Abstract
Presented is a frequency domain harmonic balance (HB) technique for modeling nonlinear unsteady aerodynamics of three-dimensional transonic inviscid flows about wing configurations. The method can be used to model efficiently nonlinear unsteady aerodynamic forces due to finite amplitude motions of a prescribed unsteady oscillation frequency. When combined with a suitable structural model, aeroelastic (fluid-structure) analyses may be performed at a greatly reduced cost relative to time marching methods to determine the limit cycle oscillations (LCO) that may arise. As a demonstration of the method, nonlinear unsteady aerodynamic response and limit cycle oscillation trends are presented for the AGARD 445.6 wing configuration. Computational results based on the inviscid flow model indicate that the AGARD 445.6 wing configuration exhibits only mildly nonlinear unsteady aerodynamic effects for relatively large amplitude motions. Furthermore, and most likely a consequence of the observed mild nonlinear aerodynamic behavior, the aeroelastic limit cycle oscillation amplitude is predicted to increase rapidly for reduced velocities beyond the flutter boundary. This is consistent with results from other time-domain calculations. Although not a configuration that exhibits strong LCO characteristics, the AGARD 445.6 wing nonetheless serves as an excellent example for demonstrating the HB/LCO solution procedure.

Nomenclature
\( \frac{A_R}{b, c} \) = aspect ratio = wing span squared/wing area
\( \frac{b, c}{\text{semi-chord and chord respectively}} \)

\( C_Q \) = vector of non-dimensional generalized forces
\( E_t \) = total energy
\( f, g, h \) = scalar functions defining unsteady grid motion in \( x, y, \) and \( z \) coordinates, respectively
\( I \) = identity matrix
\( j \) = \( \sqrt{-1} \)
\( \bar{m} \) = mass of wing
\( M_{\infty} \) = free-stream Mach number
\( M \) = number of structural modes
\( \mathcal{M} \) = generalized mass matrix
\( N_H \) = number of harmonics used in harmonic balance method
\( \bar{p}_1 \) = first harmonic unsteady pressure
\( q_{\infty} \) = free-stream dynamic pressure
\( \rho, \rho_{\infty} \) = local and free-stream density, respectively
\( \mathcal{Q} \) = vector of generalized aerodynamic forces
\( u, v, w \) = flow velocity components in \( x, y, \) and \( z \) coordinate directions, respectively
\( U_{\infty} \) = free-stream velocity
\( \nu \) = volume of a truncated cone having stream-wise root chord as lower base diameter, stream-wise tip chord as upper base diameter, and wing half span as height
\( V \) = reduced velocity or speed index \( V = U_{\infty}/\sqrt{\omega_a b} \)
\( \alpha_0 \) = airfoil or wing root steady flow angle-of-attack
\( \lambda_t \) = wing taper ratio \( \lambda_t = \alpha_t / c_r \)
\( \mu \) = mass ratio, \( \mu = \bar{m} / \rho_{\infty} \bar{v} \)
\( \psi_m \) = \( m \)th structural mode shape
\( \omega, \bar{\omega} \) = frequency and reduced frequency \( \bar{\omega} = \omega b / U_{\infty} \)
\( \omega_{\alpha} \) = wing first torsional mode natural frequency


\( \Omega \) = matrix with structural frequency ratios squared along main diagonal (e.g. \((\omega_1/\omega_\alpha)^2, ..., (\omega_M/\omega_\alpha)^2) \)

\( \xi_m, \xi = m^{th} \) modal structural coordinate and vector of modal structural coordinates

Subscripts

0,1 = zeroth and first harmonic, respectively

\( f \) = flutter onset (i.e. neutral stability) condition

1 Introduction

Recently, Hall et al. [1] presented a new and computationally efficient harmonic balance (HB) approach for modeling nonlinear periodic unsteady aerodynamics of two-dimensional turbo-machinery configurations. The HB method can be used to model nonlinear unsteady aerodynamic response due to finite amplitude motions, and it is easily implemented within the framework of a conventional iterative CFD method. Subsequently, the present authors [2; 3] devised a computational strategy based on the harmonic balance technique for modeling aeroelastic limit cycle oscillation (LCO) behavior of two-dimensional airfoil configurations.

In the following article, we now extend both the harmonic balance technique for modeling nonlinear unsteady aerodynamics, and the harmonic balance based LCO solution procedure, to three-dimensional inviscid flows about wing configurations. The main challenge in going from two to three-dimensions is that structural models in three-dimensions normally consist of many more degrees-of-freedom (DOF) other than the two “typical” pitch and plunge DOF for two-dimensional airfoil sections.

