Abstract

A review of the status of reduced order modeling of unsteady aerodynamic systems is presented. Reduced order modeling is a conceptually novel and computationally efficient technique for computing unsteady flow about isolated airfoils, wings, and turbomachinery cascades. For example, starting with either a time domain or frequency domain computational fluid dynamics (CFD) analysis of unsteady aerodynamic flows, a large, sparse eigenvalue problem is solved using the Lanczos algorithm. Then, using just a few of the resulting eigenmodes, a Reduced Order Model of the unsteady flow is constructed. With this model, one can rapidly and accurately predict the unsteady aerodynamic response of the system over a wide range of reduced frequencies. Moreover, the eigenmode information provides important insights into the physics of unsteady flows. Finally, the method is particularly well suited for use in the aeroelastic analysis of active control for flutter or gust response. As an alternative to the use of eigenmodes, Proper Orthogonal Decomposition (POD) is also explored and discussed. In general POD is an attractive alternative and/or complement to the use of eigenmodes in terms of computational cost and convenience. Balanced modes, a concept widely used in control engineering, are also briefly discussed, as are input/output models. Numerical results presented include a discussion of the effects of discretization and a finite computational domain in the CFD model on the eigenvalue distribution, the effects of the Mach number and viscosity on reduced order models and representative results from linear and nonlinear aeroelastic analysis. Recent results for transonic flows with shock waves including viscous and nonlinear effects are emphasized.

I. Introduction

The use of Computational Fluid Dynamics (CFD) models for unsteady aerodynamic flows has been a goal since the advent of the computer age. And many investigators have demonstrated the potential utility of CFD for improving the physical modeling of complex unsteady flows. However, until recently the computational cost associated with the high dimensionality of these models has precluded their use in routine applications for studying aeroelastic phenomena. Thus the research literature has been voluminous, but the applications in industry have been few in number.

In the present review, recent work on a conceptually novel and computationally efficient technique for computing unsteady flows based on the modal character of such flows is described. Eigenmode based reduced order models are given prominence in this review although other related modal descriptions may also prove useful and are discussed as well.

Why study the eigenmodes of unsteady aerodynamic flows? This is perhaps the fundamental question most often asked, although occasionally someone will express surprise that eigenmodes even exist for these flows. The reasons are several:

- Eigenvalues and eigenmodes for these flows do exist! So perhaps they can tell us something about the basic physical behavior of the flow field.

- Indeed if a relatively small number of eigenmodes are dominant, this immediately suggests a way to construct an efficient computational aerodynamic model using these dominant modes.

- Constructing the aerodynamic model in eigenmodal form is a particularly user friendly way to combine the eigenmode aerodynamic model with a structural modal model to form an aeroelastic modal model with a modest number of degrees of freedom for a given desired level of accuracy. These aeroelastic models will be especially attractive for design studies including the active control of such systems.

- Finally, as will be seen, alternative modal descriptions are available. While their usefulness is predicated on the existence of eigenmodes, these other modal descriptions seek to include more information on the flow response to enhance the accuracy of a reduced model of a given dimension or reduce the dimension for a required accuracy compared to a standard eigenmode representation. Moreover in the case of one descriptor,
the so called proper orthogonal decomposition (POD) modes, one may avoid the necessity of a tedious direct eigenvalue evaluation of the fluid dynamic equations, a major advantage of using these modes.

This article is intended to provide an overview and perspective for the future. It is based largely on the work described in Refs. [1-7]; earlier work is noted in those references. The present paper is an extension and update of Ref. [1].

II. Constructing Reduced Order Models

There are two distinct ways of going about constructing reduced order models, though there are many variations on the basic themes. One approach is to characterize the aerodynamic flow field in terms of a relatively small number of global modes. By a mode we mean a distribution of flow field variables that characterizes a gross motion of the flow. The conceptually simplest way of choosing such a set of modes is to consider the eigenmodes of the flow field. Of course, such modes form a complete set and any flow field distribution can be expressed in terms of such eigenmodes. In particular any alternative modal selection, and we shall consider several, can always be expressed in terms of such eigenmodes. Indeed it is the existence of eigenmodes that underpins any modal description of the flow. As with other simpler mechanical systems it is the hope and expectation, borne out in the results to be shown later, that a relatively small number of modes will prove adequate to describe the flow. Thus a typical CFD model which may have $10^4$ or more degrees of freedom may be reduced to considering only $10$ to $10^2$ modes for describing the pressure on an oscillating aerodynamic surface.

The second category of reduced order models does not explicitly rely on a modal description per se, but rather appeals to the idea that only a small number of inputs, i.e. structural motions or modes, and a correspondingly small number of outputs, i.e. generalized forces or specific integrals of the aerodynamic pressure distribution weighted by the structural mode shapes, will be of interest. Hence one needs to construct, for example, a transfer function matrix whose size is determined by the number of inputs and outputs. Typically this matrix will be of the order of the number of structural modes. The transfer functions are determined numerically using a systems identification technique from time simulations of the CFD code. If the number or type of inputs, i.e. structural modes, changes during an aeroelastic simulation then the aerodynamic transfer functions may need to be recalculated. On the other hand, the CFD code need not be deconstructed to determine aerodynamic modal information, thereby saving this additional effort but also foregoing the additional insight and flexibility gained by knowing the aerodynamic modes. Changes in the structural modes do not change the aerodynamic modes, of course, but changes in the structural modes may require re-calculation of the transfer functions of input/output models.

A. Linear and Nonlinear Fluid Models

Several fluid models have been considered to date. These range from classical, incompressible, potential flow models to compressible, rotational models (Euler equations) to inviscid-viscous interaction models of stall flutter in turbomachinery. Exploratory work has been completed with viscous, Navier-Stokes models, though much remains to be done there. Most results have been for two-dimensional flows over isolated airfoils and cascades of airfoils. Only for incompressible, vortex lattice models have three dimensional flows over wings been considered to date. No special conceptual difficulty is anticipated in extending other flow models to three dimensions, although the task of computing the eigenmodes becomes much more difficult using classical numerical techniques. Fortunately proper orthogonal decomposition appears to offer a viable approach for three dimensional flows.

We will defer a discussion of the details of the several fluid models to the subsequent section on results. In this section, an important conceptual categorization is discussed based upon dynamical systems ideas that transcend all fluid models, i.e., we distinguish among fully linear models, dynamically linear models and fully nonlinear models.

