Dynamics of Negative Advertisement Strategies

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Abstract

We develop a multi-period model of dynamically evolving market shares. Firms compete by choosing long-run investment strategies that determine customer retention and new customer attraction in every period. We investigate equilibrium behavior and focus on conditions for emergence of negative strategies, i.e., strategies that put more emphasis on hurting the competitor's market share than on increasing one's own market share. We show that negative strategies are more likely to emerge in a saturated market and to be triggered by firms that lag behind the market share leader, and that negative strategies are unlikely to emerge in the final period of the competition.

1 Introduction

Negative advertising and activities whose primary goal is hurting a competitor are not standard business practices. Yet, businesses occasionally engage in direct attacks on their competitors. This is best observed in consumer goods industry since negative attacks are often prominent in advertising campaigns aimed at the general public. Several advertising campaigns in the soft drink industry during the "cola wars" between Coca-Cola Company and PepsiCo during 1980s and 1990s are examples of such practices. More recent examples include the Apple and Microsoft ad war with Apples's "Get a Mac" campaign and Microsoft's "I am a PC" response, and the Verizon Wireless and AT&T ad war that started with Verizon's attack "There is a map for that" ad campaign followed by AT&T's "Set the record straight" response. (Beard, 2010, documents and discusses numerous examples.)

While businesses generally stay away from attacking competitors and tend to focus on directly helping their bottom line, negative advertising and promotional activities aimed at competitors appear to be widespread in political contests. The topic of negative advertising and negative political campaigns is well-studied in political science literature (e.g., Skaperdas and Grofman, 1995; Lau et al., 1999; Che et al., 2007; Lau et al., 2007; Lau and Rovner, 2009).

There are many attempts to explain the emergence of negative advertising strategies, ranging from emotional decision-making to intensity of competition (Beard, 2010 tries to classify motives for several prominent negative campaigns). Given the prevalence of negative campaigns in politics, it is natural to conjecture that some of the distinguishing features of political competition ought to be the features found in any environment in which negative advertising is likely to happen. The most prominent such feature is the tournament nature of political contests: the winner does not care about the number of votes (i.e., market share) as long as it has more votes than the opponents (relative market share). In addition, in some political contests there is a winner-take-all effect since the election loser is locked out of any political power. (Quelch and Jocz, 2007, use this argument to explain the discrepancy in the use of negative advertising in politics vs. business competitions.) Another prominent feature of political contests is that there is clearly defined ending time: the election day (which somewhat compares to selling perishable seasonal goods case in business competitions). If the competition has no predefined ending, engaging in negative advertising allows for a possibility of retaliation from competitors which would mitigate the incentive to engage in negative advertising in the first place. With a predefined ending time of the competition (or if there exists a small number of critical time points in the competition in which most of the utility will be realized), it is plausible that negative advertising strategies could emerge just before that moment so that no time is left for opponents to implement possible retaliatory strategies. The infamous "October surprise" in US Presidential Elections, i.e., expectations of some negativity emerging within the last couple of weeks of the campaign, fits within such an explanation.

In this paper, we develop a theoretical model with the goal of understanding conditions for negative advertising strategies to emerge as (a rational) equilibrium behavior, as well as conditions under which a decision to engage in negative advertising would be sub-optimal.

Our model, formally defined in Section 2, tracks changes in market-shares over finitely many time periods. Market-shares change as a result of the activities of all players (firms, candidates). In each time period, players implement costly investment decisions and these decisions jointly impact market-shares in the next period. Players are assumed to be utility maximizing and their utility is a time-discounted sum of utilities from each time period. The utility in each time period is an increasing linear function of the player's market share, minus an increasing linear function of the opponents' market share (if opponents' market share linear coefficient is zero, the player cares only about its own market-share in that period), minus the cost of investments the player made in that period. We study equilibrium behavior in this model.

Since players are *ex ante* symmetric in the model, we focus on a single player's activities. In order to facilitate our analysis, we make a simplifying assumption of aggregating the behavior of all opponents into a single combined opponent. Thus, the model we present and analyze has two players competing for market-shares.

The modeling choice that is of critical importance for any relevance and meaningfulness of our results is the definition of negative advertising strategies: a player's advertising strategy at a given time period t is negative if the player's decisions are aimed more at decreasing the opponent's market share than at increasing one's own market share. Our definition of negative advertising is similar to that of Skaperdas and Grofman (1995). (However, our model captures more general behavior, since Skaperdas and Grofman' framework is static, it does not involve equilibrium analysis, and it focuses on player utilities that are relevant for political competition only.) Negative advertising has been modeled and studied extensively in marketing. While the majority of marketing literature deals with empirical and experimental studies focusing on effects of negativity on consumers (e.g., Shiv et al., 1997), defining negative advertising strategies is mostly discussed as a special case of comparative advertising "in which a differentiative technique is employed" (James and Hansel, 1991). However, several theoretical models in marketing have similar features to our model. The core idea of evolving market shares through targeted investment of players is similar to that behind targeted advertising strategies (e.g., Iyer et al., 2005), while optimization behind decision-making in our dynamic multi-period model has a similarity to dynamic optimal control models in advertising (Heller and Chakrabarti, 2002; Feichtinger et al., 1994). The formulation of the dynamic game in our model and use of the Lagrange method in numerical computations is somewhat similar to the formulation and analysis of a dynamic game presented in Chow (1997)

We find that, in our model, negative advertising strategies could emerge as an equilibrium behavior both in political competition and in business competition. There are several parameters relevant for the existence of equilibria with negative advertising strategies. For example, it is not surprising to see that negative advertising strategies emerge when (i) the player cares about the opponent's market share and (ii) it is much more costly to advertise with the goal of increasing one's market share than to invest in advertising aimed at reducing opponent's market share. (We model costs to be convex.)

A parameter that turns out to be relevant is the initial market share endowment of the player relative to the market saturation (i.e, size of the market which is captured by the player or the opponent). We show that players who are lagging behind are more likely to engage in negative advertising than players who are market leaders. This fits within the examples mentioned in the opening paragraph (PepsiCo vs. Coca Cola, Apple vs. Microsoft, Verizon vs. AT&T): the company who triggered a negative advertising war was lagging behind the market leader. In terms of political competition, this result indicates that the candidate who is lagging behind is more likely to turn to negative campaigning, which is also found to hold in the models of Skaperdas and Grofman (1995) and Harrington and Hess (1996). Furthermore, we show that negative advertising is more likely to emerge when players have similar market shares in a saturated market (i.e., when the proportion of uncommitted customers/voters is small). This aligns with empirical findings of Lovett and Shachar (2011) who analyzed advertising data from several US congressional races.

The dynamic multi-period feature of our model yields some insights into the timing of the start of negative advertising activities. We find that it is nearly impossible (except for some extreme, unrealistic choices for cost functions) for negative advertising to emerge in the last period of the game, and in any period of the game in which the player's own market share is of significant importance to their utility. Under the assumption that businesses overwhelmingly focus on the present and on the immediate future, the latter can be viewed as an argument why negative advertising is not prevalent in business competition.

The paper is organized as follows. In the next section, we define the basic model. In Section 3 we discuss how the Luce choice model can be reduced to our basic model, define negative advertising strategies and discuss equilibrium existence. We then, in Section 4, discuss the two-period game model which allows us to present most of our findings. In Section 5, we discuss several generalizations of the model, including the case of more than two periods, and different forms of the cost functions. Proofs and some technical generalizations are relegated to Appendices.

2 A Dynamic Model of Market-Share Competition

There are two players in the market and the market structure is observed in finitely many discrete time periods $t = 0, 1, \ldots, T$. The players are denoted 1 and 2. The market shares at time t are given by a non-negative vector

$$V^{(t)} = (V_0^{(t)}, V_1^{(t)}, V_2^{(t)}),$$

with $V_0^{(t)} + V_1^{(t)} + V_2^{(t)} = 1$. The market share of player $i = 1, 2$ at time t is denoted by $V_i^{(t)} \geq 0$; the proportion of the market not captured by the players is $V_0^{(t)} \geq 0$.

The changes in market shares from period t to period $t+1$ $(t = 0, 1, ..., T-1)$ are defined by the transition matrix

$$P^{(t)} = \begin{bmatrix} p_{00}^{(t)} & p_{01}^{(t)} & p_{02}^{(t)} \\ p_{10}^{(t)} & p_{11}^{(t)} & p_{12}^{(t)} \\ p_{20}^{(t)} & p_{21}^{(t)} & p_{22}^{(t)} \end{bmatrix} \tag{1}$$

where $p_{ij}^{(t)} \geq 0$, and $\sum_j p_{ij}^{(t)} = 1$ for $i = 0, 1, 2$. Therefore, $p_{ij}^{(t)}$ is the proportion of customers/voters from $V_i^{(t)}$ who will transition to $V_j^{(t+1)}$. The market shares at period $t+1$ are defined by

$$V^{(t+1)} = V^{(t)} P^{(t)} \tag{2}$$

2.1 Market Share Transitions

Player 1 and Player 2 can both potentially invest into market share transitions from i to j. Let the intensity or magnitude of Player 1 and 2 investment effort aimed at transitioning consumers/voters from i to j be $m_{ij1}^{(t)}$ and $m_{ij2}^{(t)}$, respectively with $m_{ij1}^{(t)} \in [0, \overline{m}]$, $m_{ij2}^{(t)} \in [0, \overline{m}]$ and $t = 0, 1, ..., T-1$. (This magnitude could be thought of as a monetary investment aimed at transitioning customers/voters from $v_i^{(t)}$) to $V_j^{(t+1)}$, bounded by some large number $\overline{m}$.)

Market share transitions are then determined according to a generalized Luce's choice axiom (Luce, 1959),

$$p_{ij}^{(t)} = \begin{cases} \frac{h\left(m_{ij1}^{(t)}+m_{ij2}^{(t)}\right)}{h\left(m_{i01}^{(t)}+m_{i02}^{(t)}\right)+h\left(m_{i11}^{(t)}+m_{i12}^{(t)}\right)+h\left(m_{i21}^{(t)}+m_{i22}^{(t)}\right)} & \text{if } \sum_{j\in\{0,1,2\}}\left(m_{ij1}^{(t)}+m_{ij2}^{(t)}\right)\neq 0, \\ 1/3 & \text{if } \sum_{j\in\{0,1,2\}}\left(m_{ij1}^{(t)}+m_{ij2}^{(t)}\right)=0 \end{cases} \tag{3}$$

for $i, j \in \{0, 1, 2\}$,

where $h(\cdot)$ is continuous, k-homogeneous ($k > 0$), strictly increasing function with $h(0) = 0$. Note that $m_{ij1}^{(t)} + m_{ij2}^{(t)}$ is the aggregate magnitude of investment aimed at transitioning customers/voters from $V_i^{(t)}$ to $V_j^{(t+1)}$, so with h set to be the identity function, we get the classical Luce's choice axiom. Function h can be understood as an indirect utility function of a representative customer/voter given the consumption price level. The choice of $p_{ij}^{(t)} = 1/3$ in case of no investment at all targeting customers/voters from V_i^t, addition, defining $p_{ij}^{(t)} = 1/3$ when $\sum_j m_{ij1}^{(t)} + m_{ij2}^{(t)} = 0$ is needed for completeness reasons. (It follows from $\lim_{m\to 0} h(m)/3h(m)$, but is not important since such zero aggregate investment is always a dominated strategy, as will be shown later in Proposition 2.1.)

2.2 Investment costs and the utility function

The cost functions associated with magnitudes are $\widetilde{C}_{ij1}(m_{ij1}^{(t)})$ and $\widetilde{C}_{ij2}(m_{ij2}^{(t)})$. Therefore, we allow costs to depend on a particular transition from $V_i^{(t)}$ to $V_j^{(t+1)}$ that is being targeted by an investment (as well as on the identity of the player who is investing). We assume that the cost functions are continuous, increasing, and convex (although we will show that the existence of the equilibrium does not require the convexity). We also normalize costs at zero investment level: $\widetilde{C}_{ij1}(0) = \widetilde{C}_{ij2}(0) = 0$.

Player 1's lifetime utility is

$$\sum_{t=0}^{T} r^t[\delta_1^{(t)}V_1^{(t)} - \beta_1\delta_2^{(t)}V_2^{(t)}] - \sum_{t=0}^{T-1}\sum_{i,j\in\{0,1,2\}} r^t[\widetilde{C}_{ij1}(m_{ij1}^{(t)})]$$

where $r \in (0, 1]$ is the time discount factor and $\delta_i^{(t)}$ is the time-varying weight for player i

market share at time t. Therefore, $\delta_i^{(t)} V_i^{(t)}$ describes revenues drawn from player i market share at time t; since the total market volume is normalized, δ_i^t/N determines player's revenue per customer/voter in period t if there are N potential customers/voters in the whole market. The parameter β_i is the time-invariant weight that player i puts on the impact of market share of its opponent. (If opponent's market share does not directly affect player's utility, $\beta_i = 0$) Analogously, player 2's lifetime utility is

$$\sum_{t=0}^{T} r^t[\delta_2^{(t)} V_2^{(t)} - \beta_2 \delta_1^{(t)} V_1^{(t)}] - \sum_{t=0}^{T-1} \sum_{i,j \in \{0,1,2\}} r^t[\widetilde{C}_{ij2}(m_{ij2}^{(t)})]$$

This utility function attempts to capture various possibilities.

For example, setting $\delta_i^{(t)} = 0$ for $t = 0, ..., T-1$, allows only the market share in the final period T to affect players' utilities. Such a situation could occur when the market share captured by a player is converted into utility only in a predetermined consumption moment for customers. This setting could fit highly seasonal sales and sales of perishable goods and is an obvious fit for political competition since players (candidates) compete for votes cast and the market shares are only relevant on the election day.

For another example, the parameter β_i determines the impact of the opponent's revenues on the player i's utility. For $\delta_1^{(t)} = \delta_2^{(t)}$, $\beta_i = 1$ allows for the difference of market shares to impact the utility. This is similar to an important distinction of political competition in which the actual market share is secondary to the relative market share, i.e., it is important to have more votes than the opponent, regardless of the actual vote count. (For winner-takes-all political contests, it would be more appropriate to use rank functions that allocate revenues only to the market-share leader, i.e., the election winner, but such function is non-linear and not tractable for our analysis. Our formulation still captures the effect of relative standing in market-shares while keeping the utility function tractable.)

2.3 Strategy choices and equilibria

The players want to maximize their own lifetime utility by simultaneously choosing magnitudes of their investments of all transitions throughout the game.

Thus, player 1 chooses

$$m_{ij1}^{(t)} \in [0, \overline{m}], \quad i, j = 0, 1, 2; \;\; t = 0, \ldots, T-1,$$

and player 2 chooses

$$m_{ij2}^{(t)} \in [0, \overline{m}], \quad i, j = 0, 1, 2; \;\; t = 0, \ldots, T-1.$$

The equilibrium choices will be denoted by $m_{ij1}^{(t)*}$ and $m_{ij2}^{(t)*}$.

Note that all choices have to be made at time $t = 0$ when only initial market shares $V_0^{(0)} + V_1^{(0)} + V_2^{(0)} = 1$ are known. This model fits situations in which investments in particular targeted advertising activities have to be made in advance, such as having to book ad slots ahead of time. (We also note that the equilibria that we discuss in this paper are subgame perfect with respect to t, i.e., the equilibria in this simultaneous one-shot game model coincides with equilibria in a sequential game model where choices $m_{ij1}^{(t)}, m_{ij2}^{(t)}$ are made sequentially at each time period $t = 0, \ldots T-1$ with knowledge of history and $V_0^{(t)} + V_1^{(t)} + V_2^{(t)} = 1$.)

In our model, zero aggregate magnitude investment aimed at targeting customers/voters in V_i, $i = 0, 1, 2$ cannot be an equilibrium choice.

Proposition 2.1 *For any* $i = 0, 1, 2$ *and any* $t = 0, \ldots, T|!-|!1$, $\sum_{j \in \{0,1,2\}} \left(m_{ij1}^{(t)*} + m_{ij2}^{(t)*} \right) > 0$.

Proof. Suppose that $\sum_{j \in \{0,1,2\}} \left(m_{ij1}^{(t)*} + m_{ij2}^{(t)*} \right) = 0$. Then, either of the players has an incentive to deviate from such magnitude choices. Consider player 1. Player 1's utility function is continuous and increasing in $V_i^{(t+1)}$ and, thus continuous and increasing in $p_{i1}^{(t)}$.

Cost function $\widetilde{C}_{i11}$ is continuous and $\widetilde{C}_{ij1}(\epsilon)$ can be arbitrarily small for arbitrarily small ϵ. However, investing $m_{i11} = \epsilon$ instead of zero increases $p_{i1}^{(t)}$ from 1/3 to 1. Therefore, $m_{i11}^{(t)*} \neq 0$. ■

In view of Proposition 2.1, we will consider the intensity decision with positive aggregate intensities in the rest of this paper. Hence, player 1's optimization problem is

$$\max_{\{m_{ij1}^{(t)}:\ t=0,\ \ldots,\ T-1;\ i=0,\ 1,\ 2;\ j=0,\ 1,\ 2\}} \sum_{t=0}^{T} r^t[\delta_1^{(t)} V_1^{(t)} - \beta_1 \delta_2^{(t)} V_2^{(t)}] - \sum_{t=0}^{T-1} \sum_{i,j\in\{0,1,2\}} r^t[\widetilde{C}_{ij1}(m_{ij1}^{(t)})]$$

Subject to

$V^{(t+1)} = V^{(t)} P^{(t)}$,

$1 = V_0^{(t)} + V_1^{(t)} + V_2^{(t)}$,

$p_{ij}^{(t)} = h\left(m_{ij1}^{(t)} + m_{ij2}^{(t)*}\right) / \left(h\left(m_{i01}^{(t)} + m_{i02}^{(t)*}\right) + h\left(m_{i11}^{(t)} + m_{i12}^{(t)*}\right) + h\left(m_{i21}^{(t)} + m_{i22}^{(t)*}\right)\right),\quad i,j \in \{0,1,2\}$,

$V_0^{(t+1)},\ V_1^{(t+1)},\ V_2^{(t+1)} \geq 0,\ 0 \leq m_{ij2}^{(t)} \leq \overline{m}$,

$\sum_{j\in\{0,1,2\}} \left(m_{ij1}^{(t)*} + m_{ij2}^{(t)}\right) \geq \varepsilon > 0$,

$V_0^{(0)}, V_1^{(0)}, V_2^{(0)}$ are given.