For two-dimensional flows, it has been observed that a Newton-Raphson root finding technique in conjunction with the harmonic balance method proves to be a very efficient procedure for computing LCO solutions [2; 3]. However for the Newton-Raphson LCO solution procedure, one must compute gradients (or sensitivities) of the nonlinear unsteady aerodynamic loading with respect to each of the structural degrees-of-freedom. For a two-dimensional airfoil section, this requires only two gradient calculations. For three-dimensional configurations however, this step may have to be carried out for many more structural degrees-of-freedom. One focus of the research presented in this paper has been to determine to what extent one may reduce the computational effort by using a limited number of the possible structural DOF. As will be shown subsequently, using only the structural modes that dominate the flutter onset condition appears to be sufficient to model LCO behavior accurately when using the HB/LCO solution procedure.

In the following, we demonstrate the harmonic balance technique for modeling nonlinear unsteady aerodynamics, along with the harmonic balance based LCO solution procedure, for the well known AGARD 445.6 transonic wing configuration [4; 5]. We begin by presenting the aeroelastic system of equations applicable to the AGARD 445.6 wing configuration.

2 Aeroelastic Model Governing Equations

The governing aeroelastic system of equations for the AGARD 445.6 wing configuration (Fig. 1) may be written in the frequency domain as:

\[
 k_w \mathbf{M} \left( -\bar{\alpha}^2 \mu I + \frac{1}{\sqrt{2}} \Omega \right) \xi - \mathbf{C}_Q (\xi, \bar{\alpha}) = 0. \quad (1)
\]

For the present analysis, we consider a linear structural model expressed in terms of structural natural modes together with a nonlinear aerodynamic model. Structural damping is neglected. \( k_w \) is a constant dependent on the wing shape and overall mass given by

\[
k_w = \frac{\pi A R (1 + \lambda_t) (1 + \lambda_r + \lambda_t^2)}{6 \bar{\mu}}, \quad (2)
\]

and \( \mathbf{C}_Q \) is the vector of generalized aerodynamic forces, the \( m^{th} \) element of which is given by

\[
 C_{Q_m} = \frac{1}{q_{\infty} c \bar{\alpha}} \int_A \bar{p}_h \psi_m \cdot \hat{n} dA \quad (3)
\]

where the integral is evaluated over the surface of the wing.

In addition to the generalized mass matrix \( \mathbf{M} \), wing structural frequency ratio matrix \( \Omega \), and mass ratio \( \mu \), the governing aeroelastic equations also include the reduced velocity \( V \), reduced frequency \( \bar{\omega} \), and structural modal coordinates \( \xi \).

As was done for the aeroelastic LCO model of Ref. [3], one can also include the static aeroelastic equations in the mathematical model. However for the case of the AGARD 445.6 wing configuration, this is not necessary since the configuration consists of a wing based on a constant symmetric airfoil section at zero degree angle-of-attack. As such, no net steady, or static, aerodynamic load acts on the wing.

As will be discussed in the following section, in order to solve the aeroelastic system (Eq. 1), one must be able to compute the generalized aerodynamic forces \( \mathbf{C}_Q \) for finite values of the structural modal coordinates \( \xi \). This is where the harmonic balance procedure becomes an invaluable tool.

3 Harmonic Balance Method

3.1 Governing Equations

We consider the inviscid Euler equations (the Reynolds averaged Navier-Stokes equations can be treated in a similar manner),
which may be written in integral form as
\[ \frac{\partial}{\partial t} \iint_{V(t)} U \, dV + \int_{A(t)} \left( \vec{F} - \vec{U} \vec{x} \right) \cdot \hat{n} \, dA = 0 \]  
(4)

where $U$ is the vector of conservative fluid variables
\[ U = \{ \rho \, u \, v \, \rho \, p \, E_i \}^T \]  
(5)

and
\[ \vec{F} = F\hat{i} + G\hat{j} + H\hat{k} \]  
(6)

where $F$, $G$, and $H$ are the $x$, $y$, and $z$ direction component flux vectors. i.e.
\[
F = \begin{bmatrix}
\rho u \\
\rho u^2 + p \\
\rho u v \\
(\rho (E_i + p))^u
\end{bmatrix},
G = \begin{bmatrix}
\rho v \\
\rho v^2 + p \\
\rho v w \\
(\rho (E_i + p))^v
\end{bmatrix},
H = \begin{bmatrix}
\rho w \\
\rho w^2 + p \\
\rho w v \\
(\rho (E_i + p))^w
\end{bmatrix}.
\]

The unsteady motion of the control volume $\vec{x}$ is given by
\[ \vec{x} = \vec{f} + \vec{g} t + \vec{h} t^2 \]  
(7)

and this accounts for the effect of wing surface and grid motion.

