Approval Voting for Committees:

Threshold Approaches

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Abstract

When electing a committee from a pool of individual candidates, it is not enough to elicit voters’ preferences among individual candidates; one should also account for voters’ opinions about synergetic effects of candidate interactions if they are jointly elected onto a committee. We propose an approval voting method in which each voter selects a set of candidates indicating his or her approval of any committee that has sufficiently many candidates from the selected set. The committee approved by most voters is elected.

1 Introduction

Selecting a set of candidates (committee, group, team) is as commonplace as selecting a single candidate. Shareholders of a company need to elect and approve its corporate directors, consulting firms assign teams of consultants to projects, legislative bodies select committees and subcommittees, coaches select starting players on a sports team, etc. In
some cases, the selection is made by a single decision-maker, but in many cases the selection procedure has to aggregate preferences (opinions, votes) from a large number of stakeholders. In this paper, we focus on the latter aggregation problem, i.e., on voting procedures for committee selection.

A large social choice theory literature is devoted exclusively to methods of aggregating preferences into a consensus choice. Its central results are impossibility theorems showing that no aggregation procedure can simultaneously satisfy some rather reasonable properties (e.g., Arrow’s Theorem [1], the Gibbard-Satterthwaite Theorem [7, 9]). Some of the research in this field considers restrictions (e.g., restrictions on preference structures, [10]) that avoid these impossibility results. Despite the negative impossibility results, research continues on the analysis of desirable and less desirable properties of voting procedures that are used in practice, such as Borda-like methods (e.g. [8]) and approval voting [4]. The specific problem of voting a committee has been discussed in [3, 2].

These results and methods could be used to aggregate voters’ preferences over committees, but such methods are often not observed in practice. One problem is that the number of possible committees that can be constructed from even a modest pool of candidates could be prohibitively large and unmanageable. For example, there are more than two million different seven-member committees that could be constructed from a pool of 30 candidates. Identifying several most preferred candidates from a pool of 30 is feasible, but even considering a small portion of more than two million possible seven-member committees might be too much to ask from a voter.

Instead, a typical committee selection aggregates voters’ preferences over candidates and then constructs a committee consisting of the top candidates. For example, if a committee
of seven is to be selected, the seven candidates receiving the most votes are chosen. A slightly more involved method, that ensures representativeness of different types of candidates, divides candidates according to some criterion and forms a committee from the top candidates in each category. For example, five starters in an all-star NBA team consist of the top center, top two forwards, and top two guards, according to the submitted votes for individual players.

While methods of selecting a committee based on the aggregation of preferences over individual candidates are simple and intuitive, such aggregation by its design does not account for interdependencies among candidates. Consider selecting a two-member committee from the set \{1,2,3\} of three candidates, and suppose that every voter agrees that candidates 1 and 2 are better than candidate 3, but that having both 1 and 2 on the same committee is the worst possible choice (i.e., every voter prefers committees \{1,3\} and \{2,3\} to committee \{1,2\}). Clearly, every sensible procedure of ranking individual candidates ranks 1 and 2 over 3, and cannot be a basis for selecting any committee other than \{1,2\}, which is least preferred by all voters.

The purpose of this paper is to propose procedures for committee selection that consider interdependencies, but that remain simple and avoid the manageability issue of large numbers of potential committees. In particular, we adopt approval voting and modify its aggregation rule to reflect an “approval” of a committee. (A similar approach has recently be proposed in [5] but the vote aggregation procedure proposed there is fundamentally different than what we derive.) Informally, every voter selects a set of candidates, and the procedure assumes that the voter approves of every committee that has a sufficiently large intersection with his or her selected set of candidates. (As will be discussed, the definition
of “sufficiently large” intersection could depend on the size of the committee; all this is predefined and publicly known prior to the selection of the set of candidates.) While a voter can approve more than one committee, our procedure does not allow for different intensities of approvals. The selected (winning) committee is that committee that is approved of by the most voters. The procedure is defined in the next section.

2 Definitions

We assume throughout that there are $n$ voters ($i = 1, 2, \ldots, n$) and $m$ candidates ($j = 1, 2, \ldots, m$). The set of all subsets of $[m] = \{1, 2, \ldots, m\}$ is denoted by $2^{[m]}$, and the set of all potential committees with one or more candidates is $2^{[m]} \setminus \{\emptyset\}$.

We denote by $\xi$ the set of admissible committees for a particular application and assume that $\xi$ is a nonempty subset of $2^{[m]} \setminus \{\emptyset\}$. The set of all $k$-member committees is denoted by $\xi_k$. Thus, when a committee of $k$ is to be elected and all $k$-member committees are admissible, we set $\xi = \xi_k$. When all nonempty committees are admissible, $\xi = 2^{[m]} \setminus \{\emptyset\}$.

Let $V_i$ be the set of candidates approved of by voter $i$. We allow $V_i$ to be any subset of $[m]$, including the empty set. An approval voting ballot profile, or profile for short, is an $n$-tuple

$$V = (V_1, V_2, \ldots, V_n)$$

We often write $V_i$ in simplified notation, such as 356 to denote $V_i = \{3, 5, 6\}$.