We impose the Nash equilibrium concept and thus it is noticed that $m_{ij2}^{(t)*}$ for $i,j \in \{0,1,2\}$ are equilibrium values chosen by player 2. The constant$\varepsilon > 0$ can be imagined as the smallest unit of money. Player 2's problem is similar

$$\max_{\{m_{ij2}^{(t)}:\ t=0,\ \ldots,\ T-1;\ i=0,\ 1,\ 2;\ j=0,\ 1,\ 2\}} \sum_{t=0}^{T} r^t[\delta_2^{(t)} V_2^{(t)} - \beta_2 \delta_1^{(t)} V_1^{(t)}] - \sum_{t=0}^{T-1} \sum_{i,j\in\{0,1,2\}} r^t[\widetilde{C}_{ij2}(m_{ij2}^{(t)})]$$

Subject to

$V^{(t+1)} = V^{(t)} P^{(t)}$,

$1 = V_0^{(t)} + V_1^{(t)} + V_2^{(t)}$,

$p_{ij}^{(t)} = h\left(m_{ij1}^{(t)*} + m_{ij2}^{(t)}\right) / \left(h\left(m_{i01}^{(t)*} + m_{i02}^{(t)}\right) + h\left(m_{i11}^{(t)*} + m_{i12}^{(t)}\right) + h\left(m_{i21}^{(t)*} + m_{i22}^{(t)}\right)\right)$

for $i,j \in \{0,1,2\}$,

$V_0^{(t+1)},\ V_1^{(t+1)},\ V_2^{(t+1)} \geq 0,\ 0 \leq m_{ij1}^{(t)} \leq \overline{m}$,

$\sum_{j\in\{0,1,2\}} \left(m_{ij1}^{(t)} + m_{ij2}^{(t)*}\right) \geq \varepsilon$, ε is a small positive number,

$V_0^{(0)}, V_1^{(0)}, V_2^{(0)}$ are given.

Theorem 2.2 *The game has a (mixed strategy) Nash equilibrium.*

Proof. We use Glicksberg's Fixed Point Theorem (Glicksberg, 1952).

Obviously, the strategy space is not empty. Therefore, the optimization problems for player 1 and 2 are both feasible.

The strategy space of the optimization problem is a compact metric space, since $0 \leq m_{ij1}^{(t)} \leq \overline{m}$, $0 \leq m_{ij2}^{(t)} \leq \overline{m}$, $\sum_{j \in \{0,1,2\}} \left(m_{ij1}^{(t)} + m_{ij2}^{(t)} \right) \geq \varepsilon$ where ε is a small positive number, and the constraints are continuous functions.

The utility functions are continuous by definition.

Hence, the equilibrium exists by direct application of the Glicksbergs Fixed Point Theorem. ■

Remark:

1. Note that the theorem does not require the cost functions to be convex.
2. The existence of a pure strategy Nash equilibrium requires the utility function to be quasi-concave and the strategy be set to a non-empty convex, and compact set. Therefore, the only requirement that is not automatically satisfied is convexity.

Assumption 1. Player 1 sets $m_{101}^{(t)}$, $m_{121}^{(t)}$, $m_{221}^{(t)}$, and $m_{021}^{(t)}$ to zero, while Player 2 sets $m_{202}^{(t)}$, $m_{212}^{(t)}$, $m_{112}^{(t)}$, and $m_{012}^{(t)}$ to zero.

This assumption is intuitive, players will not invest in the activities that hurt themselves.

Lemma 2.3 *Let the equilibrium intensities be $m_{ij1}^{(t)*}$ and $m_{ij2}^{(t)*}$. With Assumption 1, Player 1 and Player 2's equilibrium transition probabilities will not be affected by rescaling the intensities, i.e., $(\lambda_i^*)^{1/k} m_{ij1}^{(t)*}$ and $(\lambda_i^*)^{1/k} m_{ij2}^{(t)*}$, where*

$$\lambda_1^* = 1/\left(h\left(m_{102}^{(t)*}\right) + h\left(m_{111}^{(t)*}\right) + h\left(m_{122}^{(t)*}\right)\right),$$
$$\lambda_2^* = 1/\left(h\left(m_{201}^{(t)*}\right) + h\left(m_{211}^{(t)*}\right) + h\left(m_{222}^{(t)*}\right)\right),$$
$$\text{and } \lambda_0^* = 1/\left(h\left(m_{001}^{(t)*} + m_{002}^{(t)*}\right) + h\left(m_{011}^{(t)*}\right) + h\left(m_{022}^{(t)*}\right)\right).$$

Moreover, given cost functions associated with intensities, there exists a unique new set of continuous and strictly increasing cost functions, i.e.,

$$C_{ij1}\left(\cdot\right) \triangleq \widetilde{C}_{ij1}\left(\left(\lambda_i^*\right)^k h^{-1}\left(\cdot\right)\right),$$
$$\text{and } C_{ij2}\left(\cdot\right) \triangleq \widetilde{C}_{ij2}\left(\left(\lambda_i^*\right)^k h^{-1}\left(\cdot\right)\right),$$

associated with market share transitions, among which player 1 chooses $\left\{p_{11}^{(t)}, p_{21}^{(t)}, p_{20}^{(t)}, p_{01}^{(t)}, p_{001}^{(t)}\right\}$, *player 2 chooses* $\left\{p_{22}^{(t)}, p_{12}^{(t)}, p_{10}^{(t)}, p_{02}^{(t)}, p_{002}^{(t)}\right\}$, *and* $p_{00}^{(t)} = h\left(h^{-1}\left(p_{001}^{(t)}\right) + h^{-1}\left(p_{002}^{(t)}\right)\right)$, *such that it is equivalent to model with intensities and with market share transitions.*

Proof. With Assumption 1, market transitions are

$$p_{11}^{(t)} = \frac{h\left(m_{111}^{(t)}\right)}{h\left(m_{102}^{(t)}\right)+h\left(m_{111}^{(t)}\right)+h\left(m_{122}^{(t)}\right)};$$
$$p_{12}^{(t)} = \frac{h\left(m_{122}^{(t)}\right)}{h\left(m_{102}^{(t)}\right)+h\left(m_{111}^{(t)}\right)+h\left(m_{122}^{(t)}\right)};$$
$$p_{10}^{(t)} = \frac{h\left(m_{102}^{(t)}\right)}{h\left(m_{102}^{(t)}\right)+h\left(m_{111}^{(t)}\right)+h\left(m_{122}^{(t)}\right)};$$

$$p_{22}^{(t)} = \frac{h\left(m_{222}^{(t)}\right)}{h\left(m_{201}^{(t)}\right)+h\left(m_{211}^{(t)}\right)+h\left(m_{222}^{(t)}\right)};$$
$$p_{21}^{(t)} = \frac{h\left(m_{211}^{(t)}\right)}{h\left(m_{201}^{(t)}\right)+h\left(m_{211}^{(t)}\right)+h\left(m_{222}^{(t)}\right)};$$
$$p_{20}^{(t)} = \frac{h\left(m_{201}^{(t)}\right)}{h\left(m_{201}^{(t)}\right)+h\left(m_{211}^{(t)}\right)+h\left(m_{222}^{(t)}\right)};$$

and

$$p_{00}^{(t)} = \frac{h\left(m_{001}^{(t)}+m_{002}^{(t)}\right)}{h\left(m_{001}^{(t)}+m_{002}^{(t)}\right)+h\left(m_{011}^{(t)}\right)+h\left(m_{022}^{(t)}\right)};$$
$$p_{01}^{(t)} = \frac{h\left(m_{011}^{(t)}\right)}{h\left(m_{001}^{(t)}+m_{002}^{(t)}\right)+h\left(m_{011}^{(t)}\right)+h\left(m_{022}^{(t)}\right)};$$
$$p_{02}^{(t)} = \frac{h\left(m_{022}^{(t)}\right)}{h\left(m_{001}^{(t)}+m_{002}^{(t)}\right)+h\left(m_{011}^{(t)}\right)+h\left(m_{022}^{(t)}\right)}.$$

Let the equilibrium intensities be $m_{ij1}^{(t)*}$ and $m_{ij2}^{(t)*}$. Then we can normalize the denominators by introducing

$$\lambda_1^* \triangleq 1/\left(h\left(m_{102}^{(t)*}\right)+h\left(m_{111}^{(t)*}\right)+h\left(m_{122}^{(t)*}\right)\right),$$

$$\lambda_2^* \triangleq 1/\left(h\left(m_{201}^{(t)*}\right)+h\left(m_{211}^{(t)*}\right)+h\left(m_{222}^{(t)*}\right)\right),$$

and

$$\lambda_0^* \triangleq 1/\left(h\left(m_{001}^{(t)*}+m_{002}^{(t)*}\right)+h\left(m_{011}^{(t)*}\right)+h\left(m_{022}^{(t)*}\right)\right).$$

Take $p_{11}^{(t)*}$ for example,

$$\begin{aligned} p_{11}^{(t)*} &= \frac{h\left(m_{111}^{(t)*}\right)}{h\left(m_{102}^{(t)*}\right)+h\left(m_{111}^{(t)*}\right)+h\left(m_{122}^{(t)*}\right)} \\ &= \frac{\lambda_1^* h\left(m_{111}^{(t)*}\right)}{\lambda_1^*\left(h\left(m_{102}^{(t)*}\right)+h\left(m_{111}^{(t)*}\right)+h\left(m_{122}^{(t)*}\right)\right)} \\ &= \lambda_1^* h\left(m_{111}^{(t)*}\right). \end{aligned}$$

Since $h(\cdot)$ is homogeneous degree k, and is defined on $[0,\infty)$,

$$p_{11}^{(t)*} = h\left(\left(\lambda_1^*\right)^{1/k} m_{111}^{(t)*}\right).$$

Therefore, if we define

$$\begin{aligned} m_{111}^{(t)**} &\triangleq \left(\lambda_1^*\right)^{1/k} m_{111}^{(t)*} \\ m_{102}^{(t)**} &\triangleq \left(\lambda_1^*\right)^{1/k} m_{102}^{(t)*} \\ \text{and } m_{122}^{(t)**} &\triangleq \left(\lambda_1^*\right)^{1/k} m_{122}^{(t)*}, \end{aligned}$$

the transition is just the function of the intensity, i.e, $p_{11}^{(t)*} = h\left(m_{111}^{(t)**}\right)$. It is noticed that $m_{111}^{(t)**}$, $m_{102}^{(t)**}$, and $m_{122}^{(t)**}$are not necessarily in the range $[0,\overline{m}]$ Similarly,

$$p_{12}^{(t)*} = h\left(m_{122}^{(t)**}\right),\ p_{10}^{(t)*} = h\left(m_{102}^{(t)**}\right).$$

Let the cost function associated with market share transition $p_{11}^{(t)*}$ be $C\left(p_{11}^{(t)*}\right)$. To match the utility function, we require

$$C_{11}\left(p_{11}^{(t)*}\right)=\widetilde{C}_{111}\left(m_{111}^{(t)*}\right)=\widetilde{C}_{111}\left(\left(\lambda_1^*\right)^k m_{111}^{(t)**}\right)=\widetilde{C}_{111}\left(\left(\lambda_1^*\right)^k h^{-1}\left(p_{11}^{(t)*}\right)\right).$$

$h^{-1}(\cdot)$ exists since $h(\cdot)$ is strictly increasing. Similarly,

$$C_{12}\left(p_{12}^{(t)*}\right)=\widetilde{C}_{122}\left(\left(\lambda_1^*\right)^k h^{-1}\left(p_{12}^{(t)*}\right)\right),$$

and

$$C_{10}\left(p_{10}^{(t)*}\right)=\widetilde{C}_{102}\left(\left(\lambda_1^*\right)^k h^{-1}\left(p_{10}^{(t)*}\right)\right).$$

Similarly, we get cost functions associated with market share transitions on party 2.

$$\begin{aligned}
C_{22}\left(p_{22}^{(t)*}\right)&=\widetilde{C}_{222}\left(\left(\lambda_2^*\right)^k h^{-1}\left(p_{22}^{(t)*}\right)\right),\\
C_{21}\left(p_{21}^{(t)*}\right)&=\widetilde{C}_{211}\left(\left(\lambda_2^*\right)^k h^{-1}\left(p_{21}^{(t)*}\right)\right),\\
C_{20}\left(p_{20}^{(t)*}\right)&=\widetilde{C}_{201}\left(\left(\lambda_2^*\right)^k h^{-1}\left(p_{20}^{(t)*}\right)\right),
\end{aligned}$$

For party 0, $p_{01}^{(t)*}$ and $p_{02}^{(t)*}$ are similar,

$$\begin{aligned}
C_{01}\left(p_{01}^{(t)*}\right)&=\widetilde{C}_{011}\left(\left(\lambda_0^*\right)^k h^{-1}\left(p_{01}^{(t)*}\right)\right),\\
C_{02}\left(p_{02}^{(t)*}\right)&=\widetilde{C}_{022}\left(\left(\lambda_0^*\right)^k h^{-1}\left(p_{02}^{(t)*}\right)\right).
\end{aligned}$$

However, there is slightly difference for $p_{00}^{(t)*}$ since both player 1 and player 2 will affect this probability. Define

$$\begin{aligned}
p_{001}^{(t)*}&=h\left(m_{001}^{(t)**}\right),\ m_{001}^{(t)**}\triangleq\left(\lambda_0^*\right)^{1/k} m_{001}^{(t)*},\\
p_{001}^{(t)*}&=h\left(m_{001}^{(t)**}\right),\ m_{002}^{(t)**}\triangleq\left(\lambda_2^*\right)^{1/k} m_{002}^{(t)*}.
\end{aligned}$$

Then

$$p_{00}^{(t)*}=h\left(h^{-1}\left(p_{001}^{(t)*}\right)+h^{-1}\left(p_{002}^{(t)*}\right)\right),$$

and

$$C_{001}\left(p_{001}^{(t)*}\right)=\widetilde{C}_{001}\left(\left(\lambda_0^*\right)^k h^{-1}\left(p_{001}^{(t)*}\right)\right),$$
$$C_{002}\left(p_{002}^{(t)*}\right)=\widetilde{C}_{002}\left(\left(\lambda_0^*\right)^k h^{-1}\left(p_{002}^{(t)*}\right)\right).$$

Since $h(\cdot)$ is a strictly increasing function, by the construction, there are one-to-one mapping between

$$p_{11}^{(t)*}, p_{21}^{(t)*}, p_{20}^{(t)*}, p_{01}^{(t)*}, p_{001}^{(t)*}$$

and

$$m_{111}^{(t)**}, m_{211}^{(t)**}, m_{201}^{(t)**}, m_{011}^{(t)**}, m_{001}^{(t)**},$$

respectively. Similarly, there is one-to-one mapping between

$$p_{22}^{(t)*}, p_{12}^{(t)*}, p_{10}^{(t)*}, p_{02}^{(t)*}, p_{002}^{(t)*}$$

and

$$m_{222}^{(t)**}, m_{122}^{(t)**}, m_{102}^{(t)**}, m_{022}^{(t)**}, m_{002}^{(t)**},$$

respectively.

As there is also one-to-one mapping between $m_{ij1(2)}^{(t)**}$ and $m_{ij1(2)}^{(t)*}$ given λ_i^*, the objective function of player 1 with cost function $\widetilde{C}_{ij1}\left(m_{ij1}^{(t)}\right)$, which is optimized by $\left\{m_{ij1}^{(t)*}\right\}$ given $\left\{m_{ij2}^{(t)*}\right\}$, can be optimized by $\left\{p_{11}^{(t)*}, p_{21}^{(t)*}, p_{20}^{(t)*}, p_{01}^{(t)*}, p_{001}^{(t)*}\right\}$ with the adjusted cost function $C_{ij1}(\cdot) \triangleq \widetilde{C}_{ij1}\left(\left(\lambda_i^*\right)^k h^{-1}(\cdot)\right)$, given $\left\{p_{22}^{(t)*}, p_{12}^{(t)*}, p_{10}^{(t)*}, p_{02}^{(t)*}, p_{002}^{(t)*}\right\}$. Similarly, the objective function of player 2 with cost function $\widetilde{C}_{ij2}\left(m_{ij2}^{(t)}\right)$, which is optimized by $\left\{m_{ij2}^{(t)*}\right\}$ given $\left\{m_{ij1}^{(t)*}\right\}$, can be optimized by $\left\{p_{22}^{(t)*}, p_{12}^{(t)*}, p_{10}^{(t)*}, p_{02}^{(t)*}, p_{002}^{(t)*}\right\}$ with the adjusted cost function $C_{ij2}(\cdot) \triangleq \widetilde{C}_{ij2}\left(\left(\lambda_i^*\right)^k h^{-1}(\cdot)\right)$, given $\left\{p_{11}^{(t)*}, p_{21}^{(t)*}, p_{20}^{(t)*}, p_{01}^{(t)*}, p_{001}^{(t)*}\right\}$.

On the other hand, since every $m_{ij1(2)}^{(t)**}$ can be replicated by $m_{ij1(2)}^{(t)*}$, the optimization problem with transitions can be achieved by some optimization problem with intensities.

Therefore, modeling with intensities is equivalent to modeling with market share transitions, given the adjusted cost function and $p_{00}^{(t)*}=h\left(h^{-1}\left(p_{001}^{(t)*}\right)+h^{-1}\left(p_{002}^{(t)*}\right)\right)$. ■

As a result, we will adopt the modeling where players decide on their costly investments in transitions, $p_{ij}^{(t)}$. With Assumption 1, Player 1 invests in market share transitions that could positively influence his market share (p_{01}, p_{11}, p_{21}) and/or negatively influence his opponents market share (p_{20}, in addition to p_{21}). Analogously, Player 2 invests in transitions p_{02}, p_{12}, p_{22} and/or p_{10}. (Both players could also have interest in controlling p_{00}.)

Looking at the changes of market shares in a given time period t that are result of chosen transitions, we can measure whether Player 1 invested more into increasing its own market share

$$p_{01}^{(t)} V_0^{(t)} + p_{21}^{(t)} V_2^{(t)} \tag{4}$$

or into decreasing the opponent's (i.e., Player 2) market share

$$p_{20}^{(t)} V_2^{(t)} + p_{21}^{(t)} V_2^{(t)}. \tag{5}$$

Comparing (4) and (5) and analogous quantities for Player 2, we have the following definitions for negative (positive) advertisement strategy.

Definition 2.4 *We define $d_i^{(t)}$ as **Positive Advertisement Index** at time t for player i:*

$$\begin{aligned} d_1^{(t)} &= p_{01}^{(t)} V_0^{(t)} - p_{20}^{(t)} V_2^{(t)}, && (6) \\ d_2^{(t)} &= p_{02}^{(t)} V_0^{(t)} - p_{10}^{(t)} V_1^{(t)}. && (7) \end{aligned}$$

If $d_i^{(t)} > 0$, then Player i chooses a strictly positive advertisement strategy at time t since it focuses more on increasing its market share than on decreasing the opponent's market share. Conversely, if $d_i^{(t)} < 0$, Player i chooses a strictly negative advertisement strategy since it focuses more on increasing its market share than on decreasing the opponent's market share. (Note that a direct transition between V_1 and V_2 has both a positive and negative impact, in context of labeling activities as positive or negative advertising, and cancels out in comparison of quantities (4) and (5).)