Fully linear models. Most classical aerodynamic models fall into this category, e.g., small perturbation theory in subsonic or supersonic flow that leads to a form of the convected Laplace or wave equation (with constant coefficients) for say the velocity potential. Although such models were not the primary motivation for our work on eigenmodes, it turns out the use of reduced order models and direct solution for aeroelastic eigenvalues is a powerful and computationally efficient approach for fully linear models as well.

Dynamically linear models. In transonic flow, for example, even potential flow models must be considered in their nonlinear form. Physically, this is because the variation of the mean time-independent flow field over a nonlifting airfoil has a significant effect on the unsteady time dependent lift on the airfoil when it oscillates. Said another way, the nonlifting (static) flow field is inherently nonlinear and it must be determined from the static solution of the full, nonlinear potential model. Fortunately, however, the lift due to the airfoil oscillations (for sufficiently small motion) may be treated as a linear dynamical perturbation about the nonlinear static or steady flow field.

This basic idea extends to Euler and Navier-Stokes flows as well. Hence eigenmode analysis may still be used, but the eigenmodes must be those of a small perturbation with respect to the appropriate nonlinear static flow field.

As an aside, it might be noted that this latter restriction can likely be relaxed in the following way. If we determine the eigenvalues and eigenmodes about one airfoil at one Mach number, it is likely these might be used, with good accuracy and efficiency, to form a modal series representation for another not too dissimilar airfoil at a not too removed Mach number. Of course, the coefficients of the modal expansions would be different for the two airfoils or the two different Mach numbers. When eigenmodes for one physical system are used to represent the solution of another physical system, these are usually referred to as primitive modes. The use of primitive modes is well established in
the aeroelastic literature, see Refs. [8-13], for example.

When might the use of primitive modes fail? One case might be, if the eigenmodes of a shockless flow were used to represent the flow about an airfoil with shocks. Clearly, if two flow fields are qualitatively different, using the eigenmodes of one to represent the flow field of the other is problematic as a practical matter.

It should be noted that for an elastic wing, the static wing shape may be changed by the aerodynamic flow and thus the static wing shape may vary with dynamic pressure. Therefore the aeroelastician may need to determine the aerodynamic eigenmodes for several dynamic pressures for some applications. However, in the design of an aircraft or wind tunnel model, usually the static shape itself is a design goal and hence, in that sense, it is known a priori.

**Fully nonlinear models.** In aeroelastic models, the dominant nonlinearity may be either structural or aerodynamic. Clearly if a structural nonlinearity is dominant, which is not infrequently the case, then a dynamically linear aerodynamic theory is perfectly adequate to determine not only the onset of flutter, but also the limit cycle oscillations that may exist. Note moreover that the aerodynamic eigenmode approach is equally suitable for the construction of either time domain or frequency domain aeroelastic models. Thus aerodynamic eigenmodes are particularly useful for nonlinear aeroelastic analyses when combined with nonlinear structural models. By contrast, classical aerodynamic models usually provide results in the frequency domain and CFD models normally generate results in the time domain.

Of course, in principal and with some effort in practice, the classical aerodynamic and CFD models may be used in either the frequency or time domains, though with usually less ease than the eigenmode approach.

Regarding aerodynamic nonlinearities, these most often may be due to shock waves or separated flow. Note, however, that the presence of a shock wave or separated flow does not per se dictate that the flow is dynamically nonlinear. For sufficiently small airfoil or wing motions, dynamic linearization of the aerodynamic flow is still a valid approximation. Of course, the presence of shock waves or separated flow means the steady (i.e., static) flow equilibrium is nonlinear. See, for example, Ref. [14] as an example of a dynamically linear CFD approach that includes the effects of shock waves.

It might be thought that eigenmodes could not be used for fully nonlinear models, but of course they can be with some additional work. Again, to be concrete, if an airfoil undergoes large motions, a fully nonlinear dynamic flow model must be used to describe the corresponding flow field.

How then could eigenmodes be useful? Conceptually, the idea is as follows. This procedure will be familiar to those structural dynamicists and aeroelasticians who have studied structural nonlinearities for plates and shells or helicopter blades [15, 16].

First, one considers small motions and determines the eigenvalues and, very importantly, the eigenmodes. One then forms a dynamical coordinate transformation from the (generalized coordinate) unknowns of the nonlinear dynamical system to so called normal mode coordinates, e.g., if $q$ is the vector of original unknowns, $[e^T]$ is a matrix whose columns are the (right) eigenvectors of the eigenmodes and $s$ are the normal mode coordinates. This equation [not shown] essentially defines $s$ in terms of $q$. Substituting this equation into the full nonlinear equations of fluid motion for $q$, premultiplication of the result by $[e^T]^T$ and then truncating to a small, finite number of $s$ will produce a nonlinear, reduced order model. Unlike the corresponding linear models where the equations for $s$ will be uncoupled, however, now the equations for $s$ will be coupled due to the nonlinear terms. Even so the number of equations to be solved for a given level of accuracy in determining the flow field will be much smaller than the original number of equations for the $q$ unknowns. Hence a very substantial savings in computational cost will still be realized. Indeed it is for fully nonlinear dynamical models where the full power of the eigenmode approach may be realized.

Two final points are worthy of mention. For nonlinear dynamical systems, it is possible to extend the idea of a linear eigenmode itself to nonlinear eigenmodes. This has some theoretical interest. However these nonlinear eigenmodes still lead to coupled equations, so their value in practice is often not substantially greater than that of linear eigenmodes. Even so, some day this should and will be investigated.

Finally, it is worth noting that if one determines the linear eigenmodes for say one airfoil-Mach number combination and then uses the above transformation from $q$ to $s$ for another airfoil-Mach number combination, then the corresponding equations for $s$ will also be coupled even in the linear terms. This, of course, is because we have used the eigenmodes of one fluid system to represent the dynamics of a different fluid system. That is, orthogonality of the modes only holds for eigenmodes used for the same dynamical system from which they were derived.

**B. Eigenmodes and Reduced Order Models**

Dowell, Hall, and Romanowski [1] have discussed determining the eigenmodes of such flows in some detail. Here we briefly summarize their discussion. As noted by these authors, beyond $10^4$ degrees of freedom it becomes progressively more difficult to determine the eigenmodes of such systems. Hence alternative methods have been developed and these are discussed in succeeding sections.