The set of all possible profiles is $\nu = (2^{[m]})^n$. We define $V_i S$ as the number of candidates
in committee $S$ approved of by voter $i$ so that

$$V_i S = |V_i \cap S|.$$ 

The $n$-tuple $V S = (V_1 S, V_2 S, \ldots, V_n S)$ tells how many candidates in $S$ each voter approves of.

**Example 2.1** A committee of three is to be chosen from eight candidates (1,2,\ldots,8). There are nine voters (1,2,\ldots,9) with the following approval voting ballot profile:

<table>
<thead>
<tr>
<th>Voter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>AV Ballot</td>
<td>2</td>
<td>12</td>
<td>12</td>
<td>13</td>
<td>37</td>
<td>45</td>
<td>46</td>
<td>47</td>
<td>48</td>
</tr>
</tbody>
</table>

We have $n = 9$, $m = 8$, $\xi = \xi_3$, and the profile is $V = (2,12,12,13,37,45,46,47,48)$. For committee $S = \{1,3,4\}$, $V S = (0,1,1,2,1,1,1,1,1)$. 

A threshold function is a map $t$ from $\xi$ into $\mathbb{R}^+$, the set of positive real numbers. It will become clear that the codomain of $t$ could be taken as $\{1,2,3,\ldots\}$ rather than $\mathbb{R}^+$, but we will not presume this. Let $\Upsilon$ be the set of threshold functions.

Given $\xi$, let $C$ be a map from $\nu$ to the set of nonempty subsets of $\xi$. We refer to $C$ as a choice function and to $S \in \xi \cap C(V)$ as a choice for profile $V$. Given $\xi$ and $t \in \Upsilon$, the choice function $C_t$ for threshold function $t$ is defined by

$$S \in C_t(V) \text{ if } |S \in \xi \text{ and } |\{i : V_i S \geq t(S)\}| \geq |\{i : V_i T \geq t(T)\}| \text{ for all } T \in \xi|.$$ 

Thus $S \in \xi$ is a choice for profile $V$ and threshold function $t$ if as many voters approve
of $S$ (according to $t$) as any other admissible committee. We refer to $C_t$ for $t \in \Upsilon$ as an approval voting threshold choice function, or simply TCF for short.

**Remark 2.2** When a TCF is used, one could allow all committees to be admissible by extending $t$ to all of $2^{[n]} \setminus \{\emptyset\}$ by setting $t(S) = |S| + 1$ for all $S \not\in \xi$. Obviously, with such a $t$, only $S \in \xi$ can be approved. 

**Remark 2.3** Note that a TCF does not account for the intensity of a voter’s approval of a committee. It simply records whether $V_i S$ is above the threshold $t(S)$ and not the actual value of $V_i S$. Note that the method based on approval voting ballots in which voter $i$ assigns $V_i S$ votes to a committee $S$ is equivalent to choosing a committee of candidates that maximizes the sum of individual approval voting scores of candidates from a committee:

$$\sum_{i=1}^{n} V_i S = \sum_{i=1}^{n} \sum_{j \in S} V_i\{j\} = \sum_{j \in S} \sum_{i=1}^{n} V_i\{j\} = \sum_{j \in S} |\{i : j \in V_i\}|.$$

In other words, aggregating approval voting ballots in this way is nothing else but the method of first aggregating voters preferences for individual candidates and then constructing a committee based on this aggregated ranking. As mentioned in the introduction, in this paper we explore methods for aggregating voters’ preferences that consider interdependencies.

**Example 2.4** *Example 2.1 revisited.*

Before illustrating threshold functions, we note that under ordinary approval voting for individual candidates (where voter $i$ approves of candidate $j$ if and only if $j \in V_i$, and the ranking of candidates is determined by the number of voters who approve of each of them), the three candidates with the most approval votes are 1, 2, and 4. Thus, if one were to use the
approval voting results for individual candidates to construct a three-member committee, the elected committee would be \{1, 2, 4\}. As mentioned in Remark 2.3, this outcome could be reached by assigning \(V_1S + \ldots + V_6S\) votes to committee \(S\). Committee \(\{1, 2, 4\}\) gets 10 votes, while any other three member committee gets at most nine votes.

Next consider two threshold rules.

**t \equiv 1.** Voter \(i\) “approves” of a committee \(S\) if and only if at least one of her/his “approved” candidates is among the three members of \(S\), i.e., if and only if \(V_iS \geq 1\). The only three-member committee approved of by all voters is \(\{2, 3, 4\}\). Thus \(C_t(V) = \{\{2, 3, 4\}\}\).

**t \equiv 2.** Voter \(i\) “approves” of a three-member committee \(S\) if and only if the majority (i.e., at least two) of the committee members are approved by her/him, i.e., if and only if \(V_iS \geq 2\). Under this rule, voters 2, 3 and 4 approve of committee \(\{1, 2, 3\}\), and no other committee of three has more than two approving voters. Thus \(C_t(V) = \{\{1, 2, 3\}\}\).