We can rewrite player 1's optimization problem as

$$
\begin{aligned}
&\max_{\{p_{01}^{(t)},p_{11}^{(t)},p_{21}^{(t)},p_{20}^{(t)},p_{001}^{(t)}:t=0,\ldots,T-1\}} \sum_{t=0}^{T} r^t[\delta_1^{(t)}V_1^{(t)} - \beta_1\delta_2^{(t)}V_2^{(t)}] \\
&\quad - \sum_{t=0}^{T-1} r^t[C_{01}(p_{01}^{(t)}) + C_{21}(p_{21}^{(t)}) + C_{20}(p_{20}^{(t)}) + C_{11}(p_{11}^{(t)}) + C_{001}(p_{001}^{(t)})] \\
&\text{subject to} \\
&V^{(t+1)} = V^{(t)}P^{(t)} = V^{(t)} \begin{bmatrix} p_{00}^{(t)} & p_{01}^{(t)} & p_{02}^{(t)*} \\ p_{10}^{(t)*} & p_{11}^{(t)} & p_{12}^{(t)*} \\ p_{20}^{(t)} & p_{21}^{(t)} & p_{22}^{(t)*} \end{bmatrix}, \\
&1 = V_0^{(t)} + V_1^{(t)} + V_2^{(t)}, \\
&\begin{bmatrix} p_{00}^{(t)} & p_{01}^{(t)} & p_{02}^{(t)*} \\ p_{10}^{(t)*} & p_{11}^{(t)} & p_{12}^{(t)*} \\ p_{20}^{(t)} & p_{21}^{(t)} & p_{22}^{(t)*} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\
&p_{00}^{(t)} = h\left(h^{-1}\left(p_{001}^{(t)}\right) + h^{-1}\left(p_{002}^{(t)*}\right)\right), \\
&V_0^{(t+1)}, V_1^{(t+1)}, V_2^{(t+1)} \geq 0, \\
&p_{01}^{(t)}, p_{11}^{(t)}, p_{21}^{(t)}, p_{20}^{(t)}, p_{001}^{(t)} \geq 0, \\
&V_0^{(0)}, V_1^{(0)}, V_2^{(0)} \text{ are given.}
\end{aligned}
$$

$p_{02}^{(t)*}$, $p_{10}^{(t)*}$, $p_{12}^{(t)*}$, $p_{22}^{(t)*}$, and $p_{002}^{(t)*}$ are equilibrium values chosen by Player 2. For player 2, the optimization problem is similar,

$$\begin{aligned}
&\max_{\{p_{02}^{(t)},p_{22}^{(t)},p_{12}^{(t)},p_{10}^{(t)},p_{002}^{(t)}:t=0,\ldots,T-1\}} \textstyle\sum_{t=0}^{T} r^t[\delta_2^{(t)}V_2^{(t)} - \beta_2\delta_1^{(t)}V_1^{(t)}] \\
&\qquad - \textstyle\sum_{t=0}^{T-1} r^t[C_{02}(p_{02}^{(t)}) + C_{12}(p_{12}^{(t)}) + C_{10}(p_{10}^{(t)}) + C_{22}(p_{22}^{(t)}) + C_{002}(p_{002}^{(t)})] \\
&\text{subject to} \\
&V^{(t+1)} = V^{(t)}P^{(t)} = V^{(t)} \begin{bmatrix} p_{00}^{(t)} & p_{01}^{(t)*} & p_{02}^{(t)} \\ p_{10}^{(t)} & p_{11}^{(t)*} & p_{12}^{(t)} \\ p_{20}^{(t)*} & p_{21}^{(t)*} & p_{22}^{(t)} \end{bmatrix}, \\
&1 = V_0^{(t)} + V_1^{(t)} + V_2^{(t)}, \\
&\begin{bmatrix} p_{00}^{(t)} & p_{01}^{(t)*} & p_{02}^{(t)} \\ p_{10}^{(t)} & p_{11}^{(t)*} & p_{12}^{(t)} \\ p_{20}^{(t)*} & p_{21}^{(t)*} & p_{22}^{(t)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\
&p_{00}^{(t)} = h\left(h^{-1}\left(p_{001}^{(t)*}\right) + h^{-1}\left(p_{002}^{(t)}\right)\right), \\
&V_0^{(t+1)}, V_1^{(t+1)}, V_2^{(t+1)} \geq 0, \\
&p_{02}^{(t)}, p_{22}^{(t)}, p_{12}^{(t)}, p_{10}^{(t)}, p_{002}^{(t)} \geq 0, \\
&V_0^{(0)}, V_1^{(0)}, V_1^{(2)} \text{ are given.}
\end{aligned}$$

$p_{01}^{(t)*}$, $p_{20}^{(t)*}$, $p_{21}^{(t)*}$, $p_{11}^{(t)*}$, and $p_{001}^{(t)*}$ are equilibrium values chosen by Player 1.

In the following discussions, we will consider a linear function $h(\cdot)$. Then

$$\begin{aligned}
p_{00}^{(t)} &= h\left(h^{-1}\left(p_{001}^{(t)}\right) + h^{-1}\left(p_{002}^{(t)}\right)\right) \\
&= p_{001}^{(t)} + p_{002}^{(t)}.
\end{aligned}$$

3 Linear Cost Functions

Consider linear cost functions $C(p) = c_{ij}p_{ij}$. This can be attained by assuming linear cost functions associated with intensities. We first discuss the case $T = 2$. We can express the market shares at $t = 2$ are

$$\begin{aligned} V_1^{(2)} &= p_{01}^{(1)}[p_{10}^{(0)}V_1^{(0)} + p_{20}^{(0)}V_2^{(0)} + V_0^{(0)}(p_{001}^{(0)} + p_{002}^{(0)})] \\ &\quad + p_{11}^{(1)}[p_{11}^{(0)}V_1^{(0)} + p_{21}^{(0)}V_2^{(0)} + V_0^{(0)}p_{01}^{(0)}] \\ &\quad + p_{21}^{(1)}[p_{12}^{(0)}V_1^{(0)} + p_{22}^{(0)}V_2^{(0)} + V_0^{(0)}p_{02}^{(0)}] \end{aligned}$$

and

$$\begin{aligned} V_2^{(2)} &= p_{02}^{(1)}[p_{10}^{(0)}V_1^{(0)} + p_{20}^{(0)}V_2^{(0)} + V_0^{(0)}(p_{001}^{(0)} + p_{002}^{(0)})] \\ &\quad + p_{12}^{(1)}[p_{11}^{(0)}V_1^{(0)} + p_{21}^{(0)}V_2^{(0)} + V_0^{(0)}p_{01}^{(0)}] \\ &\quad + p_{22}^{(1)}[p_{12}^{(0)}V_1^{(0)} + p_{22}^{(0)}V_2^{(0)} + V_0^{(0)}p_{02}^{(0)}]. \end{aligned}$$

The market shares at $t = 1$ are

$$V_1^{(1)} = p_{11}^{(0)}V_1^{(0)} + p_{21}^{(0)}V_2^{(0)} + V_0^{(0)}p_{01}^{(0)}$$

and

$$V_2^{(1)} = p_{12}^{(0)}V_1^{(0)} + p_{22}^{(0)}V_2^{(0)} + V_0^{(0)}p_{02}^{(0)}.$$

We first make the following cost assumptions.

Assumption 2A $c_{21} \leq c_{20}$.

Assumption 2A' $c_{12} \leq c_{10}$.

These assumptions indicate that it is cheaper to directly attract opponent's customers than to turn them away from the opponent.

Assumption 2B $c_{01} \leq c_{001}$.

Assumption 2B' $c_{02} \leq c_{002}$.

These assumptions indicate that it is cheaper to attract uncommitted customers than to focus on keeping them uncommitted.

By Theorem 2.2, since linear cost functions are also convex, there exists a pure strategy Nash Equilibrium in the subgame at $t = 1$. The following Proposition 3.1 shows the suboptimality of negative advertisement in the last period.

Proposition 3.1 *With Assumption 2A, or with Assumption 2B and there are more uncommitted customers affected by the player than opponent's customers who will become uncommitted, i.e.,*

$$p_{20}^{(1)*}V_2^{(1)*} \leq \left(1 - p_{002}^{(1)*} - p_{02}^{(1)*}\right) V_0^{(1)*}, \tag{8}$$

Player 1 *will not engage in strictly negative advertisement in the last period, i.e.,* $d_1^{(1)} \geq 0$.

Remarks:

1) Assumption 2A alone can induce no strictly negative advertisement in the last period.

2) In fact, Assumption 2A drives $p_{20}^{(1)}$ down to 0; Assumption 2B and condition (8) drive $p_{01}^{(1)}$ up to $1 - p_{002}^{(1)*} - p_{02}^{(1)*}$. Therefore, condition (8) holds if Assumption 2A holds.

Proof. Proof is by contradiction.

If $V_2^{(1)*} = 0$, it is obvious that there is no strictly negative advertisement.

If $V_2^{(1)*} \neq 0$ and $V_0^{(1)*} \neq 0$, suppose there exists a set of choices in the support such that $d_1^{(1)} < 0$. Rewriting $d_1^{(1)} < 0$ we get

$$p_{01}^{(1)*}V_0^{(1)*} < p_{20}^{(1)*}V_2^{(1)*}. \tag{9}$$

Note that $V_0^{(1)*} = p_{10}^{(0)*}V_1^{(0)} + p_{20}^{(0)*}V_2^{(0)} + V_0^{(0)}(p_{001}^{(0)*} + p_{002}^{(0)*})$ and $V_2^{(1)*} = p_{12}^{(0)*}V_1^{(0)} + p_{22}^{(0)*}V_2^{(0)} + V_0^{(0)}p_{02}^{(0)*}$ are between 0 and 1 by definition. Also, $p_{20}^{(1)*} > 0$. By (9), $p_{01}^{(1)*} < 1 - p_{002}^{(1)*} - p_{02}^{(1)*}$ if

$$p_{20}^{(1)*}V_2^{(1)*} \leq \left(1 - p_{002}^{(1)*} - p_{02}^{(1)*}\right) V_0^{(1)*},$$

which also means that the maximal value of $p_{01}^{(1)*}$ can reverse the negative advertisement. We try to find profitable deviations at $t = 1$ for $p_{20}^{(1)} \in (p_{01}^{(1)*}V_0^{(1)*}/V_2^{(1)*}, p_{20}^{(1)*}]$ and $p_{01}^{(1)} \in [p_{01}^{(1)*}, \min\left\{p_{20}^{(1)*}V_2^{(1)*}/V_0^{(1)*}, 1 - p_{002}^{(1)*} - p_{02}^{(1)*}\right\})$.

It is feasible for player 1 to decrease $p_{20}^{(1)}$ by $\varepsilon > 0$, i.e. $p_{20}^{(1)new} = p_{20}^{(1)} - \varepsilon$, or increase $p_{01}^{(1)}$ by $\varepsilon > 0$, i.e. $p_{01}^{(1)new} = p_{01}^{(1)} + \varepsilon$. To balance these changes in order to satisfy the feasible constraints, we must have $p_{21}^{(1)new} = p_{21}^{(1)} + \varepsilon$, or $p_{001}^{(1)new} = p_{001}^{(1)} - \varepsilon$. These are the choice variables that could be controlled by player 1.

Player 1's revenue is

$$
\begin{aligned}
&\delta_1^{(0)} V_1^{(0)} - \beta_1 \delta_2^{(0)} V_2^{(0)} \\
&+ r\{V_1^{(0)}[\delta_1^{(1)} p_{11}^{(0)} - \beta_1 \delta_2^{(1)} p_{12}^{(0)*}] \\
&+ V_2^{(0)}[\delta_1^{(1)} p_{21}^{(0)} - \beta_1 \delta_2^{(1)} p_{22}^{(0)*}] \\
&+ V_0^{(0)}[\delta_1^{(1)} p_{01}^{(0)} - \beta_1 \delta_2^{(1)} p_{02}^{(0)*}]\} \\
&+ r^2[\delta_1^{(2)} p_{01}^{(1)} - \beta_1 \delta_2^{(2)} p_{02}^{(1)*}][p_{10}^{(0)} V_1^{(0)} + p_{20}^{(0)*} V_2^{(0)} + (p_{001}^{(0)} + p_{002}^{(0)*}) V_0^{(0)}] \\
&+ r^2[\delta_1^{(2)} p_{21}^{(1)} - \beta_1 \delta_2^{(2)} p_{22}^{(1)*}][p_{12}^{(0)*} V_1^{(0)} + p_{22}^{(0)*} V_2^{(0)} + p_{02}^{(0)*} V_0^{(0)}] \\
&+ r^2[\delta_1^{(2)} p_{11}^{(1)} - \beta_1 \delta_2^{(2)} p_{12}^{(1)*}][p_{11}^{(0)*} V_1^{(0)} + p_{21}^{(0)*} V_2^{(0)} + p_{01}^{(0)*} V_0^{(0)}].
\end{aligned}
$$

and transition probability is determined by (3).

Case 1: Decreasing $p_{20}^{(1)}$ by $\varepsilon > 0$ is profitable if

$$
\begin{aligned}
& r^2 \delta_1^{(2)} [p_{12}^{(0)*} V_1^{(0)} + p_{22}^{(0)*} V_2^{(0)} + V_0^{(0)} p_{02}^{(0)*}] \epsilon \\
\geq \; & r[\left(C_{21}\left(p_{21}^{(1)} + \varepsilon\right) - C_{21}\left(p_{21}^{(1)}\right)\right) + \left(C_{20}\left(p_{20}^{(1)} - \varepsilon\right) - C_{20}\left(p_{20}^{(1)}\right)\right)].
\end{aligned}
$$

Let ε goes to zero, then we have

$$
\begin{aligned}
r^2 \delta_1^{(2)} [p_{12}^{(0)*} V_1^{(0)} + p_{22}^{(0)*} V_2^{(0)} + p_{02}^{(0)*} V_0^{(0)}] & \geq r[C_{21}'\left(p_{21}^{(1)}\right) - C_{20}'\left(p_{20}^{(1)}\right)] \\
& \Leftrightarrow \\
r \delta_1^{(2)} V_2^{(1)*} & \geq c_{21} - c_{20}
\end{aligned}
$$

By Assumption 2A, this obviously holds.

Case 2: Increasing $p_{01}^{(1)*}$ by $\varepsilon > 0$ is profitable if

$$\begin{aligned}
& r^2\delta_1^{(2)}[p_{10}^{(0)*}V_1^{(0)} + p_{20}^{(0)*}V_2^{(0)} + (p_{001}^{(0)} + p_{002}^{(0)*})V_0^{(0)}]\varepsilon \\
\geq \ & r[\left(C_{01}\left(p_{01}^{(1)} + \varepsilon\right) - C_{01}\left(p_{01}^{(1)}\right)\right) - \left(C_{001}\left(p_{001}^{(1)} - \varepsilon\right) - C_{001}\left(p_{001}^{(1)}\right)\right)].
\end{aligned}$$

Let ε goes to zero, then we have

$$\begin{aligned}
r^2\delta_1^{(2)}[p_{10}^{(0)*}V_1^{(0)} + p_{20}^{(0)*}V_2^{(0)} + (p_{001}^{(0)} + p_{002}^{(0)*})V_0^{(0)}] & \geq r[C'_{01}\left(p_{01}^{(1)}\right) - C'_{001}\left(p_{001}^{(1)}\right)] \\
& \Leftrightarrow \\
r\delta_1^{(2)}V_0^{(1)*} & \geq c_{01} - c_{001}.
\end{aligned}$$

By Assumption 2B, this obviously holds.

If $V_0^{(1)*} = 0$, by the definition of $d_1^{(1)}$, $d_1^{(1)} = -p_{20}^{(1)*}V_2^{(1)*} \leq 0$. There is no strictly negative advertisement iff $p_{20}^{(1)*}V_2^{(1)*} = 0$. This can be achieved by profitable deviation from Assumption 2A or $p_{20}^{(1)*}V_2^{(1)*} \leq V_0^{(1)*} = 0$. ■

Corollary 3.2 *With Assumption 2A, 2A', 2B and 2B', neither of the Players devotes to maintaining the uncommitted customers in the last period, i.e.,* $p_{01}^{(1)*} + p_{02}^{(1)*} = 1$, $p_{20}^{(1)*} = 0$, *and* $p_{10}^{(1)*} = 0$.

Remarks:

Since the equilibriums are not unique and there are at least three degrees of freedom, $p_{01}^{(1)*}$, $p_{11}^{(1)*}$ and $p_{22}^{(1)*}$ can be regarded as undetermined choices.

Proof. With Assumption 2A and Assumption 2B for Player 1, $p_{20}^{(1)*} = 0$, $p_{01}^{(1)*} = 1 - p_{002}^{(1)*} - p_{02}^{(1)*}$, $p_{001}^{(1)*} = 0$; with Assumption 2A' and Assumption 2B' for Player 2, $p_{10}^{(1)*} = 0$, $p_{02}^{(1)*} = 1 - p_{001}^{(1)*} - p_{01}^{(1)*}$, and $p_{002}^{(1)*} = 0$. Therefore, $p_{01}^{(1)*} + p_{02}^{(1)*} = 1$. Then, the uncommitted customers are completely turned into other parties. ■

Lemma 3.3 *With Assumption 2A, 2A', 2B and 2B', there exists a pure strategy subgame perfect Nash Equilibrium in the two periods game with linear cost functions.*

Proof. By Theorem 2.2, since linear cost functions are also convex functions, there exists a pure strategy Nash Equilibrium in the subgame at $t = 1$. By Corollary 3.2, the equilibrium transitions do not depend on $V_1^{(1)}$, $V_2^{(1)}$, or $V_0^{(1)}$. Hence, the value function at $t = 1$, i.e., take Player 1 for example,

$$
\begin{aligned}
& J_1\left(V_1^{(1)}, V_2^{(1)}\right) \\
& = \max_{\{p_{01}^{(1)}, p_{11}^{(1)}, p_{21}^{(1)}, p_{20}^{(1)}, p_{001}^{(1)}\}} \{\delta_1^{(1)} V_1^{(1)} - \beta_1 \delta_2^{(1)} V_2^{(1)} \\
& - \left[C_{01}(p_{01}^{(1)}) + C_{21}(p_{21}^{(1)}) + C_{20}(p_{20}^{(1)}) + C_{11}(p_{11}^{(1)}) + C_{001}(p_{001}^{(1)})\right] \\
& + r\left(\delta_1^{(2)} V_1^{(2)} - \beta_1 \delta_2^{(2)} V_2^{(2)}\right)\} \text{ for } 0 \leq t \leq T-1, \\
& \text{subject to } V^{(T)} = V^{(T-1)} P^{(T-1)} \text{ and feasibility constraints,}
\end{aligned}
$$

is linear and continuous in $V_1^{(1)}$ and $V_2^{(1)}$. Therefore, the subgame at $t = 0$ is also a continuous game and the objective function is linear in all transitions. By the above argument again, there exists a pure strategy Nash Equilibrium in the two periods game. ■

We turn to identifying situations in which Player 1 chooses strictly negative advertisement in the first period. If $V_2^{(0)} = 0$, there is no strictly negative advertisement in the first period. As a result, the following Proposition 3.4 focuses on $V_2^{(0)} \neq 0$.