The details vary from one level of fluid model to another as to how one determines the eigenvalues and eigenmodes and then constructs the reduced order model. Rather than repeat the discussion for each specific fluid model, here the discussion is presented in generic form. It is the one that most closely follows the calculation for the Euler or Navier-Stokes equations. For specific calculations for a vortex lattice model [2] or a full potential model [3, 4, 5], the reader is referred to the literature. The following discussion is taken from Ref. [6].

The Euler and Navier-Stokes equations, represent a system of highly nonlinear partial differential equations, which can be written at each point in the interior of a computational flow field domain as $q_i = Q(q_i, q_j)$ or in discrete
time as
\[
\frac{\Delta \tilde{q}_i^{n+1}}{\Delta \tau} = Q(\tilde{q}_i^n, \hat{q}_i^n) \tag{1}
\]
where \( \tilde{q}_i \) represents flow variables at the interior of the computational domain, and \( \hat{q}_i \) represents those on the exterior (boundary) of the computational domain.

There are \( N \) interior flow equations, with four equations [for a 2D Euler flow] written at each of the \( N/4 \) interior grid points. The four unknowns at each grid point are [pressure, two components of momentum and energy for a 2D Euler flow]. Additionally, there are \( M \) nonlinear algebraic boundary condition equations, also with four equations written at each of the \( M/4 \) grid points on the boundary of the computational domain.

\[
P(\tilde{q}_i^n, \hat{q}_i^n, \alpha^n) = 0 \tag{2}
\]
where there are \( L \) \( \alpha \)-variables, related to airfoil shape, or motion, etc. Equations (1) and (2) form a system of \( N + M \) equations and \( N + M \) unknowns.

Now, assuming that the unsteady flow field is a result of small dynamic perturbations about steady state, \( \tilde{q} = \tilde{q}_0 + \tilde{q}^\prime(t), \; \alpha = \alpha_0 + \alpha^\prime(t) \) and noting that \( Q(\tilde{q}_0) = P(\tilde{q}_0, \alpha_0) = 0 \), Eqs. (1) and (2) become
\[
\frac{\Delta \hat{q}_i}{\Delta \tau} = A \hat{q}_i + B \tilde{q}_i \tag{3}
\]
\[
C \hat{q}_i^n - D \hat{q}_i^n - \hat{\alpha}^n = 0 \tag{4}
\]
where
\[
A = \frac{\partial Q}{\partial \hat{q}_i}, \quad B = \frac{\partial Q}{\partial \tilde{q}_i}, \quad \text{etc.}
\]

The exterior degrees of freedom can be eliminated from Eq. (3) by first solving Eq. (4) for \( \hat{q}_i \),
\[
\hat{q}_i^n = C^{-1} D \hat{q}_i^n + C^{-1} \hat{\alpha}^n \tag{5}
\]
The following system of \( N \) coupled equations, which govern the linearized system response, is obtained by noting that the perturbation flow field at time step \( n + 1 \) is \( \hat{q}_i^{n+1} = \hat{q}_i^n + \Delta \hat{q}_i^{n+1} \), and then utilizing Eqs. (3) and (5) to give
\[
\hat{q}_i^{n+1} = A_{eq} \hat{q}_i^n + B_{eq} \hat{\alpha}^n \tag{6}
\]
where
\[
A_{eq} = I + \Delta \tau (A + BC^{-1}D) \tag{7}
\]
\[
B_{eq} = \Delta \tau BC^{-1} \tag{8}
\]
Note that the second term on the right hand side of Eq. (6) represents a known forcing term [for prescribed airfoil or wing motion]. Also, in Eq. (7), \( A_{eq} \) is shown to be a combination of conditions at the interior degrees of freedom, \( A \), plus a contribution from the boundary, while in Eq. (8), \( B_{eq} \), and therefore the external excitation of the flow, depends only on conditions associated with the exterior degrees of freedom. Finally, flow values on the airfoil boundary (for the calculation of lift, for example) can be obtained directly by Eq. (5).

Now, for zero external excitation, \( \hat{\alpha}^n \equiv 0 \), assume that the flow field at time \( n \) is related to an initial flow field in the manner that \( \hat{q}_i^n = \hat{z} \hat{q}_i^n \). Then, Eq. (6) reduces to
\[
A_{eq} \hat{q}_i = \hat{z} \hat{q}_i \tag{9}
\]
which represents an eigenvalue problem. Since \( A_{eq} \) is nonsymmetric, it will have complex eigenvalues, \( \hat{z}_1, \hat{z}_2, \hat{z}_3, \ldots, \hat{z}_N \), as well as sets of both right eigenvectors, \( [e^R] \) and left eigenvectors, \( [e^L] \), which are biorthonormal. This dynamic system will be stable if the largest eigenvalue has magnitude less than 1. The reduced coordinate \( \hat{s} \) is defined such that
\[
\hat{q}_i = [e^R]^T \hat{s} \tag{10}
\]
Now, the governing system of equations can be decoupled by substituting Eq. (10) into Eq. (6), premultiplying by \( [e^L]^T \), and taking advantage of the fact that \( [e^L]^T [e^R] = [I] \) and \( [e^L]^T [A_{eq}] [e^R] = [Z] \). Additionally, by utilizing only the first \( R \) largest magnitude eigenmodes in this transformation, a system of \( R \) decoupled equations \( (R < N) \), the reduced order approximation of the governing equations, is obtained. That is,
\[
\hat{s}^{n+1} = Z R \hat{s}^n + B_R \hat{q}_i^n \tag{11}
\]
where
\[
Z_R \equiv \mathrm{diag} \{\hat{z}_1, \hat{z}_2, \hat{z}_3, \ldots, \hat{z}_N\} \tag{12}
\]
\[
B_R \equiv [e^L]^T B_{eq} \tag{13}
\]
Flow values on the boundary can also be obtained directly from the reduced coordinate vector by using the following relationship, obtained by substituting Eq. (10) into Eq. (5):
\[
\hat{q}_i = D_R \hat{s} + \hat{\alpha} \tag{14}
\]
where
\[
D_R \equiv C^{-1} D e^R \tag{15}
\]
If Eqs. (11) and (14) are applied directly, with a small value of \( R \), the calculated system response may be considerably in error. This is because components of the forcing parallel to the omitted eigenmodes are neglected. The use of a quasi-static correction may substantially reduce this error. This technique is similar to the mode-acceleration method common to structural dynamics [9, 10].