### 3.2 Fourier Series Expansion

We consider the unsteadiness of the flow to be strictly periodic in time with period $T = 2\pi/\omega$ where $\omega$ is the fundamental unsteady frequency. Thus we can expand Eq. 4 in a Fourier series. For example,
\[ \iint_{V(t)} U \, dV = \mathcal{Q}(t) \approx \sum_{n=-N_H}^{N_H} \hat{\mathcal{Q}}_n e^{jn\omega t} \]  
(8)

so that
\[ \frac{\partial}{\partial t} \iint_{V(t)} U \, dV \approx j \omega \sum_{n=-N_H}^{N_H} n \hat{\mathcal{Q}}_n e^{jn\omega t}, \]  
(9)

and similarly
\[ \int_{A(t)} \left( \vec{F} - \vec{U} \vec{x} \right) \cdot \hat{n} \, dA = \mathcal{R}(t) \approx \sum_{n=-N_H}^{N_H} \hat{\mathcal{R}}_n e^{jn\omega t}. \]  
(10)

$N_H$ is the number of harmonics used in the Fourier expansion.

### 3.3 Fourier Coefficients

Substituting the Fourier expansions (Eqs. 9 and 10) into Eq. 4, multiplying by $e^{-jn\omega t}$, and integrating over one period, for each $m$, i.e.
\[ \int_0^T \frac{1}{T} \sum_{n=-N_H}^{N_H} (jn\hat{\mathcal{Q}}_n + \hat{\mathcal{R}}_n) e^{jn\omega t} e^{-jn\omega t} \, dt \]  
(11)

yields a system of equations for the Fourier coefficients. Namely
\[ A \hat{\mathcal{Q}} + \hat{\mathcal{R}} = 0 \]  
(12)

where
\[
A = \begin{bmatrix}
-jN_H \\
\vdots \\
jN_H
\end{bmatrix},
\hat{\mathcal{Q}} = \begin{bmatrix}
\hat{\mathcal{Q}}_{-N_H} \\
\vdots \\
\hat{\mathcal{Q}}_{N_H}
\end{bmatrix},
\hat{\mathcal{R}} = \begin{bmatrix}
\hat{\mathcal{R}}_{-N_H} \\
\vdots \\
\hat{\mathcal{R}}_{N_H}
\end{bmatrix}.
\]

### 3.4 Time Domain Variables

Next, via a Fourier transform matrix $E$, one can relate the Fourier coefficient variables $\mathcal{Q}$ to time domain solution variables stored at uniformly space sub-time levels within a given period of motion. i.e.,
\[ \hat{\mathcal{Q}} = E \mathcal{Q}, \quad \hat{\mathcal{R}} = E \mathcal{R} \]  
(13)

where
\[
\hat{\mathcal{Q}} = \begin{bmatrix}
\mathcal{Q}(t_0) \\
\mathcal{Q}(t_1) \\
\vdots \\
\mathcal{Q}(t_{2N_H})
\end{bmatrix},
\hat{\mathcal{R}} = \begin{bmatrix}
\mathcal{R}(t_0) \\
\mathcal{R}(t_1) \\
\vdots \\
\mathcal{R}(t_{2N_H})
\end{bmatrix},
\]

and
\[ t_n = \frac{2\pi n}{(2N_H + 1)\omega} \quad n = 0, 1, ..., 2N_H. \]  
(15)

More specifically,
\[ \hat{\mathcal{Q}} = \begin{bmatrix}
\iint_{V(t_0)} U(t_0) \, dV \\
\iint_{V(t_1)} U(t_1) \, dV \\
\vdots \\
\iint_{V(t_{2N_H})} U(t_{2N_H}) \, dV
\end{bmatrix}, \]  
(16)
and

\[
\mathbf{\tilde{R}} = \left\{ \begin{array}{l}
\mathcal{J}_{A(t_0)} \left( \mathcal{F}(t_0) - \mathbf{U}(t_0) \mathbf{\tilde{x}}(t_0) \right) \cdot \mathbf{\tilde{n}}(t_0) \, dt \\
\mathcal{J}_{A(t_1)} \left( \mathcal{F}(t_1) - \mathbf{U}(t_1) \mathbf{\tilde{x}}(t_1) \right) \cdot \mathbf{\tilde{n}}(t_1) \, dt \\
\vdots \\
\mathcal{J}_{A(t_{2N_H})} \left( \mathcal{F}(t_{2N_H}) - \mathbf{U}(t_{2N_H}) \mathbf{\tilde{x}}(t_{2N_H}) \right) \cdot \mathbf{\tilde{n}}(t_{2N_H}) \, dt 
\end{array} \right. ,
\]  

