions

$$\epsilon_q = \frac{2}{3} (\epsilon_1 - \epsilon_3) \dots \dots \dots (8b)$$

An experimental procedure to determine a thermodynamically consistent set of constants in these equations was given by Baldi et al. (1988). Moreover, of the experimental results confirmed that the thermomechanical behavior of overconsolidated clays quite closely satisfies requirements of the existence of the thermoelastic potential.

Plasticity

The thermoplastic strain depends on strain and temperature history, and in general cannot be uniquely expressed through effective stress and temperature. Instead, the rate of thermoplastic strain may be related to the stress rate and temperature rate. Such a relationship is proposed here in the form of a particular nonassociative flow rule. This flow rule takes into account the observed difference in the strain-rate mode generated during heating and the mode due to a mechanical loading at constant elevated temperature, obtained at the same stress and the same plastic strain state as

$$\dot{\epsilon}_{ij}^p = \Lambda_v \frac{\partial g}{\partial p'} (p', q, \epsilon_v^p, \Delta T), \quad \dot{\epsilon}_{ij}^p = \Lambda_q \frac{\partial g}{\partial q} (p', q, \epsilon_v^p, \Delta T) \dots \dots \dots (9a)$$

where the plastic multipliers Λ_v and Λ_q are functions of stress and temperature rates

$$\Lambda_v = \Lambda_v(p', \dot{q}, \dot{T}), \quad \Lambda_q = \Lambda_q(\dot{p}', \dot{q}, \dot{T}) \dots \dots \dots (9b)$$

and g = a plastic potential

$$g = g(p, q, \epsilon_v^p, T) \dots \dots \dots (10)$$

For an arbitrary reference system, the thermoplastic strain rate becomes

$$\dot{\epsilon}_{ij}^p = \frac{1}{3} \Lambda_v \frac{\partial g}{\partial p'} \delta_{ij} + \frac{3}{2} \Lambda_q \frac{\partial g}{\partial q} s_{ij} \dots \dots \dots (11)$$

If $g = f$, and $\Lambda_v = \Lambda_q$, the thermoplastic strain rate becomes associative to the yield locus.

THERMOPLASTIC NONASSOCIATIVITY

The plastic multipliers Λ_v and Λ_q are postulated to be positive at yielding, whereas they are zero otherwise. The multiplier Λ_v is derived from the requirement that the effective stress and temperature rates are always such that for any infinitesimal increment of time dt during continuing thermoplastic straining, the yielding condition is satisfied

$$f = 0 \dots \dots \dots (12a)$$

$$\dot{f} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' + \frac{\partial f}{\partial T} \dot{T} + \frac{\partial f}{\partial \epsilon_p^p} \dot{\epsilon}_p^p = 0 \dots \dots \dots (12b)$$

Substituting for the volumetric strain rate $\dot{\epsilon}_p^p$ the expression from the flow rule (Eq. 10), the following expression Λ_u is obtained:

$$\Lambda_u(q, p', T) = \frac{\left(\frac{\partial f}{\partial q}\right)\dot{q} + \left(\frac{\partial f}{\partial p'}\right)\dot{p}' + \left(\frac{\partial f}{\partial T}\right)\dot{T}}{H} \dots \dots \dots (13)$$

where H = thermoplastic hardening modulus equal to

$$H = -\frac{\partial f}{\partial \epsilon_p^p} \frac{\partial g}{\partial p'} \dots \dots \dots (14)$$

The plastic multiplier Λ_p is in general different from Λ_u . This describes why the direction of thermoplastic strain rate depends on the temperature rate at a given stress state and plastic strain amount. In particular, as discussed in the companion paper, the direction of thermoplastic strain rate during heating at constant effective stress is markedly different from that due to stress rate at a constant elevated temperature.

The plastic multiplier Λ_p is not constrained by the consistency equation (Eqs. 12), as opposed to Λ_u , because the yield condition does not depend on the plastic deviatoric strain. Thus it may be simply determined from the deviation of the thermoplastic strain rate from the mechanical plastic potential normal. We postulate that it has the following form

$$\Lambda_p = \Lambda_u + \frac{f}{H} \dot{T} \dots \dots \dots (15)$$

where

$$f = -\frac{\partial f}{\partial T} + f_{,T}(\Delta T, p', q, \epsilon_p^p) \dots \dots \dots (16)$$

In two limiting cases we have that there is no deviatoric thermoplastic strain rate during heating at constant stress, if $f_{,T} = 0$, while if $f_{,T} = \partial f / \partial T$ the strain rate direction does not depend on the temperature rate and is normal to the potential $g = 0$.