With the quasi-static correction, it is assumed that the dynamics of the system can be approximated by the first \( R \) eigenmodes. The remaining \( N - R \) modes respond in only a pseudo-static manner. Therefore, it can be shown that the time dependent perturbation flow field can be found by combining that due to the first \( R \) eigenmodes, and that due to the error in the instantaneous forcing \( \hat{q}_{err} \). The perturbation flow field due to the forcing error can easily be found from
\[
\hat{q}_{err}^{n+1} = (I - A_{eq})^{-1} B_{eq} \hat{\alpha}^n - \sum_{i=1}^{R} e^{R e_i^T} B_{eq} \hat{\alpha}^n \tag{16}
\]
As the reader can see, the basic idea is beautifully simple. However, the devil is in the details, which is why it has taken several years to convert this idea into reality as described in the subsequent sections of this paper and as fully appreciated by those who have labored toward this goal of reduced order modeling.
C. Eigenmode Computational Methodology

For the simpler (lower dimensional) fluid models, say a two-dimensional vortex lattice model of unsteady flow about an airfoil, the size the eigenvalue matrix is of the order 100 × 100. For such matrices, standard eigenvalue extraction numerical procedures may be used. We have used EISPACK, an algorithm and computer code available in most computational centers in the United States.

For more complicated fluid models, e.g., the full potential models or Euler models the order of the eigenvalue matrix may be in the range of 1000 to 10,000 squares or greater. For matrices of this size, new developments in eigenvalue extraction have been required. We have used methods based upon the Lanczos algorithm. For the full potential equation (∼ 1000 × 1000) an efficient and effective algorithm is described by Hall, Florea, and Lanzkron [4]. For the Euler equations (∼ 10⁴ × 10⁴), the paper by Dowell and Romanowski [7] will be of interest. The discussion of Mahajan et al. [5] is also recommended to the reader. As the extensions to three-dimensional and viscous flows are made, further developments in eigenvalue and eigenmode determination will likely be required or desired.

These further developments appear doable, but the amount of work should not be underestimated. Perhaps an appropriate use of primitive modes may be of help. That is, it may be possible to use the eigenmodes from a simpler fluid model as primitive modes for a more advanced fluid model. Much work remains to be done here.

A final and important point that Mahajan [5] has emphasized is that the eigenvalue problem may be formulated in either discrete or continuous time. The former allows eigenvalue extraction from existing CFD codes using pre- and post processor formats; thereby saving the considerable effort of recoding existing CFD codes.

D. Proper Orthogonal Decomposition (POD) Modes

Romanowski [17] has given a clear discussion of this approach and we follow his description here.

Given the difficulty of extracting eigenmodes for very high dimensional systems, e.g., greater than 10⁴, it is of great interest to note that a simpler modal approach is available as recently developed by Romanowski [17]. This approach adapts a methodology from the fields of nonlinear dynamics and signal processing, the POD or KL modal representation. See Ref. [17] for an introduction to the relevant literature in these fields.

Here, we quote Romanowski’s account of the essence of the method.

Karhunen-Loève Decomposition (KL Decomposition) [also called proper orthogonal decomposition (POD)] has been used for a broad range of dynamic system characterization and data compression applications. The procedure, which is briefly summarized below, results in an optimal basis for representing the given data ensemble.

The instantaneous flow field vector, \( \{ \mathbf{q}_j \} \), is retained at \( J \) discrete times, such that \( j = 1, 2, 3, \ldots, J \). A caricature flow field, \( \{ \mathbf{q}_p^c \} \), is defined as the deviation of each instantaneous flow field from the mean flow field, \( \{ \mathbf{q}_1 \} \) of the ensemble.

\[
\{ \mathbf{q}_p^c \} = \{ \mathbf{q}_j \} - \{ \mathbf{q}_1 \}
\]

A matrix \( [\Phi] \) is formed as the ensemble of the two point correlation of the caricature flow fields, such that

\[
\Phi_{j,k} = \{ \mathbf{q}_k^c \}^T \cdot \{ \mathbf{q}_j^c \}
\]

References [10] and [12] (of Ref. [17]) show that solving the eigenvalue problem

\[
[\Phi] \{ \mathbf{v} \} = \lambda \{ \mathbf{v} \}
\]

produces an optimal set of basis vectors, \( \{ \mathbf{v} \} = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_J \} \) for representing the flow field ensemble. Additionally, the magnitude of the eigenvalue, \( \lambda_j \) gives a measure of the participation of the \( j \)th KL eigenvector in the ensemble. Therefore, a reduced set of basis vectors can easily be found by limiting the set to only those KL eigenvectors corresponding to sufficiently large eigenvalues.

Since the number of time steps and thus the order of the matrix needed to compute a reasonable and useful set of KL modes is on the order of a few hundred, the determination of KL modes is computationally very inexpensive, especially as compared to determining the eigenmodes of the original fluid dynamics model. In the subsequent section results using KL modes are shown to be in excellent agreement with those obtained from the full order model and also the reduced order model based upon eigenmodes. It also might be noted that one can first use the KL decomposition to reduce the order of the original model and then do a further eigenmode analysis of the reduced order model, a technique that may be useful for some applications.

As a final comment on the POD or KL methodology, it is important to note that a similar calculation may be done in the frequency domain by assuming simple harmonic solutions and replacing the data at discrete time steps with data at discrete frequencies over a frequency interval of interest. Kim [18] has used the POD frequency domain method for a vortex lattice fluid model and Thomas, Hall and Dowell [19, 20] have done so for an Euler fluid model including shock waves at transonic conditions.

E. Balanced Modes

Baker, Mingori, and Goggin [21] have used this methodology originally developed in the controls community to develop reduced order aerodynamic models. Rule, Cox and Clark [22] have explored this method as well and we largely follow their discussion here.