Proposition 3.4 *With Assumption 2A, 2A', 2B and 2B', Player* 1 *will engage in strictly negative advertisement in the first period when* $V_2^{(0)} \neq 0$, *i.e.* $d_1^{(0)} < 0$, *if and only if*

$$\left(1 - p_{22}^{(0)*}\right) V_2^{(0)} > p_{01}^{(0)*} V_0^{(0)} \tag{10}$$

$$\text{and } V_2^{(0)} r^2 A^{(1)} \geq c_{20} - c_{21}, \tag{11}$$

where

$$A^{(1)} = \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right)\left(p_{01}^{(1)*} - p_{11}^{(1)*}\right) - \delta_1^{(1)}/r. \tag{12}$$

A sufficient condition is $p_{22}^{(0)*} < 1$, *condition (11) and*

$$V_0^{(0)} r^2 A^{(1)} \geq c_{001} - c_{01}. \tag{13}$$

Remarks:

Similar to Proposition 3.1, condition (13) drives $p_{01}^{(0)*}$ down to 0, while conditions (10) and (11) drive $p_{20}^{(0)*}$ up to $1 - p_{22}^{(0)*}$. Moreover, condition (13) leads to condition (11) if $p_{22}^{(0)*} < 1$.

Proof. The sufficient part is similar to the proof of Proposition 3.1.

If $V_0^{(0)} \neq 0$, and $V_2^{(0)} \neq 0$, we first prove the sufficient condition for $d_1^{(0)} \leq 0$. Suppose $d_1^{(0)} > 0$, i.e.,

$$V_0^{(0)} p_{01}^{(0)*} > V_2^{(0)} p_{20}^{(0)*}.$$

We try to find profitable deviations at $t = 0$ for $p_{20}^{(0)} \in [p_{20}^{(0)*}, \min\left\{p_{01}^{(0)*} V_0^{(0)*}/V_2^{(0)*}, 1\right\})$ and $p_{01}^{(0)} \in (p_{20}^{(0)*} V_2^{(0)*}/V_0^{(0)*}, p_{01}^{(1)*}]$.

Case 1: $p_{01}^{(0)*} > 0$.

Obviously, more substantially Player 1 lags behind the opponent, i.e., smaller $V_1^{(0)}/\left(V_1^{(0)} + V_2^{(0)}\right)$, larger $p_{01}^{(0)*}$ has to be. This indicates a larger space of deviation, $p_{01}^{(0)new} = p_{01}^{(0)} - \varepsilon$, and such deviation would have to be balanced by $p_{001}^{(0)new} = p_{001}^{(0)} + \varepsilon$. Looking at the utility function for Player 1, such profitable deviation exists if

$$\begin{aligned} -V_0^{(0)} r \delta_1^{(1)} \varepsilon + r^2 [\delta_1^{(2)} p_{01}^{(1)*} - \beta_1 \delta_2^{(2)} p_{02}^{(1)*}] V_0^{(0)} \\ -r^2 [\delta_1^{(2)} p_{11}^{(1)*} - \beta_1 \delta_2^{(2)} p_{12}^{(1)*}] V_0^{(0)} \geq [C'_{001}(p_{001}^{(0)}) - C'_{01}(p_{01}^{(0)})], \end{aligned}$$

for small ε. The above condition is equivalent to

$$V_0^{(0)} r^2 A^{(1)} \geq c_{001} - c_{01},$$

where

$$A^{(1)} = \delta_1^{(2)} p_{01}^{(1)*} - \beta_1 \delta_2^{(2)} p_{02}^{(1)*} - \delta_1^{(2)} p_{11}^{(1)*} + \beta_1 \delta_2^{(2)} p_{12}^{(1)*} - \frac{\delta_1^{(1)}}{r}.$$

Case 2: $p_{20}^{(0)*} < 1 - p_{22}^{(0)*}$.

A requirement for the analysis in this case is $\left(1 - p_{22}^{(0)*}\right) V_2^{(0)} \geq V_0^{(0)} p_{01}^{(0)*}$. A profitable deviation here involves $p_{20}^{(0)new} = p_{20}^{(0)} + \varepsilon$, which should be balanced by $p_{21}^{(0)new} = p_{21}^{(0)} - \varepsilon$.

Again, looking at the utility function for Player 1, such profitable deviation exists if

$$\begin{aligned} & V_2^{(0)} r^2 [\delta_1^{(2)} p_{01}^{(1)*} - \beta_1 \delta_2^{(2)} p_{02}^{(1)*} - \delta_1^{(2)} p_{11}^{(1)*} + \beta_1 \delta_2^{(2)} p_{12}^{(1)*} - \frac{\delta_1^{(1)}}{r}] \\ \geq \ & [C'_{20}(p_{20}^{(0)*}) - C'_{21}(p_{21}^{(0)*})]. \end{aligned}$$

The above condition is equivalent to

$$V_2^{(0)} r^2 A^{(1)} \geq c_{20} - c_{21}.$$

Since condition (13) drives $p_{01}^{(0)*}$ down to 0, while conditions (10) and (11) drive $p_{20}^{(0)*}$ up to $1 - p_{22}^{(0)*}$, by the definition of $d_1^{(0)}$, the sufficient condition for strictly negative advertisement is

$$\left(1 - p_{22}^{(0)*}\right) V_2^{(0)} > p_{01}^{(0)*} V_0^{(0)} \text{ and } V_2^{(0)} r^2 A^{(1)} \geq c_{20} - c_{21}$$

or

$$\begin{aligned} V_0^{(0)} r^2 A^{(1)} &\geq c_{001} - c_{01},\ p_{22}^{(0)*} < 1, \\ \text{and } V_2^{(0)} r^2 A^{(1)} &\geq c_{20} - c_{21}. \end{aligned}$$

To prove the necessary part, we will first identify the sufficient conditions for positive advertisement following the similar argument as above. There is positive advertisement in the first period if

$$V_2^{(0)} r^2 A^{(1)} \leq c_{20} - c_{21}$$

or

$$p_{20}^{(0)*} V_2^{(0)} \leq \left(1 - p_{02}^{(0)*} - p_{002}^{(0)*}\right) V_0^{(0)} \text{ and } V_0^{(0)} r^2 A^{(1)} \leq c_{001} - c_{01}.$$

Suppose either condition (10) or condition (11) does not hold, i.e.,

$$\left(1 - p_{22}^{(0)*}\right) V_2^{(0)} \leq p_{01}^{(0)*} V_0^{(0)} \tag{14}$$

or

$$V_2^{(0)} r^2 A^{(1)} < c_{20} - c_{21}. \tag{15}$$

Case 1: condition (14) holds.

Since $p_{20}^{(0)} \leq 1 - p_{22}^{(0)*}$,

$$p_{20}^{(0)} V_2^{(0)} \leq \left(1 - p_{22}^{(0)*}\right) V_2^{(0)} \leq p_{01}^{(0)*} V_0^{(0)},$$

for equilibrium choice of $p_{01}^{(0)*}$. There must be no strictly negative advertisement in the first period by any feasible $p_{20}^{(0)}$ values.

Case 2: condition (15) holds.

This is just one of the sufficient conditions for positive advertisement in the first period.

Hence, if either condition (10) or condition (11) does not hold, there is no strictly negative advertisement in the first period.

By the proof of Corollary 3.2, with Assumption 2A, 2A', 2B and 2B', we must have $p_{01}^{(1)*} + p_{02}^{(1)*} = 1$, $p_{20}^{(1)*} = 0$, and $p_{10}^{(1)*} = 0$. Then

$$\begin{aligned} A^{(1)} &= \delta_1^{(2)} p_{01}^{(1)*} - \beta_1 \delta_2^{(2)} \left(1 - p_{01}^{(1)*}\right) - \delta_1^{(2)} p_{11}^{(1)*} + \beta_1 \delta_2^{(2)} \left(1 - p_{11}^{(1)*}\right) - \frac{\delta_1^{(1)}}{r} \\ &= p_{01}^{(1)*} \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right) - p_{11}^{(1)*} \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right) - \frac{\delta_1^{(1)}}{r} \\ &= \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right) \left(p_{01}^{(1)*} - p_{11}^{(1)*}\right) - \frac{\delta_1^{(1)}}{r}. \end{aligned}$$

If $V_0^{(0)} = 0$, the above conditions are still valid.

This completes the proof. ■

The following Corollary 3.5 shows the feasibility of negative advertisement.

Corollary 3.5 *There always exist equilibrium choices at $t = 0$ satisfying condition (10). Moreover, with Assumption 2A, 2A', 2B and 2B', there exist equilibrium choices at $t = 1$*

satisfying conditions (11) or (13) if

$$V_2^{(0)} r^2 \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)} - \delta_1^{(1)}/r\right) \geq c_{20} - c_{21}$$

or

$$V_0^{(0)} r^2 \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)} - \delta_1^{(1)}/r\right) \geq c_{001} - c_{01},$$

respectively.

Remarks:

1) Consider the simple cases $c_{20} = c_{21}$ and $c_{001} = c_{01}$. There is no strictly negative advertisement in the last period. But, if

$$\delta_1^{(2)} + \beta_1 \delta_2^{(2)} \geq \delta_1^{(1)}/r,$$

it is still possible for player 1 to play strictly negative advertisement in the first period. This condition obviously holds if either $\delta_1^{(2)} = \delta_2^{(2)} = \delta_1^{(1)} = r = 1$.

2) In the political competition, i.e., $\beta_1 > 0$, these conditions are easier to satisfy.

3) If only the final market shares matter, these conditions are easier to satisfy.

Proof. By the proof of Proposition 3.4, the equilibrium choices are not unique at $t = 1$. Similarly, the equilibrium choices are not unique at $t = 0$, either.

At $t = 0$, at least $p_{22}^{(0)*}$ can be regarded as undetermined. If condition (13) holds, $p_{01}^{(0)*}$ is driven down to 0. Therefore, there exists $p_{22}^{(0)*}$ making condition (10) satisfied. If (13) does not hold, $p_{01}^{(0)*}$ can be regarded as undetermined. Hence, there still exist $p_{01}^{(0)*}$ and $p_{22}^{(0)*}$ making condition (10) satisfied.

At $t = 1$, since there are at least three degrees of freedom, $p_{01}^{(1)*}$, $p_{11}^{(1)*}$ and $p_{22}^{(1)*}$ can be regarded as undetermined equilibrium choices. It is also noticed that the maximum possible value of $A^{(1)}$ is

$$A_{\max}^{(1)} = \delta_1^{(2)} + \beta_1 \delta_2^{(2)} - \frac{\delta_1^{(1)}}{r}.$$

There exist equilibrium choices at $t = 1$ satisfying conditions (11) or (13) if

$$V_2^{(0)} r^2 \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)} - \delta_1^{(1)}/r \right) \geq c_{20} - c_{21}$$

or

$$V_0^{(0)} r^2 \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)} - \delta_1^{(1)}/r \right) \geq c_{001} - c_{01},$$

respectively. ■

Corollary 3.6 gives how initial market shares affect negative advertisement. It also sheds light on the optimal strategy of a player who is falling behind.

Corollary 3.6 *With Assumption 2A, 2A', 2B and 2B', if fixed $V_0^{(0)}$, decreasing the initial market share of Player 1, i.e., a smaller value of $V_1^{(0)}$; or fixed $V_1^{(0)}$, decreasing the initial market share of uncommitted party, i.e., a smaller value of $V_0^{(0)}$, there occur more equilibriums with Player 1's strictly negative advertisement in the first period and, moreover, Player 1's advertisement is of more negativity for some particular equilibrium.*

Proof. In order to study the effect of initial market shares on equilibriums, we consider the necessary and sufficient conditions.

With Assumption 2A, condition (11) requires that $A^{(1)} \geq 0$.

Moreover, with $p_{22}^{(0)*} = 1$ and condition (11), there is no strictly negative advertisement. Hence, conditions (10) and (11) are equivalent to

$$\begin{aligned} V_1^{(0)} &< 1 - \left(1 + p_{01}^{(0)*} / \left(1 - p_{22}^{(0)*}\right)\right) V_0^{(0)} \\ \text{and } V_1^{(0)} &\leq 1 - V_0^{(0)} - (c_{20} - c_{21}) / \left(r^2 A^{(1)}\right). \end{aligned}$$

First, it is noticed that

$$V_1^{(0)} < 1 - V_0^{(0)} - V_0^{(0)} p_{01}^{(0)*} / \left(1 - p_{22}^{(0)*}\right)$$

is equivalent to

$$0 \leq V_0^{(0)} p_{01}^{(0)*} / \left(1 - p_{22}^{(0)*}\right) < 1 - V_0^{(0)} - V_1^{(0)}.$$

Since the equilibrium choices at $t = 0$ is not unique, $p_{01}^{(0)*}$ and $p_{22}^{(0)*}$ can be regarded to be undetermined equilibrium choices. Fixed $V_0^{(0)}$, a smaller $V_1^{(0)}$, or fixed $V_1^{(0)}$, a smaller $V_0^{(0)}$ allows greater upper bounds for both $p_{01}^{(0)*}$ and $p_{22}^{(0)*}$.

Second, by the proof of Proposition 3.4, With Assumption 2A, 2A', 2B and 2B', $A^{(1)}$ is rewritten as

$$A^{(1)} = \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right) \left(p_{01}^{(1)*} - p_{11}^{(1)*}\right) - \frac{\delta_1^{(1)}}{r},$$

where $p_{01}^{(1)*}$, $p_{11}^{(1)*}$ and $p_{22}^{(1)*}$ can be regarded as undetermined equilibrium choices. It is noticed that

$$V_1^{(0)} \leq 1 - V_0^{(0)} - (c_{20} - c_{21}) / \left(r^2 A^{(1)}\right)$$

is equivalent to

$$\begin{aligned} & \left((c_{20} - c_{21}) / \left(r^2 \left(1 - V_0^{(0)} - V_1^{(0)}\right)\right) + \delta_1^{(1)}/r\right) / \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right) \\ \leq & \left(p_{01}^{(1)*} - p_{11}^{(1)*}\right). \end{aligned}$$

Hence, a smaller $V_1^{(0)}$ or a smaller $V_0^{(0)}$ allows smaller lower bound for $p_{01}^{(1)*} - p_{11}^{(1)*}$. Since $\left(p_{01}^{(1)*}, p_{11}^{(1)*}\right)$ is free within a unit square, there exist more equilibriums choices at $t = 1$ with strictly negative advertisement in the first period.

Therefore, there exist more equilibriums with strictly negative advertisement for the whole game with a smaller $V_0^{(0)}$ or $V_1^{(0)}$.

To prove the effect of initial market shares on the degree of negativity, recall the definition of positive advertisement index

$$d_1^{(0)} = p_{01}^{(0)} V_0^{(0)} - p_{20}^{(0)} V_2^{(0)}.$$

If conditions (10) and (11) hold, there is strictly negative advertisement and $p_{20}^{(0)*}$ is driven up to $1 - p_{22}^{(0)*}$. In addition, if condition (13) holds, $p_{01}^{(0)*}$ is driven down to 0; if not, $p_{01}^{(0)*}$ is regarded as undetermined equilibrium choices.

If conditions (10) or (11) does not hold, there is positive advertisement. $p_{20}^{(0)*}$ is driven down to 0 (or undetermined) or $p_{01}^{(0)*}$ is driven up to $1 - p_{02}^{(0)*} - p_{002}^{(0)*}$.

Since fixed $V_0^{(0)}$, a smaller $V_1^{(0)}$, or fixed $V_1^{(0)}$, a smaller $V_0^{(0)}$ facilitates the necessary and sufficient condition of negative advertisement, the negativity of advertisement increases at the edge of jumping from positive to negative advertisement. Within some particular equilibrium, either $p_{01}^{(0)*}$ (or $p_{20}^{(0)*}$) is driven to the corner or undetermined, so the negativity of advertisement increases when fixed $V_0^{(0)}$, decreasing $V_1^{(0)}$, or fixed $V_1^{(0)}$, decreasing $V_0^{(0)}$.

■

The following Corollary 3.7 shows the comparative statics of parameters in the utility function. We also expect to see more strictly negative advertisement with political competition.

Corollary 3.7 *With Assumption 2A, 2A', 2B and 2B', there exist more equilibriums with Player 1's strictly negative advertisement in the first period,*

1) if in political competition, i.e., $\beta_1 > 0$*, than those in business competition, i.e.,* $\beta_1 = 0$*;*

2) if Player 1 cares more about the opponent's final market share, i.e., a larger value of $\beta_1 \delta_2^{(2)}$*;*

3) if Player 1 cares more about its own final market share, i.e., a larger value of $\delta_1^{(2)}$*;*

4) if Player 1 cares less about its own market share at $t = 1$*, i.e., a smaller value of* $\delta_1^{(1)}$*;*

5) if Player 1 is more patient, i.e., a larger discount factor r*.*

More the player cares about the opponent's market share in the last period (i.e., larger $\beta_1 \delta_2^{(2)}$), as in the political competition, the negative advertising strategy in the initial period is more likely. Similarly, if player's utility puts a considerable importance on its market share at $t = 1$, i.e. if $\delta_1^{(1)}$ is significant, a strictly negative advertisement will not be likely to occur. Conversely, if $\delta_1^{(1)}$ is small (or even equal to zero), a strictly negative advertisement in the initial period is likely. These correspond to a two-shot advertising strategy in which (not very significant) market share at $t = 1$ is ignored and all of the investment in the initial period is geared towards optimal market structure for potentially large market share gains at $t = T = 2$ (when the market share matters). If a player stands to gain at $t = 2$ from large $V_0^{(1)}$ through positive advertising effort $p_{01}^{(1)}$ then it could make sense to focus on enlarging $V_0^{(1)}$ which could call for negative advertising efforts $p_{20}^{(0)}$ in the initial period.

Proof. By the proof of Proposition 3.4, with Assumption 2A, 2A', 2B and 2B',

$$A^{(1)} = \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right)\left(p_{01}^{(1)*} - p_{11}^{(1)*}\right) - \frac{\delta_1^{(1)}}{r}.$$

Obviously, $\delta_1^{(2)}$, $\delta_2^{(2)}$, $\delta_1^{(1)}$, r and β_1 affect the decision of negative advertisement at $t = 0$ only through $r^2 A^{(1)}$. However, the value of $r^2 A^{(1)}$ will not affect the multiple equilibrium choices at $t = 0$.