LOADING AND UNLOADING CRITERIA

Having specified the plastic multipliers, we may now determine conditions for thermoplastic loading and thermoelastic unloading. As opposed to isothermal plasticity, these depend not only on stress rate (or total strain rate) but on the temperature rate as well. The conditions are obtained from the consistency equation (Eqs. 12). It is thus required that for unloading either

$$f < 0 \quad \text{or} \quad f = 0 \quad \text{and} \quad \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' + \frac{\partial f}{\partial T} \dot{T} < 0 \quad \text{at} \quad \Lambda_u = 0 \dots \dots \dots (17)$$

whereas for loading

$$f = 0 \quad \text{and} \quad \dot{f} = 0 \quad \text{at} \quad \Lambda_u > 0 \dots \dots \dots (18)$$

The latter condition leads to two different cases of hardening and softening. For hardening the loading condition requires that

$$\frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' + \frac{\partial f}{\partial T} \dot{T} > 0 \dots \dots \dots (19a)$$

for the nonassociated flow rule, i.e.

$$\text{at } H > 0 \text{ i.e., } \frac{\partial g}{\partial p'} > 0 \dots \dots \dots (19b)$$

and for the associated flow rule ($\Lambda_u = \Lambda_p; g = f$) at

$$H > 0, \quad \frac{\partial f}{\partial p'} > 0 \dots \dots \dots (20)$$

For softening, the loading condition requires

$$\frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' + \frac{\partial f}{\partial T} \dot{T} < 0 \dots \dots \dots (21a)$$

$$\text{at } H < 0, \quad \frac{\partial g}{\partial p'} < 0 \dots \dots \dots (21b)$$

for nonassociated flow rule, and at

$$H < 0, \quad \frac{\partial f}{\partial p'} < 0 \dots \dots \dots (21c)$$

for associated flow rule ($\Lambda_u = \Lambda_p; g = f$).

There is a clear non-uniqueness in the rate response during softening (i.e., if $\partial g / \partial p' < 0$). In fact, the same temperature rate and stress rate (T, \dot{p}, \dot{q}) may lead either to a thermoelastic unloading or to thermoelastoplastic loading as expressed by Eqs. 17 and 21a, respectively. This is an extension to thermal conditions of the well-known property of the constitutive equations for materials exhibiting softening (see, e.g., Maier and Hueckel 1977, 1979). No such non-uniqueness occurs if strain rate is imposed with temperature rate, as will be seen later.

Moreover, in thermal conditions, the loading/unloading condition is not uniquely determined through the orientation of the stress rate vector with respect to the yield surface normal. Therefore, the loading condition (Eqs. 19), as opposed to that in isothermal plasticity, admits an inward stress rate orientation with respect to yield surface, even at hardening, $H > 0$. This occurs during the temperature change, provided that the rate of the thermal shrinking of the yield surface is sufficiently high (i.e., at sufficiently high $\partial f / \partial T > 0$ and $T > 0$), such that

$$\frac{\partial f}{\partial T} \dot{T} > -\left(\frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}'\right), \quad \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' < 0, \quad \text{at } H > 0 \dots \dots \dots (22)$$

Such a situation arises during heating of saturated clay in undrained, constant total stress conditions. Experimental evidence for such behavior is pre-

vented and discussed by Hueckel and Pellegrini (1989). It may also be deduced from earlier experiments on oil sand (Agar et al. 1986).

Analogously, in the softening range (i.e., with $H < 0$), the inequality in Eq. 21a admits an outward stress rate at cooling. This takes place when the rate of yield-surface growth at cooling, $\dot{T} < 0$ is such that

$$\frac{\partial f}{\partial T} \dot{T} < - \left(\frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' \right), \quad \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p'} \dot{p}' > 0 \quad \text{at } H < 0 \dots \dots \dots (23)$$

It should be pointed out that an outward stress rate excursion at softening is ruled out in isothermal plasticity as statically inadmissible (see, e.g., Maier and Hueckel 1977, 1979).

The aforementioned considerations hold under the condition $\partial f / \partial T > 0$ is satisfied. For all the materials investigated it was seen to be true, as shown in the companion paper. Furthermore, the neutral loading occurs for a combination of stress and temperature rates, which does not necessarily imply a stress rate tangent to the yield surface, undergoing continuing thermal softening (Naghdi 1960). Consequently, at variance with isothermal associative plasticity, at the perfectly plastic state, $H = 0$, a change in stress deviator is admissible, and equal to

$$\dot{q} = \frac{\partial f}{\partial T} \dot{T} \dots \dots \dots (24)$$

Loading and unloading criteria for imposed total strain rate and temperature rate are discussed at the end of the paper, after the inverse constitutive law has been derived.