To determine the \textit{internally balanced modes}, the eigenvalue problem of the original fluid model or its equivalent must still be solved. The authors of Ref. [21] make the point for the example treated by them that the number of \textit{internally balanced modes} needed for a given level of accuracy in a reduced order model may be less than the corresponding number of eigenmodes. They do confirm the results of Hall [2] for a potential fluid flow that a modal representation is computationally effective and accurate.

Rule, Cox and Clark [22] have also studied the internal balancing approach and made a number of useful observations. As these authors have noted, “Balanced realization is based upon the idea that a similarity transformation
exists which renders the controllability and observability Gramian matrices of the system diagonal and equal. States in the transformed space are ordered by [their] importance as transmission paths between system inputs and outputs. Values along the diagonal of the transformed Gramian matrix indicate the relative contribution of each state to the outputs of interest, and provide a simple criterion for either retaining or neglecting states to form a reduced order model.

Aerodynamic systems provide an ideal opportunity for the application of balanced realization techniques, because it is frequently the case that a large number of states must be used to transmit information from a small number of inputs (system geometry, control surface position), to a small number of outputs (net lift, moment about the elastic axis). This is especially the case with CFD schemes which require the computation of two or three components of velocity plus pressure and density at thousands of nodes around a body, only to obtain a few integrated forces. From this viewpoint, the details of the flow are unimportant; the aerodynamics are simply providing a transmission path from geometry to forces.” They further comment that, “A detailed description of an algorithm to find the balancing transformation, $R$, can be found in Ref. [23]. Note that the computational cost of finding this transformation is $O(N^3)$, which is comparable in cost to finding the eigenvalues and eigenvectors of the same system. This becomes impractical for large systems, but efforts are being made to develop more efficient methods of determining the most dominant states only [24].”

In comparing the relative merits of using eigenmodes versus balanced modes, the authors make the following point. “Eigenvalues and eigenvectors are related to the internal dynamics of the mathematical [CFD] model, and not to the physical input/output mapping between airfoil geometric states and integrated forces. Thus, the rationale for retaining or discarding modes is somewhat arbitrary [in an eigenmode analysis]; in this case [the example treated by the authors of Ref. [22] the most lightly damped modes were retained. Heavily damped modes play an important role in calculating transient system response, which accounts for the poor performance of the eigenmode method [relative to balanced modes] in this limit [of short times or high frequency].”

All results to date for balanced modes have used a vortex lattice model, but in principle other CFD models can be treated in a similar fashion.

**F. Synergy Among the Modal Methods**

In light of the above discussion, the following methodology appears to be a practical and perhaps even an optimum approach. With a given CFD model, a set of POD modes can be constructed of the order of $10^2$ to $10^3$. Then using the POD modes and the corresponding reduced order model (POD/ROM), further reduction may be obtained by extracting eigenmodes or balanced modes from the POD/ROM. For some applications where the smallest possible model is desired, e.g. design for active control of an aeroelastic system, this further reduction will be desirable and perhaps essential using an eigenmode or balanced mode ROM. However for validation studies where the identification and understanding of the most critical modes for stability is the primary issue, one may prefer to retain a POD/ROM or an eigenmode ROM.

**G. Input/Output Models**

There is a long tradition of developing aerodynamic transfer function representations from numerical data for simple harmonic motion dating from the time of R.T. Jones’ approximation to the Theodorsen function. Much of the relevant literature is summarized by Karpel [25] whose own contribution was to develop a state-space or transfer function representation of minimum order for a given level of accuracy using transfer function ideas based upon data for simple harmonic motion. Hall, Thomas, and Dowell [19] have recently discussed such models in the light of the more recent developments in aerodynamic modal representations. We follow their discussion here. See Ref. [9] and [25] for references to the original literature.

“Investigators have developed a number of techniques to reduce the complexity of unsteady aerodynamic models. R.T. Jones approximated indicial lift functions with series of exponentials in time. Such series have particularly simple Laplace transforms, i.e. rational polynomials in the Laplace variable s, making them especially useful for aeroelastic computations. Pad approximants are rational polynomials whose coefficients are found by least-squares curve fitting the aerodynamic loads computed over a range of frequencies. Vepa, Edwards, and Karpel developed various forms of the matrix Pad approximant technique. This approach reduces the number of so-called augmented states needed to model the various unsteady aerodynamic transfer functions (lift due to pitching, pitching moment due to pitching, etc.) by requiring that all the transfer functions share common poles.

...[A POD or eigenmode model] is similar in form to that obtained using a matrix Pad approximant for the unsteady aerodynamics ... and has some of the same advantages of the Pad approach. Both methods produce low degree-of-freedom models. Furthermore, both require the aerodynamic lift and moment transfer functions to share common eigenvalues (although the zeros are obviously different). This is appealing because physically the poles should be independent of the type of transfer function. However, the present [modal] approach has several advantages over the matrix Pad approximant method. The present method attempts to compute the actual aerodynamic poles, or at least the poles of a rational CFD model. The Pad approach, on the other hand, selects pole locations by some form of curve fitting [of aerodynamic data for simple harmonic motion]. In fact, some Pad techniques can produce unstable aerodynamic poles, even for stable aerodynamic systems.”

It is interesting to note that the notion of a transfer function can be extended to nonlinear dynamical systems where the counterpart is usually called a describing function. Ueda and Dowell [26] pioneered and discussed this approach. In the time domain, transfer functions can be inverted to form convolution integrals. Silva [27] has recently pioneered the extension of these ideas to nonlinear aerodynamic models using the concept of a Volterra integral.
H. Structural, Aerodynamic, and Aeroelastic Modes

Structural modes have a long and rich tradition. The novelty of much that is being discussed in the present paper is to extend these ideas to aerodynamic flows which also possess a modal character, albeit a more complex one. And finally there are aeroelastic modes one may consider.

For the determination of structural modes, one normally neglects dissipation or damping and thus only models kinetic energy (or inertia) and potential strain energy (or stiffness). The eigenvalues are real (the natural frequencies squared) as are the corresponding eigenmodes. Physically if one excites the structure with a simple harmonic oscillation at a frequency near that of an eigenvalue, the structure will perform a simple harmonic oscillation at that same frequency whose spatial distribution is given by the corresponding eigenvector.