By Corollary 3.5, there exist equilibrium choices satisfying condition (10). Therefore, in order to study the comparative statics of $\delta_1^{(2)}$, $\delta_2^{(2)}$, $\delta_1^{(1)}$, r and β_1, we focus on the necessary and sufficient conditions. Then, condition (11) require

$$A^{(1)} \geq \overline{A}^{(1)} \triangleq \frac{c_{20} - c_{21}}{V_2^{(0)} r^2} \geq 0.$$

Hence, they require

$$p_{01}^{(1)*} - p_{11}^{(1)*} \geq \left(\overline{A}^{(1)} + \delta_1^{(1)}/r\right) / \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right).$$

Moreover, since the equilibrium choices at $t = 1$ are not unique and thus $\left(p_{01}^{(1)*}, p_{11}^{(1)*}\right)$ is free within a unit square, there exist more equilibriums with Player 1's strictly negative advertisement at $t = 1$ if

$$\left(\overline{A}^{(1)} + \delta_1^{(1)}/r\right) / \left(\delta_1^{(2)} + \beta_1 \delta_2^{(2)}\right)$$

is smaller. Thus we have the conclusions. ■

We close this section by providing the simple examples as follows.

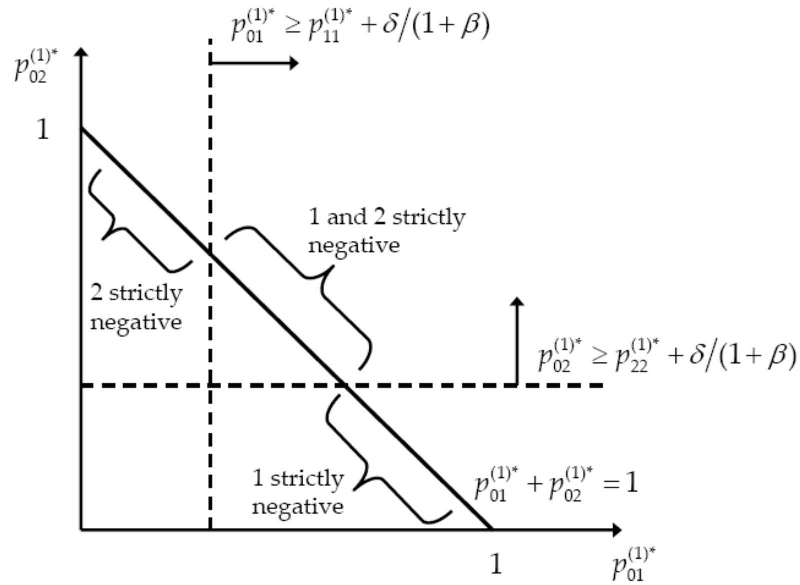
Example 3.8 *Consider the special case* $\delta_1^{(2)} = \delta_2^{(2)} = 1$ *and* $\delta_1^{(1)} = \delta_2^{(1)} = \delta$*. If* $\delta = 0$*, only the final market shares matter. Suppose there is no discount, i.e.,* $r = 1$*, all the marginal costs are equal to* c*, and* $\beta_1 = \beta_2 = \beta$*.*

By Corollary 3.2, neither of the players will adopt negative advertisement. The equilibrium choices at $t = 1$ *satisfy* $p_{01}^{(1)*} + p_{02}^{(1)*} = 1$*,* $p_{20}^{(1)*} = 0$*, and* $p_{10}^{(1)*} = 0$*.*

At $t = 0$*, by Proposition 3.4, the necessary and sufficient condition for Player 1 to adopt strict negative advertisement is*

$$\begin{aligned} \left(1 - p_{22}^{(0)*}\right) V_2^{(0)} &> p_{01}^{(0)*} V_0^{(0)} \\ \text{and } p_{01}^{(1)*} - p_{11}^{(1)*} &\geq \delta / (1+\beta) . \end{aligned}$$

Moreover, condition (13) in Proposition 3.4 is also $p_{01}^{(1)*} - p_{11}^{(1)*} \geq \delta/(1+\beta)$*, which drives* $p_{01}^{(0)*}$ *down to* 0*. Therefore, in this case, Player 1 adopts strict negative advertisement if and only if* $p_{22}^{(0)*} < 1$ *and* $p_{01}^{(1)*} - p_{11}^{(1)*} \geq \delta/(1+\beta)$*. Similarly, Player 2 adopts negative advertisement if and only if* $p_{11}^{(0)*} < 1$ *and* $p_{02}^{(1)*} - p_{22}^{(1)*} \geq \delta/(1+\beta)$*. The following figure* *shows the partitions of equilibrium choices at $t = 1$.*



With the increase of $p_{11}^{(1)*} + \delta/(1+\beta)$ and $p_{22}^{(1)*} + \delta/(1+\beta)$, the region for both Player 1 and Player 2's strictly negative advertisement shrinks and then the region for both positive advertisement appears. Moreover, the increase of β will increase the regions of strictly negative advertisement.

In order to illustrate the effect of initial market shares, we give up the equal marginal costs assumption and provide the following example.

Example 3.9 *Consider the special case $\delta_1^{(2)} = \delta_2^{(2)} = 1$ and $\delta_1^{(1)} = \delta_2^{(1)} = 0$. Suppose there is no discount, i.e., $r = 1$, and $\beta_1 = \beta_2 = \beta$. All the marginal costs are equal to c except that $c_{21} = c_{12} = c_{01} = c_{02} = (1-\gamma)c$, where $\gamma > 0$.*

Similar to Example 3.8, neither of the players will adopt negative advertisement. The equilibrium choices at $t = 1$ satisfy $p_{01}^{(1)*} + p_{02}^{(1)*} = 1$, $p_{20}^{(1)*} = 0$, and $p_{10}^{(1)*} = 0$. To focus more *on effects of the initial market shares, we consider the symmetric equilibrium, i.e.,*

$$p_{01}^{(1)*} = p_{02}^{(1)*},\ \textit{and}\ p_{11}^{(1)*} = p_{22}^{(1)*} < p_{01}^{(1)*}.$$

Denote

$$\Delta p \triangleq p_{01}^{(1)*} - p_{11}^{(1)*} = p_{02}^{(1)*} - p_{22}^{(1)*}.$$

At $t = 0$, by Proposition 3.4, the necessary and sufficient condition for Player 1 to adopt strictly negative advertisement is

$$\begin{aligned} \left(1 - p_{22}^{(0)*}\right) V_2^{(0)} &> p_{01}^{(0)*} V_0^{(0)} \\ \textit{and}\ p_{01}^{(1)*} - p_{11}^{(1)*} &\geq \gamma c / \left(r^2 (1+\beta) V_2^{(0)}\right). \end{aligned}$$

It is noticed that condition (13) will drive $p_{01}^{(0)*}$ down to 0. Similarly, Player 2 adopts strictly negative advertisement if and only if

$$\begin{aligned} \left(1 - p_{11}^{(0)*}\right) V_1^{(0)} &> p_{02}^{(0)*} V_0^{(0)} \\ \textit{and}\ p_{02}^{(1)*} - p_{22}^{(1)*} &\geq \gamma c / \left(r^2 (1+\beta) V_1^{(0)}\right). \end{aligned}$$

Therefore, we have the following partitions on initial market shares, where we make condition (11) strict.

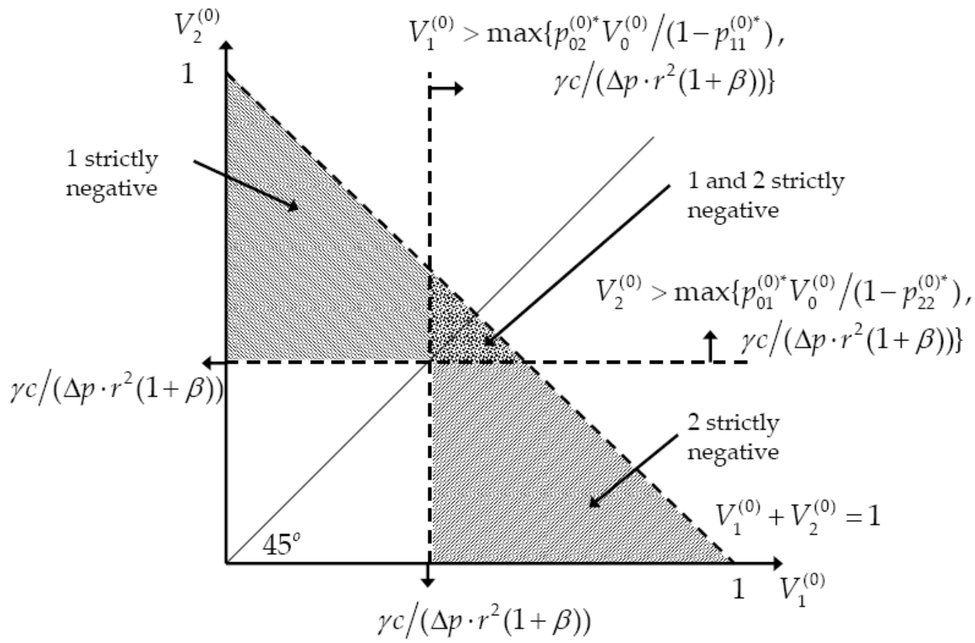
For some $p_{01}^{(0)*}$, $p_{02}^{(1)*}$, $p_{22}^{(0)*}$, and $p_{11}^{(0)*}$ Player 1 adopts strictly negative advertisement if and only if

$$V_2^{(0)} > \max\left\{p_{01}^{(0)*} V_0^{(0)} / \left(1 - p_{22}^{(0)*}\right), \gamma c / \left(\Delta p r^2 (1+\beta)\right)\right\};$$

Player 2 adopts strictly negative advertisement if and only if

$$V_1^{(0)} > \max\left\{p_{02}^{(0)*} V_0^{(0)} / \left(1 - p_{11}^{(0)*}\right), \gamma c / \left(\Delta p r^2 (1+\beta)\right)\right\}.$$

The following figure shows the partition of initial market shares.



The decrease of $V_0^{(0)}$, given $V_1^{(0)}$ increases the region of negative advertisement. Fixed $V_0^{(0)}$, smaller $V_1^{(0)}$ and larger $V_2^{(0)}$ make negative advertisement more possible.

4 Convex Cost Functions

We consider strictly convex cost functions.

The Bellman equations for player 1 are

$$J_T\left(V_1^{(T)}, V_2^{(T)}\right) = \delta_1^{(T)} V_1^{(T)} - \beta_1 \delta_2^{(T)} V_2^{(T)}, \tag{16}$$

$$
\begin{aligned}
&J_t\left(V_1^{(t)}, V_2^{(t)}\right) = \max_{\{p_{01}^{(t)}, p_{11}^{(t)}, p_{21}^{(t)}, p_{20}^{(t)}, p_{001}^{(t)}\}} \{\delta_1^{(t)} V_1^{(t)} - \beta_1 \delta_2^{(t)} V_2^{(t)} \\
&- \left[C_{01}(p_{01}^{(t)}) + C_{21}(p_{21}^{(t)}) + C_{20}(p_{20}^{(t)}) + C_{11}(p_{11}^{(t)}) + C_{001}(p_{001}^{(t)})\right] \\
&+ r J_{t+1}\left(V_1^{(t+1)}, V_2^{(t+1)}\right)\} \text{ for } 0 \leq t \leq T-1,
\end{aligned} \tag{17}
$$

where

$$
V_1^{(t+1)} = V_0^{(t)} p_{01}^{(t)} + V_1^{(t)} p_{11}^{(t)} + V_2^{(t)} p_{21}^{(t)}, \tag{18}
$$

$$
V_2^{(t+1)} = V_0^{(t)} p_{02}^{(t)*} + V_1^{(t)} p_{12}^{(t)*} + V_2^{(t)} p_{22}^{(t)*}, \tag{19}
$$

and

$$
1 = V_0^{(t)} + V_1^{(t)} + V_2^{(t)}.
$$

The feasibility constraints are

$$
\begin{aligned}
p_{001}^{(t)} + p_{002}^{(t)*} + p_{01}^{(t)} + p_{02}^{(t)*} &= 1, & (20) \\
p_{10}^{(t)*} + p_{11}^{(t)} + p_{12}^{(t)*} &= 1, & (21) \\
\text{and } p_{20}^{(t)} + p_{21}^{(t)} + p_{22}^{(t)*} &= 1. & (22)
\end{aligned}
$$

4.1 Optimal Transitions

Lemma 4.1 *With strictly convex cost functions, the subgame in the last period, i.e., $t = T-1$, has a pure strategy Nash Equilibrium.*

Remarks: The equilibrium is not unique. since the feasibility constraints are the same for both player 1 and player 2, there must be at least 3 degrees of freedom. Hence, the equilibrium is not unique.

Proof. At $t = T-1$, given $V_1^{(T-1)}, V_2^{(T-1)}$, we take play 1 for example and player 1's objective function

$$
\begin{aligned}
&\delta_1^{(T-1)} V_1^{(T-1)} - \beta_1 \delta_2^{(T-1)} V_2^{(T-1)} \\
&- \left[C_{01}(p_{01}^{(T-1)}) + C_{21}(p_{21}^{(T-1)}) + C_{20}(p_{20}^{(T-1)}) + C_{11}(p_{11}^{(T-1)}) + C_{001}(p_{001}^{(T-1)}) \right] \\
&+ r J_T \left(V_1^{(T)}, V_2^{(T)} \right)
\end{aligned}
$$

is concave in $\left\{ p_{01}^{(T-1)}, p_{11}^{(T-1)}, p_{21}^{(T-1)}, p_{20}^{(T-1)}, p_{001}^{(T-1)} \right\}$, since $J_T \left(V_1^{(T)}, V_2^{(T)} \right)$ is linear in $V_1^{(T)}$ and $V_2^{(T)}$, which are also linear in $\left\{ p_{01}^{(T-1)}, p_{11}^{(T-1)}, p_{21}^{(T-1)}, p_{20}^{(T-1)}, p_{001}^{(T-1)} \right\}$ by (18) and (19). Moreover, the strategy space is non-empty, compact and convex by (20) (21), (22) and the non-negativity constraints. By Kakutani fixed point, there exists a pure strategy Nash Equilibrium. ∎

Lemma 4.2 *For the subgame game starting at $T-2$ (take $p_{11}^{(T-1)*}$, $p_{22}^{(T-1)*}$ and $p_{02}^{(T-1)*}$ as free equilibrium choices), there exists a pure strategy subgame perfect Nash Equilibrium if marginal costs satisfy*

$$
\min \left\{ C_{01}''(p_{01}^{(T-2)}), C_{21}''(p_{21}^{(T-2)}), C_{11}''(p_{11}^{(T-2)}) \right\} \geq r J_{T-1,V_1V_1},
$$

where

$$
J_{T-1,V_1V_1} = \frac{\left(r\delta_1^{(T)} \right)^2 + r^2 \delta_1^{(T)} \delta_2^{(T)} C_{001}'' \left(p_{001}^{(T-1)*} \right) / C_{002}'' \left(p_{002}^{(T-1)*} \right)}{C_{01}'' \left(p_{01}^{(T-1)*} \right) + C_{001}'' \left(p_{001}^{(T-1)*} \right)} - \frac{\beta_1 \left(r\delta_2^{(T)} \right)^2}{C_{12}'' \left(p_{12}^{(T-1)*} \right) + C_{20}'' \left(p_{10}^{(T-1)*} \right)}.
$$

Remarks:

1) To have a simple view, let the cost functions be quadratic, i.e., $C(p) = cp_{ij}^2/2$ with equal marginal cost rate c, $r = 1$, and $\delta_1^{(T)} = \delta_1^{(T)} = 1$. Then this condition is $c^2 \geq (1 - \beta_1/2)$.

2) If β_1 is sufficiently large such that J_{T-1,V_1V_1} is negative, the condition hold.

Proof. Denote Player 1's value function as J and Player 2's value function as H.

Without loss of generality, take Player 1 for example. $J_T \left(V_1^{(T)}, V_2^{(T)} \right)$ is linear in both $V_1^{(T)}$ and $V_2^{(T)}$. By (18) and (19), which are linear in $\left\{ p_{01}^{(t)}, p_{11}^{(t)}, p_{21}^{(t)}, p_{20}^{(t)}, p_{001}^{(t)} \right\}$, $J_T \left(V_1^{(T)}, V_2^{(T)} \right)$

must also be linear in $\left\{p_{01}^{(T-1)}, p_{11}^{(T-1)}, p_{21}^{(T-1)}, p_{20}^{(T-1)}, p_{001}^{(T-1)}\right\}$. We will show Player 1's objective at $t = T-2$ is concave in the $\left\{p_{01}^{(T-2)}, p_{11}^{(T-2)}, p_{21}^{(T-2)}, p_{20}^{(T-2)}, p_{001}^{(T-2)}\right\}$,

$$
\begin{aligned}
& OBJ_1^{(T-2)} \\
= \; & \delta_1^{(T-2)} V_1^{(T-2)} - \beta_1 \delta_2^{(T-2)} V_2^{(T-2)} \\
& - \left[C_{01}(p_{01}^{(T-2)}) + C_{21}(p_{21}^{(T-2)}) + C_{20}(p_{20}^{(T-2)}) + C_{11}(p_{11}^{(T-2)}) + C_{001}(p_{001}^{(T-2)}) \right] \\
& + r J_{T-1}(p_{01}^{(T-2)} + V_1^{(T-2)} \left(p_{11}^{(T-2)} - p_{01}^{(T-2)} \right) + V_2^{(T-2)} \left(p_{21}^{(T-2)} - p_{01}^{(T-2)} \right), \\
& \qquad p_{02}^{(T-2)*} + V_1^{(T-2)} \left(p_{12}^{(T-2)*} - p_{02}^{(T-2)*} \right) + V_2^{(T-2)} \left(p_{22}^{(T-2)*} - p_{02}^{(T-2)*} \right)).
\end{aligned}
$$

Suppose the first order conditions are valid. Let $\lambda_0^{(t)}$, $\lambda_1^{(t)}$, and $\lambda_2^{(t)}$ be the Lagrangian multipliers on

$$
\begin{gathered}
p_{001}^{(t)} + p_{002}^{(t)*} + p_{01}^{(t)} + p_{02}^{(t)*} = 1, \\
p_{10}^{(t)*} + p_{11}^{(t)} + p_{12}^{(t)*} = 1, \\
\text{and } p_{20}^{(t)} + p_{21}^{(t)} + p_{22}^{(t)*} = 1,
\end{gathered}
$$

respectively. We consider the interior solution. First order conditions are as follows.

$$
\begin{aligned}
p_{01}^{(t)} &: -C_{01}' \left(p_{01}^{(t)*} \right) + r J_{t+1,V_1} V_0^{(t)} - \lambda_0^{(t)} = 0; && (23) \\
p_{11}^{(t)} &: -C_{11}' \left(p_{11}^{(t)*} \right) + r J_{t+1,V_1} V_1^{(t)} - \lambda_1^{(t)} = 0; && (24) \\
p_{21}^{(t)} &: -C_{21}' \left(p_{21}^{(t)*} \right) + r J_{t+1,V_1} V_2^{(t)} - \lambda_2^{(t)} = 0; && (25) \\
p_{20}^{(t)} &: -C_{20}' \left(p_{20}^{(t)*} \right) - \lambda_2^{(t)} = 0; && (26) \\
p_{001}^{(t)} &: -C_{001}' \left(p_{001}^{(t)*} \right) - \lambda_0^{(t)} = 0. && (27)
\end{aligned}
$$

Envelope theorem indicates

$$
J_{t,V_1} = \delta_1^{(t)} + r J_{t+1,V_1} \cdot \left(p_{11}^{(t)*} - p_{01}^{(t)*} \right) + r J_{t+1,V_2} \cdot \left(p_{12}^{(t)*} - p_{02}^{(t)*} \right), \tag{28}
$$

$$
J_{t,V_2} = -\beta_1 \delta_2^{(t)} + r J_{t+1,V_1} \cdot \left(p_{21}^{(t)*} - p_{01}^{(t)*} \right) + r J_{t+1,V_2} \cdot \left(p_{22}^{(t)*} - p_{02}^{(t)*} \right), \tag{29}
$$

where

$$J_{T+1,V_1} = J_{T+1,V_2} = 0.$$

Let $t = T-1$, first order conditions are valid. Envelope theorem indicates

$$J_{T-1,V_1} = \delta_1^{(T-1)} + r\delta_1^{(T)} \cdot \left(p_{11}^{(T-1)*} - p_{01}^{(T-1)*}\right) + r\left(-\beta_1\delta_2^{(T)}\right) \cdot \left(p_{12}^{(T-1)*} - p_{02}^{(T-1)*}\right)$$

and

$$J_{T-1,V_2} = -\beta_1\delta_2^{(T-1)} + r\delta_1^{(T)} \cdot \left(p_{21}^{(T-1)*} - p_{01}^{(T-1)*}\right) + r\left(-\beta_1\delta_2^{(T)}\right) \cdot \left(p_{22}^{(T-1)*} - p_{02}^{(T-1)*}\right).$$

Since the equilibriums are not uniuqe and there are three degrees of freedom, we can regard $p_{11}^{(T-1)*}$, $p_{22}^{(T-1)*}$ and $p_{02}^{(T-1)*}$ as freely chosen, which are independent of the state variable $V_0^{(T-1)}$, $V_1^{(T-1)}$, and $V_2^{(T-1)}$.