Before moving on to study properties of TCFs, we note that computing a TCF, and selecting the committee based on some numerical score in general, could be time-consuming. Essentially, one is unlikely to avoid having to compute the number of votes (or a score) for (almost) every admissible committee \(S \in \xi\). If \(\xi\) is large, this computational issue could become important. More formally, a TCF can be computed in time that is polynomial in \(nm + |\xi|\) (the first term corresponds to the number of bits needed to represent the approval voting ballot profile, and the second is the number of admissible committees) by simply calculating \(|\{i : V_iS \geq t(S)\}|\), but computing a TCF is NP-complete if the input is only \(nm\) (e.g., if \(\xi\) is predefined).

**Example 2.5** Consider electing a \(k\)-member committee with \(\xi = \xi_k\). Let \(t \equiv 1\), and let
$|V_i| = 2$ for all $i$. To illustrate NP-completeness, note that there is a one-to-one correspondence between all such voter profiles and graphs with vertex set $[m]$ and edges $V_1, V_2, \ldots, V_n$. Thus determining whether there exists $S \in \xi$ approved by all voters, i.e., such that $V_iS > 0$ for all $i$, is equivalent to determining whether $S$ is a vertex cover for the corresponding graph. Determining whether a graph has a vertex cover of size $k$ is one of the fundamental NP-complete problems [6]. Therefore, determining whether there exists $S \in \xi_k$ approved by all voters is also NP-complete.

The fact that computing a TCF requires comparing any admissible committee $S$ with every voter profile $V_i$ is not the source of NP-completeness. Even selecting a committee that maximizes the sum of scores of individuals in the committee (as is the case with any method that first aggregates votes for individuals and then, based on these votes chooses an admissible committee) is computationally hard: given the list of individual votes $v(j), j = 1, \ldots, m$, finding $S \in \xi$ that maximizes $\sum_{j \in S} v(j)$ is also NP-complete [6].

3 Properties of TCFs

We note three properties shared by all approval voting threshold choice functions before we consider specialized aspects of threshold functions and sets of admissible committees. The first property merely reiterates features of the definition of $C_t$ and mentions $t$ explicitly.

**Property 1: Consistency.** For all $S, T \in \xi$ and all $V \in \nu$ if $VS = VT$ and $t(S) = t(T)$ then $S \in C_t(V) \iff T \in C_t(V)$.

The other two properties do not involve $t$ explicitly. The first is a voter anonymity condition.
Property 2: Anonymity. Suppose choice function $C$ is a TCF. For all $S \in \xi$ and all $U, V \in \nu$ if $US$ is a permutation of $VS$ then $S \in C(V) \iff S \in C(U)$.

Property 1 can be strengthened to reflect anonymity. Given $V$ and $S$, let $s(V, S) = (s_0(V, S), s_1(V, S), \ldots, s_m(V, S)) = (s_0, s_1, \ldots, s_m)$, where

$$s_h = s_h(V, S) = |\{i : V_i S = h\}|,$$

the number of voters whose approval sets contain exactly $h$ members of $S$. We refer to $s(V, S)$ as the score sequence for $V$ and $S$. Obviously,

$$\sum_{h=0}^{m} s_h = n, \quad s_h = 0 \text{ for all } h > |S|.$$

Now, consistency and anonymity could be combined into a single property:

Property 1*: Consistency and Anonymity. For all $S, T \in \xi$ and all $U, V \in \nu$, if $s(V, S) = s(U, T)$ and $t(S) = t(T)$ then $S \in C_t(V) \iff T \in C_t(U)$.

Next we define a partition-consistency condition. The partition aspect refers to a division of the $n$ voters into two disjoint groups with $p$ and $n - p$ voters. Given $V = (V_1, \ldots, V_n)$ let

$$V_p^- = (V_1, \ldots, V_p, \emptyset, \ldots, \emptyset), V_p^+ = (\emptyset, \ldots, \emptyset, V_{p+1}, \ldots, V_n),$$

with $1 \leq p < n$

Property 3: Partition-consistency. If $S \in C(V_p^-)$ and $S \in C(V_p^+)$, then $S \in C(V)$.

This says, in effect, that if $S \in \xi$ is a choice for each of two disjoint groups of voters under the same type of TCF, then $S$ is also a choice for the combined group under the same
If we remove the TCF restriction from Properties 2 and 3, then the resulting conditions are necessary for a choice function $C$ to be a TCF. It is then natural to consider additional conditions on $C$ that yield a set of conditions that are necessary 	extit{and} sufficient to imply that $C$ is a TCF, i.e., that there exists some $t \in \Upsilon$ for which $C = C_t$. Although this can be done, we forego a complete specification and merely note another necessary condition for a TCF.

Property 4: independence of other candidates. If $V = (V_1, \ldots, V_n)$ and $V' = (V_1 \cap (S \cup T), \ldots, V_n \cap (S \cup T))$, then $[S \in C(V), T \not\in C(V)] \Rightarrow [S \in C(V'), T \not\in C(V')]$ and $[S, T \in C(V)] \Rightarrow [S, T \in C(V')]$.

If $T$ in Property 4 is set at $\emptyset$, we get the specialized reduction condition which says that if $S \in C(V)$ and $V' = (V_1 \cap S, \ldots, V_n \cap S)$ then $S \in C(V')$.