(17)

Thus

\[
\mathbf{A} \mathbf{\dot{Q}} + \mathbf{E} \mathbf{\tilde{R}} = 0,
\]  

(18)

and

\[
\mathbf{E}^{-1} \mathbf{A} \mathbf{\dot{Q}} + \mathbf{E}^{-1} \mathbf{E} \mathbf{\tilde{R}} = 0.
\]  

(19)

So now one can work in terms of the time domain variables, which is in general much easier to do. Note however that we take full advantage of the assumed periodic motion and thus a transient time simulation and its associated computational cost is avoided. The resulting system of equations can then be written as

\[
\mathbf{D} \mathbf{\dot{Q}} + \mathbf{\tilde{R}} = 0
\]  

(20)

where

\[
\mathbf{D} = \mathbf{E}^{-1} \mathbf{A} \mathbf{E}.
\]  

(21)

### 3.5 Pseudo Time Marching

By adding a pseudo time derivative term \( \mathbf{\dot{Q}} \) to Eq. 20, one can then develop an iterative technique for determining the harmonic balance solution \( \mathbf{Q} \). That is,

\[
\frac{\delta \mathbf{\dot{Q}}}{\delta t} + \mathbf{D} \mathbf{\dot{Q}} + \mathbf{\tilde{R}} = 0,
\]  

(22)

whereby one then simply “marches” Eq. 22 in a fictitious time-like manner until a steady state (Eq. 20) is achieved. Solution acceleration techniques such as local time-stepping, pre-conditioning, residual smoothing, and multi-grid can all be used to accelerate the convergence of the harmonic balance solution.

So for example, in the case of a finite-volume based CFD method, Eq. 22 is solved for every computational “finite-volume” comprising the computational mesh. The overall thus consists of pseudo time marching \( N \times (2N_H + 1) \) dependent variables where \( N \) is the number of mesh points times the number of dependent variables.

Modifying an existing steady CFD flow solver to implement the harmonic balance technique is thus a relatively straightforward task as the main requirement is just a re-dimensioning of the primary arrays from \( N \) elements to \( N \times (2N_H + 1) \) elements. Again since solution acceleration techniques such as local time-stepping, pre-conditioning, residual smoothing, and multi-grid can be used, the computational cost of the method in determining unsteady solutions thus scales by a factor of \( (2N_H + 1) \) times the cost of a nominal steady flow solution.

Once a harmonic balance solution has been determined for a pre-specified wing motion and frequency, one can then obtain the resulting generalized aerodynamic forces from Eq. 3.

### 4 Limit Cycle Oscillation Solution Procedure

Based on previous research efforts conducted for two-dimensional airfoil configurations, we have found that a Newton-Raphson technique in conjunction with the harmonic balance nonlinear unsteady flow solver technique provides an efficient method to determine limit cycle oscillation response.

As demonstrated in Refs. [2] and [3], the harmonic balance LCO solution method proceeds by choosing one of the structural modal coordinates to be the independent variable. For the two-dimensional airfoil case, this was chosen to be the pitch coordinate \( \tilde{\alpha} \). For three-dimensional flow about wings, we now consider \( \xi_1 \), the modal coordinate of the first (typically bending) structural mode shape, as the independent variable. We also chose \( \xi_1 \) to be real valued.

In formulating the HB/LCO solution technique, one then proceeds by dividing Eq. 1 through by \( \xi_1 \), and re-expressing the system of equations as:

\[
\mathbf{R}(\mathbf{L}, \xi_1) = k \omega M \left( -\tilde{\omega}^2 \mu I + \frac{1}{V^2} \right) \xi_1 - C \mathbf{Q} \left( \frac{\xi}{\xi_1}, \xi_1, \tilde{\omega} \right) = 0.
\]  

(23)

Considering both the real and imaginary parts of Eq. 23, the vector \( \mathbf{L} \) then represents the unknown LCO solution variables

\[
\mathbf{L} = \left\{ \begin{array}{c}
V \\
\tilde{\omega} \\
\text{Re}(\xi_2)/\xi_1 \\
\text{Im}(\xi_2)/\xi_1 \\
\vdots \\
\text{Re}(\xi_M)/\xi_1 \\
\text{Im}(\xi_M)/\xi_1 
\end{array} \right\},
\]  

(24)

which consists of the LCO reduced velocity \( V \) (this variable is customarily the independent variable in time-domain LCO solution techniques), LCO reduced frequency \( \tilde{\omega} \), and the real and imaginary parts of the ratio of each of the LCO structural modal
coordinates, for modes two and higher, to the first structural modal coordinate amplitude $\xi_1$. The real and imaginary parts of Eq. 23 as such represent a system of $2M$ equations for the $2M$ unknown LCO solution variables of $L$.