By the first order conditions and envelope theorem,

$$\begin{aligned} C'_{21}\left(p_{21}^{(T-1)*}\right) - C'_{20}\left(p_{20}^{(T-1)*}\right) &= r\delta_1^{(T)}V_2^{(T-1)}, \\ C'_{01}\left(p_{01}^{(T-1)*}\right) - C'_{001}\left(p_{001}^{(T-1)*}\right) &= r\delta_1^{(T)}V_0^{(T-1)}. \end{aligned}$$

Hence,

$$C''_{21}\left(p_{21}^{(T-1)*}\right)\frac{\partial p_{21}^{(T-1)*}}{\partial V_1^{(T-1)}} - C''_{20}\left(p_{20}^{(T-1)*}\right)\frac{\partial p_{20}^{(T-1)*}}{\partial V_1^{(T-1)}} = 0,$$

$$C''_{21}\left(p_{21}^{(T-1)*}\right)\frac{\partial p_{21}^{(T-1)*}}{\partial V_2^{(T-1)}} - C''_{20}\left(p_{20}^{(T-1)*}\right)\frac{\partial p_{20}^{(T-1)*}}{\partial V_2^{(T-1)}} = r\delta_1^{(T)},$$

and

$$C''_{01}\left(p_{01}^{(T-1)*}\right)\frac{\partial p_{01}^{(T-1)*}}{\partial V_{1(\text{or }2)}^{(T-1)}} - C''_{001}\left(p_{001}^{(T-1)*}\right)\frac{\partial p_{001}^{(T-1)*}}{\partial V_{1(\text{or }2)}^{(T-1)}} = -r\delta_1^{(T)}.$$

Moreover, since

$$
\begin{aligned}
p_{21}^{(T-1)*} + p_{20}^{(T-1)*} &= 1 - p_{22}^{(T-1)*}, \\
p_{01}^{(T-1)*} + p_{001}^{(T-1)*} &= 1 - p_{02}^{(T-1)*} - p_{002}^{(T-1)*},
\end{aligned}
$$

we have

$$\frac{\partial p_{21}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} + \frac{\partial p_{20}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} = 0,$$

and

$$\frac{\partial p_{01}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} + \frac{\partial p_{001}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} = -\frac{\partial p_{002}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}}.$$

Similarly, for Player 2, we have

$$C_{12}''\left(p_{12}^{(T-1)*}\right)\frac{\partial p_{12}^{(T-1)*}}{\partial V_2^{(T-1)}} - C_{10}''\left(p_{10}^{(T-1)*}\right)\frac{\partial p_{10}^{(T-1)*}}{\partial V_2^{(T-1)}} = 0,$$

$$C_{12}''\left(p_{12}^{(T-1)*}\right)\frac{\partial p_{12}^{(T-1)*}}{\partial V_1^{(T-1)}} - C_{10}''\left(p_{10}^{(T-1)*}\right)\frac{\partial p_{10}^{(T-1)*}}{\partial V_1^{(T-1)}} = r\delta_2^{(T)},$$

$$-C_{002}''\left(p_{002}^{(T-1)*}\right)\frac{\partial p_{002}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}} = -r\delta_2^{(T)},$$

and

$$\frac{\partial p_{12}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}} + \frac{\partial p_{10}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}} = 0,$$

$$\frac{\partial p_{002}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}} = -\frac{\partial p_{01}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}} - \frac{\partial p_{001}^{(T-1)*}}{\partial V_{2(\text{or } 1)}^{(T-1)}}.$$

Since the cost functions are strictly convex, we can get

$$\frac{\partial p_{21}^{(T-1)*}}{\partial V_1^{(T-1)}} = \frac{\partial p_{20}^{(T-1)*}}{\partial V_1^{(T-1)}} = \frac{\partial p_{12}^{(T-1)*}}{\partial V_2^{(T-1)}} = \frac{\partial p_{10}^{(T-1)*}}{\partial V_2^{(T-1)}} = 0,$$

$$\frac{\partial p_{21}^{(T-1)*}}{\partial V_2^{(T-1)}} = -\frac{\partial p_{20}^{(T-1)*}}{\partial V_2^{(T-1)}} = \frac{r\delta_1^{(T)}}{C_{21}''\left(p_{21}^{(T-1)*}\right) + C_{20}''\left(p_{20}^{(T-1)*}\right)},$$

$$\frac{\partial p_{12}^{(T-1)*}}{\partial V_1^{(T-1)}} = -\frac{\partial p_{10}^{(T-1)*}}{\partial V_1^{(T-1)}} = \frac{r\delta_2^{(T)}}{C_{12}''\left(p_{12}^{(T-1)*}\right) + C_{10}''\left(p_{10}^{(T-1)*}\right)},$$

$$\frac{\partial p_{002}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} = \frac{r\delta_2^{(T)}}{C_{002}''\left(p_{002}^{(T-1)*}\right)},$$

$$\begin{aligned}\frac{\partial p_{01}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} &= \frac{-r\delta_1^{(T)} + \left(-\partial p_{002}^{(T-1)*}/\partial V_{1(\text{or } 2)}^{(T-1)}\right) C_{001}''\left(p_{001}^{(T-1)*}\right)}{C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)} \\ &= \frac{-r\delta_1^{(T)} - r\delta_2^{(T)} C_{001}''\left(p_{001}^{(T-1)*}\right)/C_{002}''\left(p_{002}^{(T-1)*}\right)}{C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial p_{001}^{(T-1)*}}{\partial V_{1(\text{or } 2)}^{(T-1)}} &= \frac{r\delta_1^{(T)} - \left(\partial p_{002}^{(T-1)*}/\partial V_{1(\text{or } 2)}^{(T-1)}\right) C_{01}''\left(p_{01}^{(T-1)*}\right)}{\left(C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)\right)} \\ &= \frac{r\delta_1^{(T)} - r\delta_2^{(T)} C_{01}''\left(p_{01}^{(T-1)*}\right)/C_{002}''\left(p_{002}^{(T-1)*}\right)}{\left(C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)\right)}.\end{aligned}$$

Hence, the Hessian for Player 1 and Player 2's value functions are

$$\begin{aligned}J_{T-1,V_1V_1} = &\frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)} C_{001}''\left(p_{001}^{(T-1)*}\right)/C_{002}''\left(p_{002}^{(T-1)*}\right)}{C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)} \\ &-\frac{\beta_1\left(r\delta_2^{(T)}\right)^2}{C_{12}''\left(p_{12}^{(T-1)*}\right) + C_{10}''\left(p_{10}^{(T-1)*}\right)},\end{aligned}$$

$$J_{T-1,V_1V_2} = \frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)} C_{001}''\left(p_{001}^{(T-1)*}\right)/C_{002}''\left(p_{002}^{(T-1)*}\right)}{C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)}$$

$$J_{T-1,V_2V_1} = \frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)} C_{001}''\left(p_{001}^{(T-1)*}\right)/C_{002}''\left(p_{002}^{(T-1)*}\right)}{C_{01}''\left(p_{01}^{(T-1)*}\right) + C_{001}''\left(p_{001}^{(T-1)*}\right)}$$

$$J_{T-1,V_2V_2} = \frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}C''_{001}\left(p_{001}^{(T-1)*}\right)/C''_{002}\left(p_{002}^{(T-1)*}\right)}{C''_{01}\left(p_{01}^{(T-1)*}\right)+C''_{001}\left(p_{001}^{(T-1)*}\right)} + \frac{\left(r\delta_1^{(T)}\right)^2}{C''_{21}\left(p_{21}^{(T-1)*}\right)+C''_{20}\left(p_{20}^{(T-1)*}\right)},$$

$$H_{T-1,V_2V_2} = -\beta_2\left(\frac{\left(r\delta_1^{(T)}\right)^2}{C''_{21}\left(p_{21}^{(T-1)*}\right)+C''_{20}\left(p_{20}^{(T-1)*}\right)} + \frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}C''_{001}\left(p_{001}^{(T-1)*}\right)/C''_{002}\left(p_{002}^{(T-1)*}\right)}{C''_{01}\left(p_{01}^{(T-1)*}\right)+C''_{001}\left(p_{001}^{(T-1)*}\right)}\right),$$

$$H_{T-1,V_2V_1} = -\beta_2\frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}C''_{001}\left(p_{001}^{(T-1)*}\right)/C''_{002}\left(p_{002}^{(T-1)*}\right)}{C''_{01}\left(p_{01}^{(T-1)*}\right)+C''_{001}\left(p_{001}^{(T-1)*}\right)},$$

$$H_{T-1,V_1V_2} = -\beta_2\frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}C''_{001}\left(p_{001}^{(T-1)*}\right)/C''_{002}\left(p_{002}^{(T-1)*}\right)}{C''_{01}\left(p_{01}^{(T-1)*}\right)+C''_{001}\left(p_{001}^{(T-1)*}\right)},$$

and

$$H_{T-1,V_1V_1} = \frac{\left(r\delta_2^{(T)}\right)^2}{C''_{12}\left(p_{12}^{(T-1)*}\right)+C''_{10}\left(p_{10}^{(T-1)*}\right)} - \beta_2\frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}C''_{001}\left(p_{001}^{(T-1)*}\right)/C''_{002}\left(p_{002}^{(T-1)*}\right)}{C''_{01}\left(p_{01}^{(T-1)*}\right)+C''_{001}\left(p_{001}^{(T-1)*}\right)}.$$

Then, since $p_{20}^{(T-2)}$, $p_{001}^{(T-2)}$ and $\left\{p_{01}^{(T-2)}, p_{11}^{(T-2)}, p_{21}^{(T-2)}\right\}$ are separable, we have

$$\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{20}^{(T-2)}\right)^2} = -C''_{20}(p_{20}^{(T-2)}),$$

$$\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{001}^{(T-2)}\right)^2} = -C''_{001}(p_{001}^{(T-2)}),$$

$$\begin{aligned}
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{01}^{(T-2)}\right)^2} &= -C''_{01}(p_{01}^{(T-2)}) + rJ_{T-1,V_1V_1}\cdot\left(V_0^{(T-2)}\right)^2,\\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{01}^{(T-2)}p_{21}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_0^{(T-2)}V_2^{(T-2)},\\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{01}^{(T-2)}p_{11}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_0^{(T-2)}V_1^{(T-2)},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{21}^{(T-2)}\right)^2} &= -C_{21}''(p_{21}^{(T-2)}) + rJ_{T-1,V_1V_1}\cdot\left(V_2^{(T-2)}\right)^2, \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{21}^{(T-2)} p_{01}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_2^{(T-2)}V_0^{(T-2)}, \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{21}^{(T-2)} p_{11}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_2^{(T-2)}V_1^{(T-2)},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{11}^{(T-2)}\right)^2} &= -C_{11}''(p_{11}^{(T-2)}) + rJ_{T-1,V_1V_1}\cdot\left(V_1^{(T-2)}\right)^2, \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{11}^{(T-2)} p_{01}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_1^{(T-2)}V_0^{(T-2)}, \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{11}^{(T-2)} p_{21}^{(T-2)}} &= rJ_{T-1,V_1V_1}V_1^{(T-2)}V_2^{(T-2)}.
\end{aligned}
$$

Therefore, $OBJ_1^{(T-2)}$ is concave in $\left\{p_{01}^{(T-2)}, p_{11}^{(T-2)}, p_{21}^{(T-2)}, p_{20}^{(T-2)}, p_{001}^{(T-2)}\right\}$ iff

$$
\begin{bmatrix}
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{01}^{(T-2)}\right)^2}, & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{01}^{(T-2)} p_{21}^{(T-2)}}, & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{01}^{(T-2)} p_{11}^{(T-2)}} \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{21}^{(T-2)} p_{01}^{(T-2)}}, & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{21}^{(T-2)}\right)^2}, & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{21}^{(T-2)} p_{11}^{(T-2)}} \\
\frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{11}^{(T-2)} p_{01}^{(T-2)}} & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial p_{11}^{(T-2)} p_{21}^{(T-2)}} & \frac{\partial^2 OBJ_1^{(T-2)}}{\partial\left(p_{11}^{(T-2)}\right)^2}
\end{bmatrix}
$$

is negative semidefinite. The requirement is

$$
\begin{aligned}
rJ_{T-1,V_1V_1}\cdot\left(V_0^{(T-2)}\right)^2 &\leq C_{01}''(p_{01}^{(T-2)}), \\
rJ_{T-1,V_1V_1}\cdot\left(V_2^{(T-2)}\right)^2 &\leq C_{21}''(p_{21}^{(T-2)}), \\
rJ_{T-1,V_1V_1}\cdot\left(V_1^{(T-2)}\right)^2 &\leq C_{11}''(p_{11}^{(T-2)}),
\end{aligned}
$$

$$
\begin{aligned}
rJ_{T-1,V_1V_1} &\leq C''_{01}(p_{01}^{(T-2)})C''_{21}(p_{21}^{(T-2)})/\left(\left(V_0^{(T-2)}\right)^2 C''_{21}(p_{21}^{(T-2)}) + \left(V_2^{(T-2)}\right)^2 C''_{01}(p_{01}^{(T-2)})\right), \\
rJ_{T-1,V_1V_1} &\leq C''_{01}(p_{01}^{(T-2)})C''_{11}(p_{11}^{(T-2)})/\left(\left(V_0^{(T-2)}\right)^2 C''_{11}(p_{11}^{(T-2)}) + \left(V_1^{(T-2)}\right)^2 C''_{01}(p_{01}^{(T-2)})\right), \\
rJ_{T-1,V_1V_1} &\leq C''_{11}(p_{01}^{(T-2)})C''_{21}(p_{21}^{(T-2)})/\left(\left(V_1^{(T-2)}\right)^2 C''_{21}(p_{21}^{(T-2)}) + \left(V_2^{(T-2)}\right)^2 C''_{11}(p_{11}^{(T-2)})\right),
\end{aligned}
$$

and

$$
rJ_{T-1,V_1V_1} \leq \frac{C''_{01}(p_{01}^{(T-2)})C''_{21}(p_{21}^{(T-2)})C''_{11}(p_{01}^{(T-2)})}{\left(V_1^{(T-2)}\right)^2 C''_{01}(p_{01}^{(T-2)})C''_{21}(p_{21}^{(T-2)}) + \left(V_2^{(T-2)}\right)^2 C''_{01}(p_{01}^{(T-2)})C''_{11}(p_{11}^{(T-2)}) + \left(V_0^{(T-2)}\right)^2 C''_{21}(p_{21}^{(T-2)})C''_{11}(p_{11}^{(T-2)})}.
$$

Then a sufficient condition is

$$
rJ_{T-1,V_1V_1} \leq \min\left\{C''_{01}(p_{01}^{(T-2)}), C''_{21}(p_{21}^{(T-2)}), C''_{11}(p_{11}^{(T-2)})\right\}.
$$

Similarly, Player 2's objective is concave in $\left\{p_{02}^{(T-2)}, p_{22}^{(T-2)}, p_{12}^{(T-2)}, p_{10}^{(T-2)}, p_{002}^{(T-2)}\right\}$ if

$$
rH_{T-1,V_2V_2} \leq \min\left\{C''_{02}(p_{02}^{(T-2)}), C''_{12}(p_{12}^{(T-2)}), C''_{22}(p_{22}^{(T-2)})\right\},
$$

which obviously holds since $H_{T-1,V_2V_2} \leq 0$.