INCREMENTAL STRAIN-STRESS-TEMPERATURE EQUATIONS

The overall strain rate in a thermoelastoplastic process is a sum of the thermoelastic and thermoplastic strain rates (Eqs. 6, 7, and 10) as follows:

$$\begin{Bmatrix} \dot{\epsilon}_v \\ \dot{\epsilon}_q \end{Bmatrix} = \begin{bmatrix} D_{vp} & D_{vq} \\ D_{qp} & D_{qq} \end{bmatrix} \begin{Bmatrix} \dot{p}' \\ \dot{q} \end{Bmatrix} + \begin{Bmatrix} A_v \\ A_q \end{Bmatrix} \dot{T} \dots \dots \dots (25)$$

where

$$D_{vp} = K_a^{-1} \frac{1}{p'} + \frac{1}{H} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial p'}, \quad D_{vq} = \frac{1}{H} \frac{\partial f}{\partial q} \frac{\partial g}{\partial p'} \dots \dots \dots (26)$$

$$D_{qp} = \frac{1}{H} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial q}, \quad D_{qq} = \frac{1}{3G} + \frac{1}{H} \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} \dots \dots \dots (27)$$

and

$$A_v = A_T + \frac{1}{H} \frac{\partial f}{\partial T} \frac{\partial g}{\partial p'} \dots \dots \dots (28)$$

$$A_T = \alpha_0^* + 2\alpha_2 \Delta T + (\alpha_1 + 2\alpha_3 \Delta T) \ln \frac{p'}{P_0} \dots \dots \dots (29a)$$

$$A_q = \frac{1}{H} \frac{\partial f}{\partial T} \frac{\partial g}{\partial q} \dots \dots \dots (29b)$$

$$K_a = \frac{K_1}{1 + e_0} + (\alpha_1 + \alpha_3 \Delta T) \Delta T \dots \dots \dots (30)$$

or briefly

$$\dot{\epsilon} = D \dot{\sigma}' + A \dot{T} \dots \dots \dots (31)$$

with an obvious identification of symbols.

Due to the nonassociativity of the flow rule, the constitutive matrix **D** is nonsymmetric. It becomes symmetric when the potential *g* coincides with the yield surface $f = 0$ and $A_v = A_q$ and thus the flow rule becomes associative

$$D_{vq} = D_{qp} = \frac{1}{H} \frac{\partial f}{\partial q} \frac{\partial f}{\partial p'} \dots \dots \dots (32)$$

INCREMENTAL STRESS-STRAIN-TEMPERATURE LAW (INVERSE)

A relationship inverse to Eqs. 25 and 31 may be obtained, by expressing the stress rate components \dot{q} and \dot{p}' in Eq. 12 in terms of $\dot{\epsilon}_q$, $\dot{\epsilon}_v$, and \dot{T} , so that the multiplier A_v becomes now

$$A_v = \frac{\left(\frac{\partial f}{\partial p'} p' K_a^{-1} \right) \dot{\epsilon}_v + 3 \frac{\partial f}{\partial q} G \dot{\epsilon}_q + \left(\frac{\partial f}{\partial T} - A_T K_a^{-1} p' \frac{\partial f}{\partial p'} \right) \dot{T}}{H - H_{cr}} \dots \dots \dots (33)$$

where

$$H_{cr} = - \left(\frac{\partial g}{\partial p'} p' K_a^{-1} \frac{\partial f}{\partial p'} + 3G \frac{\partial g}{\partial q} \frac{\partial f}{\partial q} \right) \dots \dots \dots (34)$$

Thus the inverse rate stress-strain-temperature equation reads

$$\begin{Bmatrix} \dot{p}' \\ \dot{q} \end{Bmatrix} = \begin{bmatrix} C_{pv} & C_{pq} \\ C_{qv} & C_{qq} \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_v \\ \dot{\epsilon}_q \end{Bmatrix} + \begin{Bmatrix} B_p \\ B_q \end{Bmatrix} \dot{T} \dots \dots \dots (35)$$

where

$$C_{pv} = K_a^{-1} \left(1 - \frac{1}{H - H_{cr}} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial p'} \right) p' \dots \dots \dots (36)$$

$$C_{pq} = - \frac{3G}{H - H_{cr}} \frac{\partial f}{\partial q} \frac{\partial g}{\partial p'} \dots \dots \dots (37)$$

$$C_{qv} = - \frac{K_a^{-1} p'}{H - H_{cr}} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial q} \dots \dots \dots (38)$$

$$C_{qq} = 3G \left(1 - \frac{1}{H - H_{cr}} \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} \right) \dots \dots \dots (39)$$

$$B_q = \frac{1}{H - H_{\sigma'}} \left(\frac{\partial f}{\partial T} - A_T K_a^{-1} p' \frac{\partial f}{\partial p'} \frac{\partial g}{\partial g} \right) \dots \dots \dots (40)$$