As observed in Refs. [2] and [3], an efficient method for solving Eq. 23 is to use a simple Newton-Raphson root finding technique. This is an iterative method for solving for the unknown LCO variables $L$ whereby one “marches” the vector equation

$$L^{n+1} = L^n - \left[ \frac{\partial R(L^n)}{\partial L} \right]^{-1} R(L^n),$$

until a suitable level of convergence is achieved.

We have also observed that one can use simple forward finite-differencing to compute the column vectors of $\frac{\partial R(L)}{\partial L}$. That is,

$$\begin{bmatrix} \frac{\partial R(L)}{\partial \nu} \\ \frac{\partial R(L)}{\partial \sigma} \\ \frac{\partial R(L)}{\partial Re(\xi_\beta/\xi_1)} \\ \frac{\partial R(L)}{\partial Im(\xi_\beta/\xi_1)} \end{bmatrix} \text{ etc.}$$

(26)

where for example

$$\frac{\partial R(L)}{\partial \nu} \approx \frac{R(L, V + \epsilon) - R(L, V)}{\epsilon},$$

$$\frac{\partial R(L)}{\partial \sigma} \approx \frac{R(L, \omega + \epsilon) - R(L, \omega)}{\epsilon},$$

etc. for a small $\epsilon$. Determining the column vectors of $\frac{\partial R}{\partial L}$ in this manner thus requires numerous computations of $R(L)$ for various perturbations to $L$. This in turn means several computations of the unsteady aerodynamic loading $C_D$ for the different perturbations of $L$, and this is where the harmonic balance solver comes into play. This is also the most computationally expensive aspect of the LCO solution methodology. However, the overall method does lend itself to a simple computational parallelization strategy as each of the gradient approximations can be calculated on an individual computer processor.

### 4.1 The AGARD 445.6 Transonic Wing Configuration

As noted in the introduction, in order to demonstrate the harmonic balance nonlinear aerodynamic and LCO solution procedures, the the AGARD model 445.6 transonic wing configuration [4; 5] is chosen as an example. This is a 45 degree quarter chord swept wing based on the NACA 64A004 airfoil section, which has an aspect ratio of 3.3 (for the full span), and a taper ratio of 2/3. Figure 1 illustrates the computational mesh employed for this configuration with Fig. 1a showing a close-up of the wing surface and symmetry boundary grids, and Fig. 1b showing the outer boundary grid. The grid, which in this instance is the “medium” resolution mesh, is based on an “O-O” topology that employs 49 (mesh $i$ coordinate) computational nodes about the wing in the stream-wise direction, 33 (mesh $j$ coordinate) nodes normal to the wing, and 33 (mesh $k$ coordinate) nodes along the semi-span. The total number of fluid dynamic DOF for this CFD mesh is thus 266,805 (i.e. 5 dependent flow variables $i_{max}(49) \times j_{max}(33) \times k_{max}(33)$).
Finally, Fig 2 shows the first three computed (via a finite element analysis) structural mode shapes and natural frequencies for the AGARD 445.6 “weakened” wing configuration as presented in [4]. The first mode shape is seen to consist of a first bending type of motion, the second mode shape then being a first twisting type of motion, and finally the third mode being a second bending motion. Two additional mode shapes, second twisting and third bending, are also provided in [4]. These mode shapes have all been mapped to the computational meshes used in this study using a sixth-order multi-dimensional least squares curve fitting technique.