Therefore, by Kakutani fixed point, there exists a pure strategy Nash Equilibrium at $t = T - 2$ as well. ■

4.2 Two period case $T = 2$

At $t = 1$, by the first order conditions,

$$
\begin{aligned}
C'_{21}\left(p_{21}^{(1)*}\right) - C'_{20}\left(p_{20}^{(1)*}\right) &= r\delta_1^{(2)}V_2^{(1)} \\
C'_{01}\left(p_{01}^{(1)*}\right) - C'_{001}\left(p_{001}^{(1)*}\right) &= r\delta_1^{(2)}V_0^{(1)}.
\end{aligned}
$$

Assume quadratic cost function, i.e., $C(p) = c_{ij}p_{ij}^2/2$. With quadratic cost functions, first order conditions lead to

$$
\begin{array}{c}
p_{11}^{(1)*} = 1 - p_{12}^{(1)*} - p_{10}^{(1)*}; \\
p_{01}^{(1)*} = \frac{1}{c_{01}+c_{001}} \left(c_{001} \left(1 - p_{002}^{(1)*} - p_{02}^{(1)*} \right) + r\delta_1^{(2)} V_0^{(1)} \right); \\
p_{001}^{(1)*} = \frac{1}{c_{01}+c_{001}} \left(c_{01} \left(1 - p_{002}^{(1)*} - p_{02}^{(1)*} \right) - r\delta_1^{(2)} V_0^{(1)} \right); \\
p_{21}^{(1)*} = \frac{1}{c_{20}+c_{21}} \left(c_{20} \left(1 - p_{22}^{(1)*} \right) + r\delta_1^{(2)} V_2^{(1)} \right); \\
p_{20}^{(1)*} = \frac{1}{c_{20}+c_{21}} \left(c_{21} \left(1 - p_{22}^{(1)*} \right) - r\delta_1^{(2)} V_2^{(1)} \right).
\end{array}
$$

The following assumptions ensure that the first order conditions are valid (the interior solutions).

Assumption 3A $c_{21} \geq r\delta_1^{(2)} V_2^{(1)}$

If $c_{21} < r\delta_1^{(2)} V_2^{(1)}$, $p_{20}^{(1)*} = 0$. This indicates no strictly negative advertisement in the last period.

Assumption 3B $c_{01} \geq r\delta_1^{(2)} V_0^{(1)}$

If $c_{01} < r\delta_1^{(2)} V_0^{(1)}$, $p_{001}^{(1)} = 0$. However, $p_{01}^{(1)*}$ can still be the interior solution.

We have the following proposition as the necessary and sufficient condition for no negative advertisement in the last period.

Proposition 4.3 *With quadratic cost functions, Assumption 3A and 3B, Player 1 will not play strictly negative advertisement in the last period if and only if*

$$
\begin{array}{c}
\Delta < 0 \text{ } or \\
V_2^{(1)} \leq \frac{c_{01}c_{21}\left(1-p_{22}^{(1)*}\right)-\sqrt{\Delta}}{2c_{01}\delta_1^{(2)} r} \text{ } or \text{ } V_2^{(1)} \geq \frac{c_{01}c_{21}\left(1-p_{22}^{(1)*}\right)+\sqrt{\Delta}}{2c_{01}\delta_1^{(2)} r} \text{ } for \text{ } \Delta \geq 0,
\end{array}
$$

where

$$
\Delta = c_{01}^2 c_{21}^2 \left(1 - p_{22}^{(1)*} \right)^2 - 4c_{01} \left(c_{20} + c_{21} \right) \delta_1 r V_0^{(1)} \left(c_{001} p_{001}^{(1)*} + r\delta_1^{(2)} V_0^{(1)} \right).
$$

Moreover, fixed $V_0^{(1)}$, smaller Player 1's market share is, more negative the advertisement *is iff Player 2 is not too strong, i.e.,*

$$V_2^{(1)} \leq c_{21}\left(1 - p_{22}^{(1)*}\right) / \left(2\delta_1 r\right).$$

Proof. The condition is just a direct result from the equilibrium solution,

$$\begin{array}{c} V_0^{(1)}\left(c_{001}p_{001}^{(1)*} + r\delta_1^{(2)}V_0^{(1)}\right)/c_{01} \\ -V_2^{(1)}\left(c_{21}\left(1 - p_{22}^{(1)*}\right) - r\delta_1^{(2)}V_2^{(1)}\right)/\left(c_{20} + c_{21}\right) > 0. \end{array}$$

■

To understand the proposition 4.3, The following corollary provides a sufficient condition of no negative advertisement.

Corollary 4.4 *With quadratic cost functions and Assumption 3B, Player 1 will not play strictly negative advertisement in the last period if*

$$V_0^{(1)} \geq \sqrt{c_{01}c_{21}^2 / \left(4\left(c_{20} + c_{21}\right) r^2 \left(\delta_1^{(2)}\right)^2\right)},$$

or

$$V_2^{(1)} \geq c_{21} / \left(r\delta_1^{(2)}\right).$$

Remarks: To have a simple view, let $c_{01} = c_{21} = c_{20} = c$, $r = 1$ and $\delta_1^{(2)} = 1$. Then the sufficient conditions above are

$$V_0^{(1)} \geq c\sqrt{1/8} \text{ or } V_2^{(1)} \geq c.$$

Proof. From the proof of proposition 4.3, with quadratic cost functions, Assumption 3A and 3B the positive advertisement strategy index is

$$
\begin{aligned}
d_1^{(1)} &= p_{01}^{(1)*} V_0^{(1)} - p_{20}^{(1)*} V_2^{(1)} \\
&= V_0^{(1)} \left(c_{001} p_{001}^{(1)*} + r\delta_1^{(2)} V_0^{(1)} \right) / c_{01} \\
&\quad - V_2^{(1)} \left(c_{21} \left(1 - p_{22}^{(1)*} \right) - r\delta_1^{(2)} V_2^{(1)} \right) / \left(c_{20} + c_{21} \right) \\
&\geq \left(V_0^{(1)} \right)^2 r\delta_1^{(2)} / c_{01} - V_2^{(1)} \left(c_{21} - r\delta_1^{(2)} V_2^{(1)} \right) / \left(c_{20} + c_{21} \right).
\end{aligned}
$$

A sufficient condition of no strictly negative advertisement is

$$
\left(V_0^{(1)} \right)^2 r\delta_1^{(2)} / c_{01} \geq V_2^{(1)} \left(c_{21} - r\delta_1^{(2)} V_2^{(1)} \right) / \left(c_{20} + c_{21} \right).
$$

The maximal value of the right hand side is achieved when

$$
V_2^{(1)} = c_{21} / \left(2r\delta_1^{(2)} \right),
$$

and thus the maximal value is

$$
c_{21}^2 / \left(4 \left(c_{20} + c_{21} \right) r\delta_1^{(2)} \right).
$$

Therefore, this requires

$$
\left(V_0^{(1)} \right)^2 \geq c_{01} c_{21}^2 / \left(4 \left(c_{20} + c_{21} \right) r^2 \left(\delta_1^{(2)} \right)^2 \right).
$$

Another sufficient condition of no negative advertisement is

$$
0 \geq V_2^{(1)} \left(c_{21} - r\delta_1^{(2)} V_2^{(1)} \right) / \left(c_{20} + c_{21} \right).
$$

This requires

$$
V_2^{(1)} \leq 0 \text{ or } V_2^{(1)} \geq c_{21} / \left(r\delta_1^{(2)} \right).
$$

These are contradicted with the feasibility constraint and Assumption 3A if $V_2^{(1)} > c_{21}/\left(r\delta_1^{(2)}\right)$. However, we know if $c_{21} < r\delta_1^{(2)}V_2^{(1)}$, $p_{20}^{(1)*} = 0$, there is no strictly negative advertisement. Therefore, $V_2^{(1)} \geq c_{21}/\left(r\delta_1^{(2)}\right)$ is valid for no strictly negative advertisement. ∎

The following proposition consider $t = 0$. By the first order conditions and envelope theorem, we have

$$\begin{aligned} C'_{20}\left(p_{20}^{(0)*}\right) - C'_{21}\left(p_{21}^{(0)*}\right) &= r^2 A^{(1)} V_2^{(0)}, \\ C'_{001}\left(p_{001}^{(0)*}\right) - C'_{01}\left(p_{01}^{(0)*}\right) &= r^2 A^{(1)} V_0^{(0)}, \end{aligned}$$

where $A^{(1)}$ is defined as (similar in the linear cost functions),

$$A^{(1)} = \delta_1^{(2)} p_{01}^{(1)*} - \beta_1 \delta_2^{(2)} p_{02}^{(1)*} - \delta_1^{(2)} p_{11}^{(1)*} + \beta_1 \delta_2^{(2)} p_{12}^{(1)*} - \frac{\delta_1^{(1)}}{r}.$$

Assume quadratic cost functions, we have

$$\begin{aligned} p_{20}^{(0)*} &= \frac{1}{c_{20}+c_{21}}\left(c_{21}\left(1-p_{22}^{(0)*}\right) + r^2 A^{(1)} V_2^{(0)}\right); \\ p_{21}^{(0)*} &= \frac{1}{c_{20}+c_{21}}\left(c_{20}\left(1-p_{22}^{(0)*}\right) - r^2 A^{(1)} V_2^{(0)}\right); \\ p_{001}^{(0)*} &= \frac{1}{c_{001}+c_{01}}\left(c_{01}\left(1-p_{002}^{(0)*}-p_{02}^{(0)*}\right) + r^2 A^{(1)} V_0^{(0)}\right); \\ p_{01}^{(0)*} &= \frac{1}{c_{001}+c_{01}}\left(c_{001}\left(1-p_{002}^{(0)*}-p_{02}^{(0)*}\right) - r^2 A^{(1)} V_0^{(0)}\right). \end{aligned}$$

In the following discussion, we assume conditions in Lemma 4.2 hold and thus first order conditions at $t = 0$ are valid. With quadratic cost functions, the conditions in Lemma 4.2 are

$$\min\{c_{01}, c_{21}, c_{11}\} \geq r J_{T-1,V_1V_1},$$

where

$$J_{T-1,V_1V_1} = \frac{\left(r\delta_1^{(T)}\right)^2 + r^2\delta_1^{(T)}\delta_2^{(T)}c_{001}/c_{002}}{c_{01}+c_{001}} - \frac{\beta_1\left(r\delta_2^{(T)}\right)^2}{c_{12}+c_{10}}.$$

At $t = 0$, the following assumptions ensure that the first order conditions are valid (the interior solutions).

Assumption 4A $c_{21} \geq -r^2 A^{(1)} V_2^{(0)}$ and $c_{20} \geq r^2 A^{(1)} V_2^{(0)}$.

If $c_{21} < -r^2 A^{(1)} V_2^{(0)}$, $p_{20}^{(0)*} = 0$. This indicates no strictly negative advertisement in the first period.

Assumption 4B $c_{01} \geq -r^2 A^{(1)} V_0^{(0)}$ and $c_{001} \geq r^2 A^{(1)} V_0^{(0)}$.

If $c_{001} < r^2 A^{(1)} V_0^{(0)}$, $p_{01}^{(0)*} = 0$. There is negative advertisement (but may not be strict) in the first period.

Remarks: To have a simple view, let $c_{01} = c_{21} = c_{20} = c_{001} = c$, $r = 1$, $\delta_1^{(1)} = 0$, and $\delta_1^{(2)} = \delta_1^{(2)} = 1$, if

$$c < V_0^{(0)} \left(p_{01}^{(1)*} - p_{11}^{(1)*} + \beta_1 p_{12}^{(1)*} - \beta_1 p_{02}^{(1)*} \right),$$

Player 1 devotes to deterring the opponent, i.e., $p_{01}^{(0)*} = p_{21}^{(0)*} = 0$ and $p_{20}^{(0)*} = 1 - p_{22}^{(0)*}$. There is strictly negative advertisement in the first period if $p_{22}^{(0)*} < 1$.

Proposition 4.5 *With quadratic cost functions, Assumption 4A and 4B, Player 1 will play strictly negative advertisement in the first period if and only if*

$$\begin{aligned} & \left(\frac{1}{c_{001} + c_{01}} \left(V_0^{(0)} \right)^2 + \frac{1}{c_{20} + c_{21}} \left(V_2^{(0)} \right)^2 \right)^2 A^{(1)} \\ > \; & \frac{c_{001} \left(1 - p_{002}^{(0)*} - p_{02}^{(0)*} \right)}{c_{001} + c_{01}} V_0^{(0)} - \frac{c_{21} \left(1 - p_{22}^{(0)*} \right)}{c_{20} + c_{21}} V_2^{(0)}. \end{aligned}$$

Moreover, fixed $V_0^{(0)}$, if $A^{(1)} \geq 0$, smaller Player 1's market share is, more negative the advertisement is. However, if $A^{(1)} < 0$, it requires that Player 2 is not too strong, i.e.,

$$V_2^{(0)} \leq -c_{21} \left(1 - p_{22}^{(0)*} \right) / \left(2r^2 A^{(1)} \right).$$

Remarks: This condition is similar to conditions in Proposition 3.4, which also indicates a larger $A^{(1)}$ will trigger a strictly negative advertisement in the first period.

Proof. By the definition of positive advertisement strategy index,

$$
\begin{aligned}
d_1^{(0)} &= p_{01}^{(0)*} V_0^{(0)} - p_{20}^{(0)*} V_2^{(0)} \\
&= \frac{1}{c_{001}+c_{01}} \left(c_{001} \left(1 - p_{002}^{(0)*} - p_{02}^{(0)*} \right) - r^2 A^{(1)} V_0^{(0)} \right) V_0^{(0)} \\
&\quad - \frac{1}{c_{20}+c_{21}} \left(c_{21} \left(1 - p_{22}^{(0)*} \right) + r^2 A^{(1)} V_2^{(0)} \right) V_2^{(0)} \\
&= - \left(\frac{1}{c_{001}+c_{01}} \left(V_0^{(0)} \right)^2 + \frac{1}{c_{20}+c_{21}} \left(V_2^{(0)} \right)^2 \right)^2 r^2 A^{(1)} \\
&\quad + \frac{c_{001} \left(1 - p_{002}^{(0)*} - p_{02}^{(0)*} \right)}{c_{001}+c_{01}} V_0^{(0)} - \frac{c_{21} \left(1 - p_{22}^{(0)*} \right)}{c_{20}+c_{21}} V_2^{(0)}.
\end{aligned}
$$

Therefore, $d_1^{(0)} < 0$ if and only if

$$
\begin{aligned}
& \frac{c_{001} \left(1 - p_{002}^{(0)*} - p_{02}^{(0)*} \right)}{c_{001}+c_{01}} V_0^{(0)} - \frac{c_{21} \left(1 - p_{22}^{(0)*} \right)}{c_{20}+c_{21}} V_2^{(0)} \\
< & \left(\frac{1}{c_{001}+c_{01}} \left(V_0^{(0)} \right)^2 + \frac{1}{c_{20}+c_{21}} \left(V_2^{(0)} \right)^2 \right)^2 r^2 A^{(1)}.
\end{aligned}
$$

If $A^{(1)} \geq 0$, then fixed $V_0^{(0)}$, more behind Player 1 is lagging, more negative the advertisement is iff Player 2 is strong enough, i.e.,

$$
V_2^{(0)} \geq -c_{21} \left(1 - p_{22}^{(0)*} \right) / \left(2r^2 A^{(1)} \right),
$$

which is consistent with the result $A^{(1)} = 0$. This obviously holds since $V_2^{(0)}$ is non-negative. Similarly, if $A^{(1)} < 0$, the condition is

$$
V_2^{(0)} \leq -c_{21} \left(1 - p_{22}^{(0)*} \right) / \left(2r^2 A^{(1)} \right).
$$

■

To understand the Proposition, we provide sufficient condition as follows.

Corollary 4.6 *With quadratic cost functions, Assumption 4A and 4B, Player 1 will play* *strictly negative advertisement in the first period if*

$$V_2^{(0)} > \sqrt{(c_{20}+c_{21})/(c_{001}+c_{01})}c_{001}/\left(2r^2A^{(1)}\right) \;\; \textit{if } A^{(1)} > 0;$$

$$\begin{aligned} & \min\left\{0, \left(c_{21}\left(1-p_{22}^{(0)*}\right)+r^2A^{(1)}\right)\right\} \\ > \;\; & (c_{20}+c_{21})\left(c_{001}-r^2A^{(1)}\right)/(c_{001}+c_{01}) \;\; \textit{if } A^{(1)} \leq 0. \end{aligned}$$

Proof. With quadratic cost functions, Assumption 4A and 4B, from the proof of Proposition 4.5, we know that

$$\begin{aligned} d_1^{(0)} &= \frac{1}{c_{001}+c_{01}}\left(c_{001}\left(1-p_{002}^{(0)*}-p_{02}^{(0)*}\right)-r^2A^{(1)}V_0^{(0)}\right)V_0^{(0)} \\ &\quad -\frac{1}{c_{20}+c_{21}}\left(c_{21}\left(1-p_{22}^{(0)*}\right)+r^2A^{(1)}V_2^{(0)}\right)V_2^{(0)} \\ &\leq \frac{1}{c_{001}+c_{01}}\left(c_{001}-r^2A^{(1)}V_0^{(0)}\right)V_0^{(0)} \\ &\quad -\frac{1}{c_{20}+c_{21}}\left(c_{21}\left(1-p_{22}^{(0)*}\right)+r^2A^{(1)}V_2^{(0)}\right)V_2^{(0)}. \end{aligned}$$

Hence, a sufficient condition for the occurrence of negative advertisement is

$$\begin{aligned} & \frac{1}{c_{001}+c_{01}}\left(c_{001}-r^2A^{(1)}V_0^{(0)}\right)V_0^{(0)} \\ < \;\; & \frac{1}{c_{20}+c_{21}}\left(c_{21}\left(1-p_{22}^{(0)*}\right)+r^2A^{(1)}V_2^{(0)}\right)V_2^{(0)}. \end{aligned}$$

If $A^{(1)}=0$, the condition is

$$\frac{c_{001}(c_{20}+c_{21})}{c_{21}(c_{001}+c_{01})}V_0^{(0)} < \left(1-p_{22}^{(0)*}\right)V_2^{(0)}.$$

If $A^{(1)}>0$, the maximum value of left hand side is

$$c_{001}^2/\left(4(c_{001}+c_{01})r^2A^{(1)}\right),$$

which is achieved by

$$V_0^{(0)} = c_{001} / \left(2r^2 A^{(1)} \right).$$

Since right hand side is greater than

$$\frac{1}{c_{20} + c_{21}} r^2 A^{(1)} \left(V_2^{(0)} \right)^2,$$

a sufficient condition is

$$V_2^{(0)} > \sqrt{\left(c_{20} + c_{21} \right) / \left(c_{001} + c_{01} \right)} c_{001} / \left(2r^2 A^{(1)} \right)$$

If $A^{(1)} < 0$, the maximum value of left hand side is

$$\left(c_{001} - r^2 A^{(1)} \right) / \left(c_{001} + c_{01} \right),$$

which is achieved when $V_0^{(0)} = 1$. The minimum value of right hand side is

$$\min \left\{ 0, \left(c_{21} \left(1 - p_{22}^{(0)*} \right) + r^2 A^{(1)} \right) / \left(c_{20} + c_{21} \right) \right\}.$$

Hence, a sufficient condition is

$$\begin{aligned} & \min \left\{ 0, \left(c_{21} \left(1 - p_{22}^{(0)*} \right) + r^2 A^{(1)} \right) \right\} \\ > \ & \left(c_{20} + c_{21} \right) \left(c_{001} - r^2 A^{(1)} \right) / \left(c_{001} + c_{01} \right). \end{aligned}$$

This include the case when $A^{(1)} = 0$. Hence, we have the sufficient conditions in the corollary.