For aeroelastic modes (and also for aerodynamic modes) the physical interpretation as well as the mathematical determination of the eigenvalues and eigenvectors or eigen modes is more subtle and difficult, but still rewarding! First of all the eigenvalues are complex with real and imaginary parts of the eigenvalue giving the oscillation frequency and rate of growth or decay (damping) of the eigenmode. As for a structural mode, if one is clever enough to excite only a single aeroelastic eigenmode then an oscillation will occur whose spatial distribution is given by the corresponding eigenvector. However the eigenvalues of an aerodynamic flow are closely spaced together, typically much closer than the eigenvalues for structural modes. Indeed if the aerodynamic computational domain were extended to infinity the eigenvalues would no longer be discrete but rather form a continuous distribution for most aerodynamic flows. Thus exciting only a single aerodynamic mode experimentally is a difficult feat. For some turbomachinery flows with bounded flows between blades in a cascade, discrete well-spaced eigenvalues are possible that have a resonant character. This is also true for some aerodynamic eigenmodes in a wind tunnel, of course. And these have been observed experimentally.

Aeroelastic modes are those that exist when the structural and aerodynamic modes are fully coupled, i.e. oscillations of a fluid mode excite all structural modes and vice versa. In general these aeroelastic modes also have complex eigenvalues and eigenvectors. At low speeds (well below the flutter speed, for example) one may usually identify the structural and aerodynamic eigenvalues separately since the structural/aerodynamic coupling is weak. However as the flutter speed is approached, the eigenvalues may change substantially and the modes are more strongly coupled. It is even possible for a mode that is aerodynamic in origin at low speeds to become the critical flutter mode at higher speeds, although normally it is one or more of the structural modes that becomes unstable as the flow of velocity approaches the flutter speed.

Winther, Goggin and Dykeman [28] have suggested using aeroelastic modes to reduce the total number of modes to be used in a simulation of overall aircraft motion. This seems like an idea worth exploring, although aeroelastic modes by definition vary with flow condition, e.g. dynamic pressure and Mach number, and thus the aeroelastic modes at one flight condition will not be the aeroelastic modes at another. Of course, if one uses a sufficient number of aeroelastic modes they will be able to describe accurately the system dynamics at any flight condition, but that tends to defeat the purpose of minimizing the number of modes in the representation.

Also it should be noted that the particular implementation of aeroelastic modes in Ref. [28] does not include aerodynamic states or modes per se, which limits that particular approach when the aerodynamic modes themselves are active and couple strongly with the structural modes. This is probably the exceptional case, but it can happen.

III. Results

In an earlier summary of work and results on this topic [1], we have discussed 1) comparisons of the reduced order model to classical unsteady incompressible aerodynamic theory, 2) reduced order calculations of compressible unsteady aerodynamics based on the full potential equation, 3) reduced order calculations of unsteady flow about an isolated airfoil based on the Euler equations, 4) reduced order calculations of unsteady viscous flows associated with cascade stall flutter, and 5) linear flutter analyses using reduced order model.

In the present paper, recent results for transonic flows with shock waves including viscous and nonlinear effects are emphasized. Before turning to these however, we consider some fundamental results concerning the effects of spatial discretization and a finite computational domain.

A. The Effects of Spatial Discretization and a Finite Computational Domain

For simplicity we use a classical CFD model, the vortex lattice method for an incompressible potential fluid, to illustrate the points we wish to make. Compressible potential flow models and Euler flow models have provided numerical results consistent with those obtained from the vortex lattice models in this regard. The results discussed here are from Heeg and Dowell [29].

In CFD there are two approximations that are nearly universal to all such models. One is the construction of a computational grid that determines the limits of spatial resolution of the computational model. The second is the approximation of an infinite fluid domain by a finite domain. It is a principal purpose of the present discussion to note that the computational grid not only determines the spatial resolution obtainable by the CFD model, but also the frequency or temporal resolution that can be obtained. Further, as will be shown, the finiteness of the computational domain determines the resolution of the eigenvalue distribution for a CFD model. Both of these observations have important ramifications for assessing the CFD model and its ability to provide an adequate approximation to the original fluid model on which it is founded as well as being helpful in constructing and understanding reduced order models.

In the following discussion we shall consider both discrete time as well as continuous time eigenvalues. Even in a high dimensional system such as usually encountered with CFD, the relationship between any dynamical variable such
as vortex strength, velocity potential, flow velocity, density, pressure, etc. and its time evolution as expressed for the determination of eigenvalues and eigenvectors is a simple one. For a given dynamic variable, \( q \), which changed with time, \( t \), the eigenvalue relationship is

\[ q = Ae^{\lambda t} \]  

(20)

where \( \lambda \) is the continuous eigenvalue. For a discrete time representation where the time step is \( \Delta t \), we define the discrete time eigenvalue, \( z \), as the ratio of \( q \) to its value one time step earlier. It is easily seen then that

\[ z = e^{\lambda t} \]  

(21)

or

\[ \lambda = \log(z)/\Delta t \]  

(22)

It will be useful in our discussion to consider both \( \lambda \) and \( z \).

Here we use the vortex lattice model, because 1) it is one of the simplest CFD models, 2) it has been widely used and 3), among practitioners, it is thought to be well understood in terms of its capability and limitations. As noted earlier, similar results are obtained from more elaborate CFD models which include the effects of flow compressibility, rotationality and/or viscosity.

As an example, we consider the flow over an airfoil with a certain number of vortex elements on the airfoil and in the wake. Initially, we select 20 elements on the airfoil and 360 elements in the wake. The finite length of the wake extends 18 chord lengths. The eigenvalues and eigenmodes of the flow can be computed by now well-established methods.

The eigenvalue distribution for \( \lambda \) is shown in Fig. 1. Note that the real part of the eigenvalue is the damping and the imaginary part is the frequency of the eigenvalue. We now study the effects of 1) refining the grid or vortex lattice spacing and 2) changing the extent of the wake length. Note that in Fig. 1, the baseline configuration’s eigenvalue with the largest imaginary part has a value of 10\( \pi \). If we now halve the number of grid points or double the grid spacing, while maintaining the same total wake length, the total number of eigenvalues remains constant. The frequency range of the original eigenvalues has doubled and the eigenvalues exhibit increased damping. Thus we see that refining the grid has led to increasing the frequency range of the eigenvalue distribution. By contrast the spacing of the eigenvalues in the eigenvalue distribution is little changed by a change in the computational grid spacing.