4 Threshold Functions

We use TF henceforth as an abbreviation of threshold function. We refer to $t \in \Upsilon$ as a cardinal TF if, for every $S \in \xi$, $t(S)$ depends only on $|S|$, and as a constant TF is $t(S) = t(T)$ for all $S, T \in \xi$. A constant TF is clearly a cardinal TF, and if $\xi \subseteq \xi_k$ for some $k$, then a cardinal TF is also a constant TF. (Example 2.4 provides an example of a threshold function.) For larger admissible sets, for example $\xi = 2^{[m]} \setminus \{\emptyset\}$, a cardinal TF is nondecreasing if, for all $S, T \in \xi$, $|S| < |T|$ implies $t(S) \leq t(T)$. Similarly, a cardinal TF is nonincreasing if, for all $S, T \in \xi$, $|S| < |T|$ implies $t(S) \geq t(T)$. Apart from constant TF’s, we will not pay much attention to nonincreasing TF’s because it seems odd to have $t(S) > t(T)$ when $|S| < |T|$. The following lemma provides a connection to choice functions.
Lemma 4.1 If $t$ is a nonincreasing cardinal TF then, for all $S, T \in \xi$ for which $S \subset T$, $S \in C_t(V) \Rightarrow T \in C_t(V)$.

Proof. When $t$ is nonincreasing and $S \subset T$, we have $V_i S \leq V_i T$ and $t(S) \geq t(T)$, so if a voter approves of $S$, i.e., if $V_i S \geq t(S)$, then that voter also approves of $T$, since $V_i T \geq V_i S \geq t(S) \geq t(T)$. It follows that if $S \in C_t(V)$ then $T \in C_t(V)$.

Because Lemma 4.1 applies to constant TF’s, such functions favor larger committees as choices when $\xi$ includes committees of various sizes. For example, if $t(S) = 1$ for all $S \in \xi$, and $\xi = 2^{|m|} \setminus \{\emptyset\}$, then $[m]$, the committee of the whole, is a choice for every $V \in \nu$. We shall therefore consider constant functions primarily in restricted settings such as $\xi = \xi_k$.

Note that cardinal TFs satisfy neutrality: for every permutation $\pi$ of $[m]$ and every $S = \{j_1, \ldots, j_k\}$, $t(S) = t(\{\pi(j_1), \ldots, \pi(j_k)\})$, i.e., they treat candidates equally. This feature may make cardinal TFs inappropriate for some contexts. For example, suppose a committee of five is to be chosen from 18 candidates who divide naturally into three distinct groups. If it is desirable but not mandatory to have at least one candidate from each group on the chosen committee, then $t$ could be biased in favor of representativeness. With $\xi_5^r$ the set of committees in $\xi_5$ that represent precisely $r$ of the three groups, one might take $t(S) = 2$ for $S \in \xi_5^3$, $t(S) = 3$ for $S \in \xi_5^2$, and $t(S) = 5$ for $S \in \xi_5^1$. However, if only the committees in $S_5^3$ are admissible, we could let $\xi = \xi_5^3$ and take $t$ constant there.

Other restrictions on $t$ could be considered for larger admissible sets such as, for all $S, T \in \xi$, (i) $1 \leq t(S) \leq |S|$, (ii) $S \subset T \Rightarrow t(S) + 1 \leq t(T)$, and/or (c) $S \cap T = \emptyset \Rightarrow t(S \cup T) \leq t(S) + t(T)$. For example, when $\xi = 2^{|m|} \setminus \{\emptyset\}$, (i) and (ii) imply that $t(S) = |S|$ for all $S \in \xi$. 

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We focus henceforth on cardinal TF’s. (Thus, when $\xi \subseteq \xi_k$, a cardinal TF is a constant TF.)

The following lemma for this case provides a counterpart to Lemma 4.1.

**Lemma 4.2** If $\xi = 2^m \setminus \{\emptyset\}$ and $t(S) = |S|$ for all $S \in \xi$ then, for all $S, T \in \xi$ for which $S \subset T$, $T \in C_t(V) \Rightarrow S \in C_t(V)$.

**Proof.** Under the lemma’s hypotheses, including $S \subset T$, every voter who approves of $T$ also approves of $S$, and it follows that if $T \in C_t(V)$ then $S \in C_t(V)$.

Lemma 4.2 shows that some TF’s in the general case could favor smaller committees as choices from $\xi$. If necessary, this could be counteracted by adopting a secondary choice rule that only larger members of $C_t(V)$ are “acceptable”. It could also be counteracted by adopting an entirely different threshold function that is biased towards committees of a certain size.

A related idea that might tend to elect moderate-sized committees concerns majority threshold functions. We define the **majority TF** by $t(S) = |S|/2$ for all $S \in \xi$, and the **strict majority TF** by $t(S) = (|S| + 1)/2$ for all $S \in \xi$. The definitions apply to all admissible sets, including $\xi_k$, where the majority and strict majority TF’s are equivalent for $C$ when $k$ is odd. In Example 2.4 we have already seen the (strict) majority threshold function ($t \equiv 2$) when $\xi = \xi_3$. The following example illustrates the strict majority TF with $\xi = 2^m \setminus \{\emptyset\}$.