4.2 Unsteady Grid Motion Treatment

As noted in section 3.4, for the harmonic balance method, the flow solution variables are stored at $2N_H + 1$ sub-time levels over a period of one cycle of motion. This means that the unsteady deformed shape of the wing is thus required at $2N_H + 1$ sub-time levels over the same period of motion. In the following analysis, we consider a linear structural model whereby we approximate the finite-amplitude motions as a superposition of a limited number of the natural mode shapes of the structure. That is,

$$\mathbf{x}_{i,j,k}(t_n) = \mathbf{x}_{0,i,j,k} + \phi_{i,k}(j) \text{Re} \left( \sum_{m=1}^{M} \xi_m \psi_m e^{j\omega t_n} \right)$$

where $\mathbf{x}_{0,i,j,k}$ is the nominal stationary grid and $\phi_{i,k}(j)$ is a blending function, which is equal to one at the wing surface boundary, and which decays to zero at the outer far-field grid boundary. In this instance, we use the following blending function

$$\phi_{i,k}(j) = 1 - \frac{s_{i,k}(j)}{s_{i,k}(j_{\text{max}})}$$

where $s_{i,k}(j)$ is the distance, in the $j$th mesh coordinate, along a grid line curve emanating from the $(i,k)$th mesh point on the surface of the wing.

This procedure for modeling the finite amplitude wing surface and grid motion is very simple and efficient. However, it can eventually lead to negative volumes when the amplitude of the motion is too great. This is in part due to the “O-O” grid topology, which has the tendency to have a rather highly skewed mesh near the wing tip.

5 Nonlinear Unsteady Aerodynamics

We next compute the nonlinear aerodynamic response for a $M_{\infty}$=0.960 background steady flow about the AGARD 445.6 wing configuration due to various finite amplitudes of unsteady wing motion. We choose a reduced frequency of $\tilde{\omega} = 0.1$ since this is near the reduced frequency at which the wing flutters.
5.1 Mesh Convergence

First, we consider the quality of the mesh. To get a sense of required mesh resolution, we consider a small, in effect linear, amplitude motion of the first modal coordinate, i.e. $\xi_1 = 0.0001$ for a reduced frequency $\tilde{\omega} = 0.1$. Of course grid resolution is also an important issue for larger amplitude motions. However for larger motion amplitudes, the number of harmonics one uses in the HB method, in addition to the grid resolution, plays a role in overall solution accuracy. As such, a comprehensive study of model resolution becomes quite a bit more involved. For this reason, we have chosen to examine a small amplitude motion. Future grid convergence studies will consider larger amplitude motions and higher harmonics.

Figure 3 shows the computed steady and unsteady flow surface pressure distributions at 70% span for three different grid resolutions. As can be seen, there is not much difference in the solutions amongst the three different grid resolutions. As such, we have chosen to employ 49x33x33 grid for all subsequent calculations. As can be seen, since the AGARD 445.6 wing uses only a 4% thick symmetric airfoil section, the pressure distribution is also symmetric about the upper and lower surfaces, and because the section is so thin, no definitive shock is readily apparent at even this high transonic Mach number.

<table>
<thead>
<tr>
<th>Structural Mode Shape</th>
<th>$\xi_{\text{max}}$</th>
<th>$\xi_{1\text{max}}/(b/2)$</th>
<th>$\xi_{1\text{max}}/\tilde{\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - First Bending</td>
<td>0.0120</td>
<td>0.2416</td>
<td>0.3309</td>
</tr>
<tr>
<td>2 - First Twisting</td>
<td>0.0025</td>
<td>0.0500</td>
<td>0.0685</td>
</tr>
<tr>
<td>2 - Second Bending</td>
<td>0.0030</td>
<td>0.0547</td>
<td>0.0749</td>
</tr>
</tbody>
</table>

Table 1. AGARD 445.6 Wing Configuration Maximum Modal Coordinate Amplitudes - 49(i) x 33(j) x 33(k) Grid.

5.2 Finite Amplitude Motion Limits

As mentioned in section 4.2, negative computational cell volumes can occur once the amplitude of the wing motion becomes too large. Table 1 shows the maximum value for each of the first three structural modal coordinates, when taken separately, at which point negative volume problems start to occur for the 49x33x33 mesh. Also listed is the corresponding maximum unsteady vertical displacement of the wing surface $\tilde{z}_{1\text{max}}$ during an interval of motion.

5.3 Harmonic Convergence

Next, Figs. 4 and 5 show computed real and imaginary parts, respectively, of the first three generalized force coefficients ($C_{Q_1}$, $C_{Q_2}$, and $C_{Q_3}$) normalized by the each modal coordinate ($\xi_1$, $\xi_2$, and $\xi_3$) as a function of the magnitude of each modal coordinate ($\xi_1$, $\xi_2$, and $\xi_3$). Shown are results when using one, two, and three harmonics in the harmonic balance solution procedure. As can be seen, the amplitude of the modal coordinate has an effect on the generalized force, and particularly so for the real part. Also evident is how one should use more harmonics when considering ever increasing modal coordinate amplitudes. However, in this example, the use of only two harmonics can be
seen to produce a very high level of harmonic convergence for even the largest modal amplitudes. It has been our experience that two harmonics in many cases are all that are necessary for achieving a good level of accuracy as long as the amplitudes are not too large, say for example, a maximum of five degrees in pitch amplitude for the case of an airfoil section.