Now consider what happens as the extent of the wake length is increased while the grid spacing is held constant. In Fig. 2, the baseline configuration is compared to an aerodynamic model that has half as many aerodynamic elements in the wake. Now we see the spacing between eigenvalues has increased by about a factor of two, but the largest imaginary part of the eigenvalue distribution (frequency) is unchanged. Hence, the effect of extending the wake length (for a fixed grid resolution) is to refine the resolution of the eigenvalue distribution, but not to change the maximum frequency of the eigenvalue distribution. A more in depth interpretation of this behavior is given in Heeg and Dowell [29] along with further details and numerical examples.

B. The Effects of Mach Number and Steady Angle of Attack: Subsonic and Transonic Flows

Here we examine some recent results from Thomas, Hall and Dowell [19, 20] and also Florea, Hall and Dowell [30]. The former used an Euler equation flow model with a frequency domain POD method and the latter a transonic potential flow model with a now standard eigenvalue, eigenmode formulation. Although using different flow models and modal representations, the results of these two studies lead to similar conclusions regarding the nature of the flow and the efficacy of a modal representation of the aerodynamic flow.

In Fig. 3, the full eigenvalue spectrum for a flat plate airfoil using coarse computational grid is shown to elucidate the effects of Mach number from \( M = 0 \) to 1.1. These results are from Thomas, et al. [20] Interestingly the eigenvalue spectrum changes notably over this range. For \( M = 0 \), all the eigenvalues are real and negative. Hence none of the eigenmodes have an oscillatory character. For

![Figure 1: Influence of varying the size of the aerodynamic elements of vortex-lattice model. Continuous time eigenvalues, \( \lambda \).](image1)

![Figure 2: Influence of simultaneously varying the size and number of aerodynamic elements in the wake of vortex-lattice model, maintaining a constant wake length. Continuous time eigenvalues, \( \lambda \).](image2)
any Mach greater than zero, however, eigenvalues that are complex conjugate appear along with real eigenvalues. The eigenvalue pattern continues to evolve as the Mach number increases, with another significant change in character in the transonic range from $M = 0.9$ to $1.1$. The corresponding eigenmodes have also been determined including the characteristic pressure distributions on the airfoil. Typically the eigenmode that corresponds to the smallest negative real eigenvalue has a pressure distribution similar to that for steady flow at a constant angle of attack.

As an aside, it is very interesting to note that the eigenvalues for $M = 0$ are distributed along the real axis in Fig. 3, while in Fig. 1 they are distributed along the imaginary axis. In both cases these represent discrete approximations to a branch cut. It is well known from Theodorsen’s theory for $M = 0$ that the branch cut can be placed along any line emanating from near the origin of the complex plane. The results of Fig. 1 and Fig. 3 indicate that different CFD models for the same physical flow may place this branch cut along distinctly different rays from the origin.

In Fig. 4, results are shown for a flat plate at $M = 0.2$ and $0.9$ and a NACA 0008 airfoil at $M = 0.85$. For the latter, a shock is present. Results are shown for a finer mesh typical of CFD calculations and results are shown from the full eigenspectrum and those eigenvalues of the flow obtained using 100 POD modes to construct a reduced order model (ROM). The POD modes were determined using solutions at discrete values, often called “snapshots” in the POD literature, computed at uniformly distributed frequencies in the range $-1.0 < \text{Im}(\lambda) < 1.0$. The dominant eigenmodes are well approximated by the POD/ROM model. Note the characteristic distribution of the eigenvalues as a function of Mach number including with and without shock.

Further results have been obtained for a NASA 64A006 airfoil. The CFD grid is shown in Fig. 5 and the steady flow pressure distribution is shown in Fig. 6. Note a shock is distinctly present for $M > 0.86$. For this airfoil a bending/torsion flutter analysis is conducted over the Mach number range $M = 0.5$ to $0.9$. The flutter boundary is shown in Fig. 7. Root loci for the two dominant aerelastic modes (which originate in the plunging and pitching structural modes at low Mach number) are shown in Fig. 8 for Mach numbers in the range $M = 0.86$ to $0.9$. These root loci show that, in the Mach number range where the position of the shock on the airfoil moves appreciably, the critical eigenmode for flutter changes from the plunging mode to the pitching mode. There is a corresponding and sharp change in the flutter boundary, c.f. Fig. 8. One of the benefits of a reduced order modal representation of the aerodynamic flow is the capability and ease of constructing such root loci which provide a significantly improved understanding of transonic flutter.

We now turn to some complementary results from Florea, et al. [30] who have studied a NACA 0012 airfoil and a MBB A3 airfoil. We present results for only the former airfoil here.

The grids used for the CFD models are shown in Fig. 9. In addition to the basic grid, a refined grid in the vicinity of
the shock wave was also considered. In Fig. 10 the steady flow pressure distribution is shown for $M = 0.75$ and several steady angles of attack. The corresponding eigenvalue distributions are shown in Fig. 11. Somewhat surprisingly perhaps the eigenvalue distribution does not change radically with angle of attack changes even though the flow at zero angle of attack is shockless, while that a $2^\circ$ angle of attack has a strong shock.

Another comparison of eigenvalue distributions is shown in Fig. 12 where the angle of attack is held at zero, but a range of Mach numbers is considered. While over the full range of Mach number the eigenvalue distribution does change, there is no radical change in the high subsonic, transonic range per se.

Finally in Fig. 13, a comparison is shown between the...
results of the full CFD model (over 5000 degrees of freedom) and those from a reduced order model. Two different versions of the reduced order are used with 67 and 160 degrees of freedom respectively. Good correlation is obtained between the full CFD model and the ROM for lift and moment on an oscillating airfoil over a wide range of reduced frequency.

C. The Effects of Viscosity

Epureanu, Hall and Dowell [31] have considered the effects of viscosity using the POD methodology in the frequency domain. Earlier results by Florea, Hall and Cizmas using the direct eigenvalue approach were discussed in Ref. [32]. The results of Epureanu et al. are for a cascade of airfoils. The basic flow model uses a potential description in the outer inviscid region and a simplified boundary layer model in the inner region. The solution domain is shown in Fig. 14 and Fig. 15. A comparison of results from this model with a Navier-Stokes solver has shown good agreement. Comparisons have also been made with experimental data showing reasonable correlation.