**Example 4.3** Suppose $n = 12$, $m = 8$, and

\[ V = (123, 15, 16, 278, 23, 24, 2578, 34, 347, 46, 567, 568). \]
Under the strict majority TF, the maximum approvals for a 1-member committee is 4, for a 2-member committee it is 2 (34,23,56,57,58,78), for a 3-member committee it is 5 (only for 234), for a 4-member committee it is 3 (5678), and for a 5-member committee it is 4 (15678). Hence \{2,3,4\} is the only member of \(C_t(V)\).

The Appendix provides the axiomatization of approval voting threshold functions.

5 Top individuals might not be top team players

In this section we demonstrate that a TCF procedure aggregates voters preferences over teams; i.e., that top individuals might not be selected for a winning committee.

Under ordinary approval voting, the candidates that appear in the most \(V_i\)'s constitute the “best” committees. This is not generally true for the majority and strict majority TFs, as already demonstrated by Example 2.4.

**Proposition 5.1** When \(\xi = 2^{|m|} \setminus \{\emptyset\}\) and \(t\) is the majority or the strict majority TF, there exist \(n, m,\) and \(V \in \nu\) such that candidate 1 is in a majority of the \(V_i\) and in more \(V_i\)'s than any of other candidates, but candidate 1 is in no \(S \in C_t(V)\).

**Proof.** First consider the majority TF. Suppose \(n = 6, m = 5,\) and \(V = (123, 124, 135, 145, 25, 34).\) Note that \(C_t(V) = \{2345\}\) since every voter approves of the 4-member committee 2345, and no other committee has unanimous approval. However, candidate 1 is in the most \(V_i\)'s (four), while every other candidate is in three \(V_i\)'s.

Next consider the strict majority TF. Suppose \(n = 10, m = 6,\) and

\[V = (1234, 1236, 1245, 1256, 1345, 1346, 1456, 235, 246, 356).\]
Because every voter approves of three of the five candidates in \{2,3,4,5,6\}, this 5-member committee has unanimous approval. No other committee has unanimous approval (the closest being 12356 with 9 approvals), so \(C_t(V) = \{23456\}\). But candidate 1 is in seven \(V_i\)’s, while the others are in six \(V_i\)’s.

The examples from the preceding proof can be modified by adding a few more voters and many more candidates without changing the conclusion of Proposition 5.1. For example, if we add three voters to the latter example and take \(V_{11} = V_{12} = V_{13} = \{7,8,\ldots,100\}\) then \(C_t(V)\) is still \{2,3,4,5,6\} and candidate 1 is in more \(V_i\)’s (a majority) than any other candidate.

Note that the proposition also holds for almost any threshold function \(t\):

**Proposition 5.2** Let \(t\) be a threshold function such that \(t(S) \geq 2\) for every \(S \in \xi\) for which \(1 \in S\). Suppose that there exist \(T \in \xi\) such that \(1 \notin T\), \(t(T) \leq |T|\) and such that \(AT < t(A)\) for every \(A \neq T\). Then, for every \(n \geq 3\), there exists a \(V \in \nu\) such that candidate 1 is in a majority of the \(V_i\)’s and in more \(V_i\)’s than any of other candidates, but candidate 1 is in no \(S \in C_t(V)\).

**Proof.** Take \(V_n = T\) and \(V_1 = V_2 = \ldots = V_{n-1} = \{1\}\). Then, \(C_t(V) = \{T\}\) since \(T\) is the only committee approved by voter \(n\), and none of the other voters approves of any admissible committee. On the other hand, 1 is approved by \(n - 1\) voters, while any other candidate is approved by at most one voter.

The proof assumes that there are voters who choose not to approve of any committee by approving candidate 1 only. This is not the case in the next proposition but the same result still holds.
Proposition 5.3 Let $S', S'', T \in \xi$, $\{1\} = S' \cap S''$, $(S' \cup S'') \cap T = \emptyset$. Let $t(S') \leq |S'|$, $t(S'') \leq |S''|$, $t(T) \leq |T|$. Suppose that for every $A$ such that $1 \in A$, (i) $AT < t(A)$ and (ii) either $S'A < t(A)$ or $S''A < t(A)$. Then, for every $n \geq 9$, there exists a $V \in \nu$ such that candidate 1 is in a majority of the $V_i$ and in more $V_i$’s than any of other candidates, but candidate 1 is in no $S \in C_1(V)$.

Proof. First observe that for every $n \geq 9$ (and for $n = 7$), there exists positive integers $p_1, p_2, p_3$ such that $p_1 + p_2 + p_3 = n$, $p_1 + p_2 > p_3$ and $p_3 > \max\{p_1, p_2\}$. Let

$$V_i = \begin{cases} S' & \text{if } 1 \leq i \leq p_1 \\ S'' & \text{if } p_1 < i \leq p_1 + p_2 \\ T & \text{if } p_1 + p_2 < i \leq n \end{cases}$$

Our assumptions say that in approval voting for individual candidates, 1 is approved by $p_1 + p_2$ voters while no other candidate is approved by more than $p_3 < p_1 + p_2$ voters. Note that any set $A$, such that $1 \in A$ is approved by at most $\max\{p_1, p_2\}$ candidates and thus no such set is in $C_1(V)$ because $T$ is approved by $p_3 > \max\{p_1, p_2\}$ voters. 