■

5 Competition in Finitely Many Periods

In this section, we generalize the results in linear cost functions to arbitrarily finite $T \geq 2$ periods.

The following Proposition generalizes Proposition 3.1.

Proposition 5.1 *With Assumption 2A, or with Assumption 2B and there are more uncommitted customers than opponent's customers who will become uncommitted, i.e.,*

$$p_{20}^{(T-1)*} V_2^{(T-1)} \leq V_0^{(T-1)} \left(1 - p_{002}^{(T-1)*} - p_{02}^{(T-1)*}\right),$$

Player 1 *will not engage in strictly negative advertisement in the last period, i.e.,* $d_1^{(T-1)} \geq 0$.

Proof. The proof is similar to the one in Proposition 3.1. We only give the revenue part of the utility here,

$$\begin{aligned}
&\sum_{t=0}^{T-1} r^t [\delta_1^{(t)} V_1^{(t)} - \beta_1 \delta_2^{(t)} V_2^{(t)}] \\
&+ r^T [\delta_1^{(T)} p_{01}^{(T-1)*} - \beta_1 \delta_2^{(T)} p_{02}^{(T-1)*}] V_0^{(T-1)} \\
&+ r^T [\delta_1^{(T)} p_{21}^{(T-1)*} - \beta_1 \delta_2^{(T)} p_{22}^{(T-1)*}] V_2^{(T-1)} \\
&+ r^T [\delta_1^{(T)} p_{11}^{(T-1)*} - \beta_1 \delta_2^{(T)} p_{12}^{(T-1)*}] V_1^{(T-1)}.
\end{aligned}$$

Suppose $d_1^{(T-1)} < 0$, which means $p_{01}^{(T-1)} V_0^{(T-1)} < p_{20}^{(T-1)} V_2^{(T-1)}$.

By the similar argument in the proof of Proposition 3.1, we obtain the results. ■

Similar to Proposition 3.4, we identify situations in which Player 1 chooses a strictly negative advertising strategy in any previous period $t = 0, ..., T-2$.

Proposition 5.2 *Player* 1 *will engage in strictly negative advertisement in the period of*

$t = 0, ..., T-2$, *when* $V_0^{(0)} \neq 0$, *and* $V_2^{(0)} \neq 0$, *i.e.* $d_1^{(0)} \leq 0$, *if and only if*

$$\left(1 - p_{22}^{(t)*}\right) V_2^{(t)} > p_{01}^{(t)*} V_0^{(t)} \tag{30}$$

$$\textit{and } V_2^{(t)} r^2 A^{(t+1)} \geq c_{20} - c_{21}, \tag{31}$$

where

$$A^{(t+1)} = \delta_1^{(t+2)} p_{01}^{(t+1)*} - \beta_1 \delta_2^{(t+2)} p_{02}^{(t+1)*} - \delta_1^{(t+2)} p_{11}^{(t+1)*} + \beta_1 \delta_2^{(t+2)} p_{12}^{(t+1)*} - \frac{\delta_1^{(t+1)}}{r}.$$

A sufficient condition is $p_{22}^{(t)*} < 1$, *condition (30), and*

$$V_0^{(t)} r^2 A^{(t+1)} \geq c_{001} - c_{01} \tag{32}$$

Proof. The proof is similar with Proposition 3.4. We give the revenue part of the utility function.

$$\begin{aligned}
&\sum_{\tau=0}^{t} r^\tau [\delta_1^{(\tau)} V_1^{(\tau)} - \beta_1 \delta_2^{(\tau)} V_2^{(\tau)}] + \sum_{\tau=t+3}^{T} r^\tau [\delta_1^{(\tau)} V_1^{(\tau)} - \beta_1 \delta_2^{(\tau)} V_2^{(\tau)}] \\
&+ r^{t+1} [\delta_1^{(t+1)} p_{01}^{(t)*} - \beta_1 \delta_2^{(t+1)} p_{02}^{(t)*}] V_0^{(t)} \\
&+ r^{t+1} [\delta_1^{(t+1)} p_{21}^{(t)*} - \beta_1 \delta_2^{(t+1)} p_{22}^{(t)*}] V_2^{(t)} \\
&+ r^{t+1} [\delta_1^{(t+1)} p_{11}^{(t)*} - \beta_1 \delta_2^{(t+1)} p_{12}^{(t)*}] V_1^{(t)} \\
&+ r^{t+2} [\delta_1^{(t+2)} p_{01}^{(t+1)*} - \beta_1 \delta_2^{(t+2)} p_{02}^{(t+1)*}] V_0^{(t+1)} \\
&+ r^{t+2} [\delta_1^{(t+2)} p_{21}^{(t+1)*} - \beta_1 \delta_2^{(t+2)} p_{22}^{(t+1)*}] V_2^{(t+1)} \\
&+ r^{t+2} [\delta_1^{(t+2)} p_{11}^{(t+1)*} - \beta_1 \delta_2^{(t+2)} p_{12}^{(t+1)*}] V_1^{(t+1)}.
\end{aligned}$$

Suppose $d_1^{(t)} > 0$, which means $p_{01}^{(t)} V_0^{(t)} > p_{20}^{(t)} V_2^{(t)}$. By the similar trick, we can prove the results. ■

Similar to the two period case, we have the following results.

Corollary 5.3 *There exist equilibrium choices at time t satisfying condition (30). More-*

over, with Assumption 2A or Assumption 2B, there may exist equilibrium choices at time $t+1$ *satisfying conditions (31) or (32) if*

$$V_2^{(t)} r^2 \left(\delta_1^{(t+2)} + \beta_1 \delta_2^{(t+2)} - \delta_1^{(t+1)}/r\right) \geq c_{20} - c_{21}$$

or

$$V_0^{(t)} r^2 \left(\delta_1^{(t+2)} + \beta_1 \delta_2^{(t+2)} - \delta_1^{(t+1)}/r\right) \geq c_{001} - c_{01},$$

respectively.

Corollary 5.4 *Decreasing the (predetermined) relative market share of a player in period* t *before the end of the game does not decrease the probability of the equilibrium strictly negative advertisement at that period. Smaller the player's market share is, conditions for the player to use strictly negative advertising strategy in the that period are easier to satisfy, i.e.,*

$$\begin{aligned} V_1^{(t)} &< 1 - \left(1 + p_{01}^{(t)*} / \left(1 - p_{22}^{(t)*}\right)\right) V_0^{(t)} \\ \text{and } V_1^{(t)} &\leq 1 - V_0^{(t)} - \left(c_{20} - c_{21}\right) / \left(r^2 A^{(t+1)}\right). \end{aligned}$$

Corollary 5.5 *If Player 1 utility depends on the opponents market share, i.e., if* $\beta_1 \delta_2^{(t+2)} > 0$, *then engaging in strictly negative advertisement is more likely if it is anticipated that the opponent will in the next round focus more on attracting customers from the player than on attracting non-committed customers, i.e., if* $p_{12}^{(t+1)*} > p_{02}^{(t+1)*}$.

Example 5.6 *Consider the special case when only final market share matters, i.e.,* $\delta_1^{(T)} \neq 0$, $\delta_2^{(T)} \neq 0$, *and* $\delta_1^{(t)} = \delta_2^{(t)} = 0$ *for* $t = 0, ..., T-1$. *All the marginal costs are equal to* $c > 0$ *except that* $c_{21} = c_{12} = c_{01} = c_{02} = (1-\gamma)\, c$, *where* $\gamma > 0$.

For $t = T-1$, *there is no strictly negative advertisement.*

For $t = T-2$, *it similar to our two periods case, and*

$$A^{(T-1)} = \left(\delta_1^{(T)} + \beta_1 \delta_2^{(T)}\right) \left(p_{01}^{(T-1)*} - p_{11}^{(T-1)*}\right)$$

the conditions are similar in Proposition 3.4. It is possible to have negative advertisement.

However, for $t \leq T-3$, the necessary and sufficient condition is reduced to

$$V_0^{(t)} p_{01}^{(t)} < V_2^{(t)} \left(1 - p_{22}^{(t)*}\right) \text{ and } c_{20} \leq c_{21}.$$

Since $c_{21} = (1-\gamma)\, c < c_{20}$, there is no strictly negative advertisement in the period $t = 0, ..., T-3$.

6 Market Share Related Cost Functions

Here we allow more generalized cost functions, which involve both transition probabilities and market shares. To maintain the simplicity, we assume that cost functions are homogenous degree one in the market share. The utility function of player 1 is rewritten as

$$\sum_{t=0}^{T} r^t [\delta_1^{(t)} V_1^{(t)} - \beta_1 \delta_2^{(t)} V_2^{(t)}] -$$
$$\sum_{t=0}^{T-1} r^t [V_0^{(t)} C_{01}(p_{01}^{(t)}) + V_2^{(t)} C_{21}(p_{21}^{(t)}) + V_2^{(t)} C_{20}(p_{20}^{(t)}) + V_1^{(t)} C_{11}(p_{11}^{(t)}) + V_0^{(t)} C_{001}(p_{001}^{(t)})].$$

This utility function can be further expressed as

$$\sum_{t=0}^{T} r^t [\widetilde{\delta}_1^{(t)} V_1^{(t)} - \beta_1 \widetilde{\delta}_2^{(t)} V_2^{(t)}] - \sum_{t=0}^{T-1} r^t [C_{01}(p_{01}^{(t)}) + C_{001}(p_{001}^{(t)})], \tag{33}$$

where

$$\widetilde{\delta}_1^{(t)} = \begin{cases} \delta_1^{(t)} + C_{01}(p_{01}^{(t)}) - C_{11}(p_{11}^{(t)}) + C_{001}(p_{001}^{(t)}) & 0 \leq t \leq T-1 \\ \delta_1^{(t)} & t = T \end{cases} \tag{34}$$

and

$$\widetilde{\delta}_2^{(t)} = \begin{cases} \delta_2^{(t)} + \frac{1}{\beta_1} C_{21}(p_{21}^{(t)}) + \frac{1}{\beta_1} C_{20}(p_{20}^{(t)}) - \frac{1}{\beta_1} C_{01}(p_{01}^{(t)}) - \frac{1}{\beta_1} C_{001}(p_{001}^{(t)}) & 0 \leq t \leq T-1 \\ \delta_2^{(t)} & t = T \end{cases} . \tag{35}$$

$\widetilde{\delta}_1^{(t)}$ and $\widetilde{\delta}_2^{(t)}$ can be regarded as the adjusted weights that Player 1 puts on its own and its opponent's market share, respectively. Moreover, in the last period $t = T$, these adjusted weights are the same with ones in the benchmark case.

Since the utility functions are continuous as well, we can prove existence of the equilibrium as well.

Theorem 6.1 *The game with market share related cost functions has a mixed strategy Nash equilibrium.*

Consider linear $C(\cdot)$ function, we can prove the following two propositions with similar tricks.

Proposition 6.2 *In equilibrium of the finite T period game with market share related cost functions, with Assumption 2A, or with Assumption 2B and there are more uncommitted customers than opponent's customers who will become uncommitted, i.e.,*

$$p_{20}^{(T-1)*} V_2^{(T-1)} \leq V_0^{(T-1)} \left(1 - p_{002}^{(T-1)*} - p_{02}^{(T-1)*}\right),$$

Player 1 will not engage in strictly negative advertisement in the last period, i.e., $d_1^{(T-1)} \geq 0$.

Proof. Proof is by contradiction analogous to the proof of Proposition 5.1. ■

Compared to the proof of Proposition 5.1, the only differences are the canceling of the market share $V_2^{(T-1)}$ or $V_0^{(T-1)}$. Since $0 \leq V_2^{(T-1)}, V_0^{(T-1)} \leq 1$, it is obvious that conditions in the proof of Proposition 5.1 are much easier to satisfy.

Proposition 6.3 *In equilibrium of the finite T period game with market share related cost functions, Player 1 will engage in strictly negative advertisement in the period of $t = 0, ..., T-2$ when $V_0^{(t)} \neq 0$ and $V_2^{(t)} \neq 0$, i.e. $d_1^{(t)} < 0$, if and only if*

$$\left(1 - p_{22}^{(t)*}\right) V_2^{(t)} > p_{01}^{(t)*} V_0^{(t)} \tag{36}$$

$$\text{and } r^2 \widetilde{A}^{(t+1)} \geq c_{20} - c_{21}, \tag{37}$$

where

$$\widetilde{A}^{(t+1)} = \widetilde{\delta}_1^{(t+2)} p_{01}^{(t+1)*} - \beta_1 \widetilde{\delta}_2^{(t+2)} p_{02}^{(t+1)*} - \widetilde{\delta}_1^{(t+2)} p_{11}^{(t+1)*} + \beta_1 \widetilde{\delta}_2^{(t+2)} p_{12}^{(t+1)*} - \frac{\widetilde{\delta}_1^{(t+1)}}{r}.$$

A sufficient condition is $p_{22}^{(t)*} < 1$*, condition (36), and*

$$r^2 \widetilde{A}^{(t+1)} \geq c_{001} - c_{01}. \tag{38}$$

Proof. Proof is by contradiction similar to the one in Proposition 5.2. ■

Similar with Proposition 6.2, $V_0^{(t)}$ and $V_2^{(t)}$ are canceled in conditions (38) and (37), respectively, due to the market share related cost functions. Since $\widetilde{\delta}_1^{(t+2)}$ and $\widetilde{\delta}_2^{(t+2)}$ are the same with $\delta_1^{(t+2)}$ and $\delta_2^{(t+2)}$ at time $t = T-2$, compared to Proposition 5.2, market related cost functions facilitates the use of negative advertising strategy if

$$\widetilde{\delta}_1^{(t+2)} p_{01}^{(t+1)*} - \beta_1 \widetilde{\delta}_2^{(t+2)} p_{02}^{(t+1)*} - \widetilde{\delta}_1^{(t+2)} p_{11}^{(t+1)*} + \beta_1 \widetilde{\delta}_2^{(t+2)} p_{12}^{(t+1)*} - \frac{\widetilde{\delta}_1^{(t+1)}}{\delta} \geq 0.$$

Example 6.4 *Similar with Example 5.6, we consider the special case where only the final market shares matter, i.e.,* $\delta_1^{(T)} \neq 0$*,* $\delta_2^{(T)} \neq 0$*, and* $\delta_1^{(t)} = \delta_2^{(t)} = 0$ *for* $t = 0, ..., T-1$*. However, since the adjusted weights may not be zero at* $t = 0, ..., T-1$*, the market shares at* $t = 0, ..., T-1$ *affect the results.*

If Player 1 utility depends on the opponents market share, i.e., if $\beta_1 \delta_2^{(t+2)} > 0$, then whether to engage in negative advertising strategies depends not only on the sign of $p_{12}^{(t+1)*} - p_{02}^{(t+1)*}$ as discussed in Corollary 5.5, but also on the sign of $\widetilde{\delta}_2^{(t+2)}$ for $t = 0, ..., T-2$, which relies on β_1 and the cost functions except for $t = T-2$.

7 Infinite Horizon Game

We address the extension of our model to infinitely many periods. The underlying game is not a repeated game, as transition probabilities are chosen at the beginning and the market-shares evolve dynamically. Since the set of strategies in the infinite game is still a continuous game, it is easy to verify the existence of the Nash equilibrium. Given no ending point in the infinite horizon framework, we have a straightforward generalization of Proposition 5.2 for linear cost functions.

Proposition 7.1 *Player* 1 *will engage in strictly negative advertisement in period* t *when* $V_2^{(0)} \neq 0$, *i.e.* $d_1^{(t)} < 0$, *if and only if*

$$\begin{aligned} \left(1 - p_{22}^{(t)*}\right) V_2^{(t)} &> p_{01}^{(t)*} V_0^{(t)} \\ \text{and } V_2^{(t)} r^2 A^{(t+1)} &\geq c_{20} - c_{21}, \end{aligned} \tag{39}$$

where

$$A^{(t+1)} = \delta_1^{(t+2)} p_{01}^{(t+1)*} - \beta_1 \delta_2^{(t+2)} p_{02}^{(t+1)*} - \delta_1^{(t+2)} p_{11}^{(t+1)*} + \beta_1 \delta_2^{(t+2)} p_{12}^{(t+1)*} - \frac{\delta_1^{(t+1)}}{r}.$$

A sufficient condition is $p_{22}^{(t)*} < 1$, *condition (39), and*

$$V_0^{(t)} r^2 A^{(t+1)} \geq c_{001} - c_{01}.$$

8 One Shot Game and Commitment

Failure to commit is a potential problem that will destroy the previous equilibria. In the finite T period game, each player make a complete plan about the future at $t = 0$. However, each of them has the incentive not to commit to what they have planned after the realization of market shares at that period. Therefore, if no commitment becomes common knowledge, we need to consider the equilibrium without commitment. Intuitively, the game without commitment mimics the one shot game ($T = 1$), taking the volumes in the previous period as given, since future means nothing to both players.

In the one shot game, the strategy set is convex. Therefore, there exits a pure strategy Nash equilibrium. With simple calculus, we also can have the following proposition for linear cost functions.

Proposition 8.1 *With Assumption 2A, or with Assumption 2B and there are more uncommitted customers than opponent's customers who will become uncommitted, i.e.,*

$$p_{20}^{(0)*}V_2^{(0)} \leq V_0^{(0)}\left(1 - p_{002}^{(0)*} - p_{02}^{(0)*}\right),$$

Player 1 *will not engage in strictly negative advertisement, i.e.,* $d_1^{(0)} \geq 0$.

In the finitely many period game without commitment, the players play one shot game in each period. Therefore, $V_0^{(0)}$, $V_1^{(0)}$, and $V_0^{(0)}$ should be explained as market shares at the beginning of each period.

9 Concluding Remarks

We examine the effect of negative advertisement strategy on the market share. In particular, negative advertisement focuses on reducing opponent's market share by enticing opponent's customers to the undecided group, while positive advertisement focuses on increasing one's own market-share by attracting undecided customers. This modeling approach considers both voters' choices and candidates' choices. However, the cost functions for investment in transition probabilities could consist of both pure financial costs as well as costs due to consumer behavior assumptions that one might want to capture.

In the political competition players care about the relative market-shares which means that both player's own market share and its opponent's market share are relevant. Thus, it is not surprising that negative advertising strategies are emerging. Even when players only care about their own market share, negative advertisement strategies are possible, especially when market-share in the final period is important. In such situations, negative strategies in early periods can turn to be beneficial when attracting customers in the last period.

More player lags behind the opponent in market-share, more likely are negative advertisement strategies.

Negative advertisement strategies are not likely to emerge in the final period, except for very special and unrealistic cost functions. For example, there will be no equilibrium negative strategies in the last period when cost functions are linear and have equal marginal cost.

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