Representative comparisons are shown between the full CFD model and a POD/ROM in Fig. 16 for a pressure distribution at fixed interblade phase angle and frequency, in Fig. 17 for lift versus interblade phase angle for a fixed frequency, and in Fig. 18 for lift versus reduced frequency for a fixed interblade phase angle. The results are in generally good agreement given the complexity of the flow. A reduction in degrees of freedom by two orders of magnitude or more is realized for this example.

D. Nonlinear Aeroelastic Reduced Order Models

One of the remaining challenges is to construct nonlinear aerodynamic reduced order models. An example of a shock wave undergoing large oscillations in a one-dimensional channel has been treated by Hall et al. [33]. However, no results from reduced order models for flows about an airfoil undergoing large motions have yet been reported in the literature.

On the other hand, an example wing problem has been examined with a linear ROM vortex lattice aerodynamic model and a nonlinear structural model for a delta wing. The details are presented by Tang and Dowell [34]. Physically we are examining a low Mach number, small angle of attack flow about a plate-like structure undergoing oscillations on the order of the plate thickness. For a plate, oscillations of this magnitude give rise to a strong geometric structural nonlinearity.

The consequence of this structural nonlinearity is that once the flutter speed is exceeded the wing goes into a limit cycle oscillation (LCO) of bounded amplitude. Of course, a purely linear aeroelastic model would predict an exponentially growing oscillation for flow conditions beyond the flutter boundary.

In Fig. 19, the delta wing geometry is shown along with the aerodynamic grid used for the vortex lattice model. In the aeroelastic ROM, seven aerodynamic modes are combined with ten structural modes for a total of seventeen aeroelastic degrees of freedom. The structural modes are
extracted from a finite element (FE) code which has on the order of one thousand degrees of freedom. There are a similar number of degrees of freedom in the vortex lattice fluid model. Solutions to the aeroelastic ROM are obtained by time marching the equations of motion. Were we to attempt to time march the original CFD plus FE codes, the computational would be increased by several orders of magnitude. Some solutions were obtained using the original CFD vortex lattice code combined with a structural modal model to check the results of our aeroelastic reduced order model. The agreement is excellent.

In Fig. 20, we show the results of our aeroelastic ROM for the flutter boundary as a function of sweep angle for the delta wing. Experimental results from an earlier study by Doggett and Soistmann [35] are also shown and provide good correlation.

In Fig. 21 results are shown for the LCO amplitude and frequency as a function of flow velocity. Experimental correlation is provided from a test conducted in the Duke Wind Tunnel. Again good correlation between theory and experiment is shown.
IV. Concluding Remarks and Directions for Future Research

So where do we stand and where might we go? Where we stand is that powerful new approaches to modeling unsteady aerodynamic flows have been developed. They provide a level of accuracy and computation efficiency not previously available. In particular, construction of reduced order models based upon rigorous fluid dynamical theory is now possible to 1) calculate true damping and frequency for all aeroelastic modes at all parameter conditions; 2) provide a practical approach for constructing highly efficient, accurate aerodynamic models suitable for designing control laws and hardware for aeroelastic systems; and 3) make the use of CFD models routine in aeroelastic analysis.

This is a considerable achievement. What might the future bring?

1) **Fully (dynamically) nonlinear models.** We should be able to develop rigorous reduced order models that will accurately model large and violent aircraft motions.

2) **Aeroacoustics.** The eigenmode-reduced order model should work well here also, but far field boundary conditions will need special attention for this (or any other) approach. See Ref. [36] for a discussion of the present state-of-the-art in computational aeroacoustics.

3) **Turbulence and turbulence models.** If we use a standard turbulence model, e.g., k-e, etc., then the present method formally goes through. However it is possible that the real value of the eigenmodal reduced order model approach will be to encourage the development of better tur-
Stagnation Point

Figure 15: Solution domain used to calculate the viscous flow. Special local analytic solution is used at the stagnation point. The system of coordinates along the airfoil surface is indicated by $\zeta$ and $\eta$. A typical displacement thickness is sketched along the airfoil and wake.

Figure 16: Real part of the coefficient of pressure $C_P$ obtained using 25 POD modes when the interblade phase angle $\sigma$ is 90°, the reduced frequency $k$ is 0.85, and the upwind far field Mach number $M$ is 0.5.

Figure 17: Real part of the coefficient of lift $C_L$ obtained using 25 POD modes when the reduced frequency $k$ is 0.85, and the upwind far field Mach number $M$ is 0.5.

Figure 18: Real part of the coefficient of lift $C_L$ obtained using 25 POD modes when the interblade phase angle $\sigma$ is 90°, and the upwind far field Mach number $M$ is 0.5.

Is it possible that one could attack the full Navier-Stokes equations using the eigenmode-reduced order model methodology? The answer is that in some sense such work has already begun. The classical hydrodynamic stability theory is based upon the boundary layer approximation combined with a highly simplified geometry, a flat plate of infinite extent. However that work per se, now some fifty to seventy years ago in its origins, did not lead to advances beyond the limitations of the classical infinite geometry. Already, models with an outer inviscid model combined with viscous boundary layer theory have been developed, and some encouraging preliminary results are emerging[32]. Thus one might hope to overcome that classical geometrical limitation and treat the larger scale viscous motions about an airfoil or wing. With these large scale motions determined, it might even be possible to refine the eigenmode representation to determine local flow behavior. Clearly this is only a hypothesis, but a very intriguing one. Also, the recent work of Tang, Graham and Peraire [37] on reduced order models in low Reynolds number flows using KL modes is interesting to note in this regard.

Finally, for those (including the authors) who still find fascination and challenge in the classical models, it would be very interesting to explore the question of in what sense do discrete, but closely spaced, eigenvalues represent a branch cut in two or three dimensional, fully linearized potential flow.

In five or ten years, some of these questions should have definitive answers and very likely others will supersede them. If the reader is encouraged or inspired to explore eigenmodes for unsteady aerodynamic flows, the present paper will have served its purpose.

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Figure 19: Aeroelastic model of delta wing using vortex-lattice aerodynamic model.

Figure 20: Variation of flutter velocity index and flutter frequency ratio with sweep angle.

Figure 21: Theoretical and experimental nondimensional transverse velocity (top) and frequency (bottom) of the limit cycle oscillation of a plastic wing model.

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