There can be even more candidates that are top individual choices and that do not belong to any winning committee. For example, when $t \equiv 1, k = 3$ and $(n,m) = (3,4)$, profile $V = (123, 123, 4)$ has $C_1(V) = \{124, 134, 234\}$. In this case no $S \in C_1(V)$ contains all three of the most popular candidates. The following theorem notes a stronger result.

Theorem 5.4 Suppose $1 \leq t \equiv \alpha < k$. Then there are $n, m$ and a profile $V$ with $C_\alpha(V) = \{S\}$ such that at least $k$ candidates not in $S$ each appears in more $V_i$’s than any candidates in $S$. 

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**Proof.** Suppose $t \equiv 1 < k$. When $k = 2$, profile

$$V = \{1, 2, 134, 134, 234, 234\}$$

has $C_1(V) = \{12\}$, and each of 3 and 4 is in more $V_i$'s than either of 1 or 2. When $k \geq 3$, let $n = m = 2k$ with

$$V = (1, 2, \ldots, k, \{1, k+1, k+2, \ldots, 2k\}, \ldots, \{k, k+1, k+2, \ldots, 2k\}).$$

Then $C_1(V) = \{1, 2, \ldots, k\}$, each $j \leq k$ is in two $V_i$’s and each $j > k$ is in $k > 2$ $V_i$’s.

Suppose $2 \leq \alpha < k$. Let $L$ be a list of the $\binom{k}{\alpha}$ subsets of $\{1, 2, \ldots, k\}$ that have $\alpha$ members. Let $m = 2k$, and let $L'$ be a list of the $\binom{k}{\alpha}$ subsets of $\{1, 2, \ldots, k\}$ with $\alpha$ members in union with $\{k+1, k+2, \ldots, 2k\}$. For example, when $\alpha = 2$ and $k = 3$, $L = (12, 13, 23)$ and $L' = (12456, 13456, 23456)$. Let $r$ be a positive integer greater than $\alpha/(k - \alpha)$, let $n = (r+1)\binom{k}{\alpha}$, and let

$$V = (L, rL'),$$

where $rL'$ denotes $r$ repetitions of $L'$. Then $\{1, 2, \ldots, k\}$ has unanimous approval under $t \equiv \alpha$ and is the only member of $\xi_k$ with this property, for any other member of $\xi_k$ lacks the approval of at least one of the first $\binom{k}{\alpha}$ voters. Hence $C_\alpha(V) = \{1, 2, \ldots, k\}$. The number of voters with candidate $j > k$ in their approval sets is $\binom{k}{\alpha}$, and the number with candidate
\[ j \leq k \text{ in their approval sets is} \]
\[ (r + 1) \left[ \binom{k}{\alpha} - \binom{k - 1}{\alpha} \right] = (r + 1) \binom{k}{\alpha} \frac{\alpha}{k}. \]

Because our choice of \( r \) ensures that \( r > (r + 1) \frac{\alpha}{k} \), each candidate in \( \{k + 1, \ldots, 2k\} \) is in more approval sets than each candidate in \( \{1, 2, \ldots, k\} \). \]

**Remark 5.5** An interesting combinatorial adjunct of Theorem 5.4 is to determine the minimum \( n \) for each \( k \geq 2 \) which admits a \( V \) that satisfies the conclusion of the theorem. The minimum \( n \) for \( k = 2 \) is easily seen to be \( n = 4 \) with \( V = (1, 2, 134, 234) \). Then \( C_1(V) = \{\{1, 2\}\} \) and \( C_2(V) = \{\{3, 4\}\} \). The minimum \( n \) for \( k = 3 \) is \( n = 6 \) with \( V = (1, 3, 256, 346, 2789, 145789) \), in which case \( C_1(V) = \{123\} \), \( C_2(V) = \{456\} \) and \( C_3(V) = \{789\} \). We are not certain of the minimum \( n \) for \( k = 4 \) but know that it is no greater than 10 because the 10-voter profile

\[ V = (1, 2, 3, 4, \{1, 5, 8, 9, 11, 12\}, \{2, 6, 7, 10, 11, 12\}, \{3, 5, 7, 9, 10, 11\}, \{4, 6, 8, 9, 10, 12\}, \{1, 5, 6, 13, 14, 15, 16\}, \{2, 7, 8, 13, 14, 15, 16\}) \]

has \( C_1(V) = \{1234\} \), \( C_2(V) = \{5678\} \), \( C_3(V) = \{\{9, 10, 11, 12\}\} \) and \( C_4(V) = \{\{13, 14, 15, 16\}\} \).

It should be noted that the behavior exhibited by the preceding propositions and the theorem shows only what is possible and not necessarily what is probable. We would ordinarily expect that the most popular candidates will be contained in some of the committees
chosen by the majority and and strict majority TFs. However, one can envision situations in which top individual performers are not considered suitable for committee work.

The following proposition shows that, when all committees are acceptable, there is only one cardinal TF which ensures that every candidate in any selected committee (i.e., every member of $C_t$) is approved by at least one voter.