Next, Fig. 6 shows surface pressure distributions for the zeroth harmonic (Fig. 6a) (i.e. mean flow), along with the real (Fig. 6b) and imaginary (Fig. 6c) parts of the first harmonic (unsteady flow), again normalized by the modal coordinate amplitude $\xi_1$ at a location approximately 70% of wing semi-span. In this instance, we consider four different modal coordinate ampli-
Figure 6. Mean and Unsteady Flow Surface Pressure Distribution Dependence on Mode Shape Amplitude: $\alpha_0 = 0.0$ (deg), $M_\infty = 0.960$, $\phi = 0.1$, and $N_{H} = 3$.

Finally, Fig. 7 shows normalized zeroth and first harmonic surface pressure distributions for the largest modal coordinate amplitudes ($\xi_1 = 0.0001, 0.0020, 0.0050, 0.0100$) for the case of three harmonics used in the harmonic balance technique. Readily apparent are nonlinear effects due to the ever increasing modal coordinate amplitudes. In fact, this is surprisingly so for the zeroth harmonic of the solution (Fig. 6a).

Figure 7. Mean and Unsteady Flow Surface Pressure Distribution Dependence on Number of Harmonic Used in HB Solver: $\alpha_0 = 0.0$ (deg), $M_\infty = 0.960$, $\phi = 0.1$, and $\xi_1 = 0.0120$. 

amplitude ($\xi_1=0.0120$) while varying the number of harmonics used in the harmonic balance technique. Two harmonics can be seen to produce good harmonic convergence.

6 Nonlinear Unsteady Aeroelasticity

6.1 Flutter Onset Trend for the AGARD 445.6 Wing Configuration

Figure 8 shows the computed Mach number flutter onset reduced velocity trend for the AGARD 445.6 wing configuration. Computational model results are shown together with experimental results. The six Mach numbers are the same as those of the experimental investigation of Ref. [5]. The reduction of the flutter onset velocity in the transonic region is readily apparent. The flutter onset conditions are determined using the proper orthogonal decomposition (POD) reduced order model (ROM) time-linearized unsteady aerodynamic approach of Ref. [7]. The POD/ROM approach has been demonstrated to provide a good approximation of the flutter onset conditions, which can then be used as initial conditions for the iterative HB/LCO solution procedure. The agreement between theory and experiment is generally good except at the supersonic free-stream Mach numbers. Other inviscid computational studies by investigators such as Lee-Rausch and Batina [8] and Gordnier and Melville [9] have demonstrated similar differences between theory and experiment for the supersonic Mach numbers.

Table 2 shows computed values for the reduced velocity $V$, reduced frequency $\bar{\omega}$, and the shape of the unsteady motion of the wing at the flutter onset condition, which is presented in terms of the ratio of the amplitude of each of structural modal coordinates normalized to the amplitude of first structural modal coordinate. As can be seen, the first bending structural motion dominates the flutter onset unsteady motion for the six Mach numbers considered, particularly in the transonic region. In fact, beyond the first three structural modes, higher modes only contribute a fraction of a percent to the overall unsteady motion at the flutter onset condition. As will be shown, one can typically neglect these higher numbered modes in the HB/LCO solution procedure.

6.2 LCO Behavior Trends for the AGARD 445.6 Wing Configuration

Based on the LCO solution procedure presented in the Section 4, Fig. 9 shows computed LCO behavior trends for the AGARD 445.6 wing configuration for the six different Mach numbers. Unfortunately, Ref. [5] does not provide any specific experimental LCO data. Plotted in Fig. 9 is the LCO amplitude of the first structural modal coordinate, $\xi_1$, versus the reduced velocity $V$ for each of the six different Mach numbers. The data points for what appear to be zero LCO amplitude actually correspond to a very small LCO amplitude of $\xi_1 = 0.0001$.

In this instance, only the first three structural modes are used for the HB/LCO solution process. The convergence of the HB/LCO solution procedure is also very rapid. Only one or two iterations of Eq. 25 are required. The LCO amplitudes in the transonic region go to larger values than those in the subsonic and high supersonic region since the first bending mode shape very much dominates in the transonic region. As such, negative volume problems with the unsteady grid are less of an issue in the transonic region.