$$B_p = \left[A_T K_a^{-1} p' \left(1 - \frac{1}{H - H_{\sigma'}} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial g} \right) + \frac{1}{H - H_{\sigma'}} \frac{\partial f}{\partial T} \frac{\partial g}{\partial p'} \right] \dots \dots \dots (41)$$

or briefly
 $\dot{\sigma}' = C\dot{\epsilon} + B\dot{T}$ (42)

with an obvious identification of symbols.
 Having inverted the constitutive law, we may now formulate criteria for loading and unloading for imposed total strain and temperature rates. These criteria are alternative to those discussed before. The criteria may be obtained by substituting into the unloading criterion (Eq. 17) expressions for \dot{q} and \dot{p}' corresponding to thermoelastic unloading. The unloading criterion thus reads

$$U(\dot{\epsilon}_q, \dot{\epsilon}_v, \dot{T}) = 3 \frac{\partial f}{\partial q} G \dot{\epsilon}_q + \frac{\partial f}{\partial p'} (p' K_a^{-1}) \dot{\epsilon}_v + \left(\frac{\partial f}{\partial T} - A_T p' K_a^{-1} \right) \dot{T} \leq 0 \dots \dots \dots (43)$$

The criterion for loading is determined by exclusion and thus it requires that
 $U(\dot{\epsilon}_q, \dot{\epsilon}_v, \dot{T}) > 0 \dots \dots \dots (44)$

Note that these criteria uniquely define unloading and reloading both in the strain-hardening and strain-softening range. However, in the presence of temperature rate, they are not any more unique in the superposed stress-strain rate space, as opposed to isothermal plasticity (Maier and Hueckel 1977, 1979).

CONCLUDING REMARKS

The foregoing constitutive equations combine the critical state theory and thermoplasticity theory with particular properties of soils under thermomechanical conditions. The strain hardening via thermoplastic volumetric strain employed here allowed us to make reference to a well-known class of mechanical hardening/softening models, such as "modified Cam-clay" (Roscoe and Burland 1968). However, soils and shales do not always behave mechanically as predicted by these models, and often even the form of the yield surface departs from the standard shapes like ellipsoidal or bi-ellipsoidal. Therefore, for the sole mechanical part of soil behavior, more sophisticated models should be chosen to match the behavior of a specific soil (e.g., Nova and Hueckel 1981, Nova 1982). Still, even with the aforementioned simple volumetric strain hardening, characteristic effects in soil behavior under thermal loading, like those of variation of the elastic domain with temperature, thermoplastic consolidation, thermoplastic strain rate non-associativity, and undrained failure due to heating can be fairly well captured. The aforementioned effects are described using the presented model and discussed against experimental evidence in the companion paper (Hueckel and Baldi 1990).

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APPENDIX II. NOTATION

The following symbols are used in this paper:

A_u, A_r, A_q, B_q, B_p	=	auxiliary functions;
$C_{pq}, C_{pq}, C_{qv}, C_{qq}$	=	stiffness matrix and compliance matrix;
C, D	=	auxiliary functions;
$D_{pq}, D_{pq}, D_{qp}, D_{qq}$	=	auxiliary functions;
e_{ij}	=	deviatoric strain tensor;
f	=	yield function;
f, f_{ir}	=	auxiliary functions;
G	=	shear modulus of elasticity;
g	=	plastic potential;
H	=	plastic hardening modulus;
H_c	=	critical hardening modulus;
K_a	=	thermoelastic bulk modulus (swelling);
K_i	=	isothermal bulk modulus (swelling) of elasticity;
p'	=	effective mean stress;
p_s	=	p' at geostatic state;
q	=	stress deviator invariant;
q_a	=	q at geostatic state;
s_{ij}	=	stress deviator;
T	=	temperature;
T_s	=	temperature at geostatic state;
U	=	unloading index;
α	=	thermal expansion coefficient (function);
$\alpha_0 \dots \alpha_3$	=	coefficients;
ϵ	=	strain;
ϵ_v	=	volumetric strain;
ϵ_p	=	thermoplastic volumetric strain;
ϵ_v^e	=	thermoelastic volumetric strain;
ϵ_v^p	=	thermoelastic volumetric strain;
Ω'	=	isotropic thermoelastic complementary energy;
A_u	=	plastic multiplier (volumetric);
A_q	=	plastic multiplier (deviatoric);
σ'_{ij}	=	effective stress tensor; and
()	=	rate.

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ϵ_p	=	thermoelastic volumetric strain;
ϵ_{ij}^e	=	thermoelastic volumetric strain;
ϵ_{ij}^p	=	isotropic thermoelastic complementary energy;
Ω'	=	plastic multiplier (volumetric);
Λ_v	=	plastic multiplier (deviatoric);
Λ_g	=	plastic multiplier (deviatoric);
σ'_{ij}	=	effective stress tensor, and
$(\)$	=	rate.