**Proposition 5.6** Suppose $t$ is a cardinal TF with $1 \leq t(S) \leq |S|$ for every $S \in \xi = 2^{|m|} \setminus \{\emptyset\}$, where $m \geq 2$. Then $t(S) = |S|$ for all $S \in \xi$ if and only if, for every $V \in \nu$ that has $|V_i| > 0$ for at least one $i$, every $S \in C_t(V)$ contains only candidates with one or more votes.

**Proof.** If $t(S) = |S|$ for all $S \in \xi$ and $V$ is not $(\emptyset, \ldots, \emptyset)$, then every $S \in C_t(V)$ is a subset of some $V_i$ and therefore has positive support for every candidate therein.

Suppose $t$ is not equivalent to the preceding TF. Then there is a $k$ between 2 and $m$ and an integer $\alpha$ such that $1 \leq t(S) = \alpha \leq k - 1$ for all $S \in \xi_k$. With $K = \{1, 2, \ldots, \alpha\}$, the constant profile $V = (K, K, \ldots, K)$ has unanimous approval for every $S \in \xi_k$ that includes $K$ and $k - \alpha$ other candidates from $[m] \setminus K$ with no votes. 

As the preceding proposition shows, when cardinal TFs are used, the only way when there is a guaranteed link between approval of individual candidates and committees is that approval of all members of the committee is needed before a committee is approved.

Finally, we note that the choice of an appropriate threshold has important consequences on the committee that will be chosen. More precisely, Given $\xi_k$ with $k \geq 2$, a voter’s approval set may depend on the constant TF used to determine choice, i.e., on the value of $\alpha$. However, if $V$ is fixed and we vary $\alpha$, $C_\alpha(V)$ might vary widely for the different $\alpha$’s. An extreme possibility is noted in our final theorem.
Theorem 5.7 Given $k \geq 2$, there are $n, m$ and a corresponding $V$ such that $C_{\alpha}(V) = \{S_{\alpha}\}$ for $\alpha = 1, 2, \ldots, k$ with the $S_{\alpha}$ mutually disjoint.

Proof. Given $k \geq 2$, let $S_{\alpha}$ be a $k$-element set for each $\alpha$ in $\{1, 2, \ldots, k\}$ with the $S_{\alpha}$ mutually disjoint. Let $\bigcup S_{\alpha}$ be the candidate set, so $m = k^2$. With $\lambda_1, \lambda_2, \ldots, \lambda_k$ as-yet-unspecified positive integers, set

$$n = \sum_{\alpha=1}^{k} \lambda_{\alpha} \binom{k}{\alpha}$$

and for each $\alpha$ let $V \in \nu$ have $\lambda_{\alpha}$ copies of each $\alpha$-member subset of $S_{\alpha}$. We will choose the $\lambda_{\alpha}$ so that $C_{\alpha}(V) = \{S_{\alpha}\}$.

Set $\lambda_k = 1$. Then $C_k(V) = \{S_k\}$ because $S_k$ is the only $V_i$ that contains at least $k$ candidates.

Set $\lambda_{k-1} = 2$. Note that the only $V_i$’s with $|V_i| \geq k - 1$ are $S_k$ and the two copies of each $(k-1)$-member subset of $S_{k-1}$. This ensures that $C_{k-1} = \{S_{k-1}\}$. (For example, when $k = 2$, $V = (1, 1, 2, 2, 34)$ with $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$.)

When $k \geq 3$, let $\lambda_{k-2}, \ldots, \lambda_1$ be increasingly larger integers so that $C_{\alpha}(V) = \{S_{\alpha}\}$ for $\alpha = k-2, \ldots, 1$. For any such $\alpha$, the number of $V_i$ which are $\alpha$-member subsets of $S_{\alpha}$ equals $\lambda_{\alpha} \binom{k}{\alpha}$, and a suitably large $\lambda_{\alpha}$ will make this greater than the number of $V_i$ that contain $\alpha$ or more of any given $k$-member subset of $S_{\alpha+1} \cup \ldots \cup S_k$. ■

6 Concluding Remarks

The main motivation of this paper is to point out that procedures for electing a team or committee could and should take into account voters’ assessments of the team quality,
and not just voters’ assessments of the candidates as individuals. We suggested a method that modifies the way approval voting ballots are counted so that a voter approves of a team if and only if the team includes a sufficient number of candidates he or she voted for. The ballots are counted using threshold functions which have natural interpretations. Constant threshold functions simply allow voters to approve of any committee that has at least some fixed number of members that are approved by the voter. Similarly, cardinal threshold functions (such as the majority threshold functions) allow voters to approve of any committee that has, e.g., some percentage (e.g., majority) of members that are approved by the voter.

The simplicity of the proposed procedure is important since overly-complicated procedures that demand too much information from voters have slim chances of being implemented. By having to select only the set of candidates they approve of, voters avoid the potentially daunting burden of having to report their preference ranking over all possible committees. Furthermore, requiring voters to select a set of candidates instead of requiring that they rank individual candidates, is aligned with the purpose of the aggregation procedure (selecting a committee, i.e., a set of candidates). Thus, approval voting, in contrast to other widespread voting methods for selecting individual candidates, seems to be an obvious starting point for developing an aggregation method for committee selection.