Note how the LCO behavior trend curves are nearly vertical. This is indicative of very linear aeroelastic LCO response behavior, which is not all that surprising for this particularly thin wing that has a maximum thickness ratio of 4%. A recent time-
Table 2. AGARD 445.6 Wing Configuration Mach Number Flutter Onset Conditions.

<table>
<thead>
<tr>
<th>Mach Number, $M_{\infty}$</th>
<th>0.499</th>
<th>0.678</th>
<th>0.901</th>
<th>0.960</th>
<th>1.072</th>
<th>1.141</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>0.479</td>
<td>0.450</td>
<td>0.368</td>
<td>0.300</td>
<td>0.452</td>
<td>0.664</td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
<td>0.211</td>
<td>0.141</td>
<td>0.0929</td>
<td>0.0771</td>
<td>0.0664</td>
<td>0.0790</td>
</tr>
<tr>
<td>Re$(\xi_2)/\xi_1$</td>
<td>-0.312</td>
<td>-0.243</td>
<td>-0.115</td>
<td>-0.0635</td>
<td>-0.102</td>
<td>-0.171</td>
</tr>
<tr>
<td>Im$(\xi_2)/\xi_1$</td>
<td>0.115</td>
<td>0.0624</td>
<td>0.0253</td>
<td>0.0137</td>
<td>0.00854</td>
<td>0.0142</td>
</tr>
<tr>
<td>Re$(\xi_3)/\xi_1$</td>
<td>0.0492</td>
<td>0.0360</td>
<td>0.0179</td>
<td>0.0106</td>
<td>0.0348</td>
<td>0.0005</td>
</tr>
<tr>
<td>Im$(\xi_3)/\xi_1$</td>
<td>0.00379</td>
<td>0.00202</td>
<td>0.000389</td>
<td>-0.000558</td>
<td>-0.00246</td>
<td>-0.000591</td>
</tr>
<tr>
<td>Re$(\xi_4)/\xi_1$</td>
<td>-0.00557</td>
<td>-0.00477</td>
<td>-0.00161</td>
<td>-0.0000110</td>
<td>0.0007</td>
<td>0.0117</td>
</tr>
<tr>
<td>Im$(\xi_4)/\xi_1$</td>
<td>0.00495</td>
<td>0.00346</td>
<td>0.00203</td>
<td>0.00104</td>
<td>-0.00103</td>
<td>-0.00166</td>
</tr>
<tr>
<td>Re$(\xi_5)/\xi_1$</td>
<td>0.00102</td>
<td>0.00101</td>
<td>0.00117</td>
<td>0.000116</td>
<td>0.000437</td>
<td>-0.000361</td>
</tr>
<tr>
<td>Im$(\xi_5)/\xi_1$</td>
<td>-0.00127</td>
<td>-0.00067</td>
<td>-0.000368</td>
<td>-0.000486</td>
<td>-0.000498</td>
<td>-0.000574</td>
</tr>
</tbody>
</table>

Figure 10. Computed AGARD 445.6 Wing Configuration LCO Behavior Trends When Using Different Numbers of Structural Modes in HB/LCO Solutions Procedure.

can be seen, only the first few structural mode shapes are necessary to produce converged LCO solutions. Again, this is most likely due to the fact that the first bending mode tends to dominate the aeroelastic flutter motion for the AGARD 445.6 wing.

It is interesting to note that the strongest nonlinear LCO behavior occurs for $M_{\infty}=1.072$ and $M_{\infty}=1.141$ as shown in Fig. 9. For these Mach numbers, the nonlinear effect is to decrease the reduced velocity at which LCO may occur below the flutter onset condition. This may be at least a partial explanation for the observed differences between theory and experiment in Fig. 8. There the calculated (linear) reduced velocity at the onset of flutter is compared to the reduced velocity at which flutter was observed experimentally. As shown in Fig. 8, the flutter/LCO observed experimentally does indeed occur at a reduced velocity below that predicted theoretically for the onset of flutter.

7 Conclusions

A harmonic balance method for modeling nonlinear periodic unsteady three-dimensional inviscid transonic flows about wing configurations is presented. Demonstrated is the ability of the method to model nonlinear aerodynamic effects due to large amplitude motions for a well known aeroelastic configuration. The use of only two harmonics in the procedure is shown to be quite sufficient for producing harmonic convergence. A limit cycle oscillation solution methodology is also presented and demonstrated for the same benchmark transonic configuration. A weak LCO is observed, which is consistent with results from time-domain calculations by Gordnier and Melville [9]. Nevertheless, it is suggested that nonlinear aerodynamic effects may at least
partially explain some of the previously observed differences between experiment and linear aeroelastic theory.

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References