While social choice theory impossibility theorems eliminate the possibility of the existence of a single best voting procedure for committee selection, it would nevertheless be interesting to propose new committee selection methods that take into account interdependencies among individuals within a committee and then to identify strengths and weaknesses of these methods. Such a task goes beyond the scope of this paper.
It would also be interesting to analyze strategic behavior of the voters for any proposed committee selection procedure. Even though voters submit ballots in the same form as in the approval voting procedure for selecting an individual winner, they could strategize in a different way when a committee is being selected. For example, depending on a threshold rule, a voter could approve of a small set of candidates that he or she would like to see jointly on the committee, thereby expressing an opinion about the interdependencies among those candidates. Of course, strategic considerations with respect to voter’s assessments of the opinions and preferences of other voters also need to be analyzed.

References


Appendix:

Axiomatization of Approval Voting Threshold Functions

Basics:

- $n \geq 1$ voters, $m \geq 2$ candidates, $[m] = \{1, 2, \ldots, m\}$.

- $\xi \subseteq 2^m \setminus \{\emptyset\}$ is a set of admissible committees. We will place restrictions on $\xi$ to avoid anomalous cases. For non-triviality, we always assume $|\xi| \geq 2$.

- $V_i$ is the set of candidates approved by voter $i$, and $\nu_i$ is the collection of feasible $V_i$.
  
  We assume that $\nu_i = 2^m$, so all votes are allowed.

- $V = (V_1, \ldots, V_n)$ is a voter profile, and $\nu = (2^m)^n$ is the set of all possible profiles.

- $V_iS = |V_i \cap S|$ for all $V_i \in \nu_i$ and all $S \in \xi$.

- $V S = (V_1S, \ldots, V_nS)$, the number of candidates in $S$ approved by the voters.

- $C$, a choice function, is a mapping from $\nu$ into the family of subsets of $\xi$. We require $C(V) \neq \emptyset$ for every profile $V \in \nu$. If $S \in C(V)$ then $S$ is a choice for $V$.

- A threshold function is a map $t : \xi \to \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and $\Upsilon$ is the set of all possible threshold functions. A $t \in \Upsilon$ is integral if every $t(S)$ is a positive integer.

- $C$ is a threshold function (TCF) is there is a $t \in \Upsilon$ such that, for all $V \in \nu$ and $S \in \xi$,

$$S \in C(V) \text{ iff } |\{i : V_iS \geq t(S)\}| \geq |\{i : V_iT \geq t(T)\}| \quad \forall T \in \xi$$
We write \( C = C_t \) when \( C \) is a TCF for \( t \in \Upsilon \). By the definition, when \( C = C_t \), we can always presume that \( t \) is integral, and we do this in what follows.

Let \( \alpha_t(S, V) = |\{i : V_i S \geq t(S)\}| \), the number of voters who meet or exceed the threshold \( t(S) \) for \( S \). Hence

\[
S \in C_t(V) \text{ iff } \alpha_t(S, V) \geq \alpha_t(T, V) \quad \forall T \in \xi.
\]

Our aim is to give conditions on a choice function \( C \) that are necessary and sufficient (under structural conditions on \( \xi \) given below) for \( C \) to be a TCF.

**Initial Axioms:**

We begin with four axioms that are necessary conditions for \( C = C_t \) for some integral \( t \in \Upsilon \). The first two apply to any \( n \geq 1 \): we omit their simple proofs of necessity. The second two assume \( n \geq 2 \), so they apply only when there are two or more voters. It is understood that the axioms apply to all \( U, V \in \nu \) and all \( p \) with \( 1 \leq p < n \).

**P1 (Nullity).** \( C(\emptyset, \ldots, \emptyset) = \xi \)

**P2 (Anonymity).** If \( U \) is a permutation of \( V \) then \( C(U) = C(V) \).

The following two-part profile partitions are used in P3 and P4:

\[
V_p^- = (V_1, \ldots, V_p, \emptyset, \ldots, \emptyset), \quad V_p^+ = (\emptyset, \ldots, \emptyset, V_{p+1}, \ldots, V_n);
\]
\[
V_i^* = (V_1, \ldots, V_{i-1}, \emptyset, V_{i+1}, \ldots, V_n), \quad V_i^\circ = (\emptyset, \ldots, \emptyset, V_i, \emptyset, \ldots, \emptyset).
\]

In the latter partition, \( i \) is any integer in \( \{1, 2, \ldots, n\} \).

**P3 (Partition Consistency).** If \( C(V_p^-) \cap C(V_p^+) \neq \emptyset \) then \( C(V) = C(V_p^-) \cap C(V_p^+) \).

**P4 (Partition Inclusivity).** If \( C(V_i^*) \cap C(V_i^\circ) = \emptyset \) for \( i = 1, \ldots, n \), then \( C(V) = \emptyset \).
Axiom P3 says that if a committee would be chosen in both parts of a two-part partition of the electorate, then all such committees - and only those - will be in the choice set for the whole electorate. P4 says that if P3 does not apply for each of the noted \( n \) two-part partitions of 1 and \( n - 1 \) voters (i.e. \( C(V^*_i) \cap C(V^o_i) \) is empty), then a committee is in the choice set of the whole electorate if and only if it is in the choice set of at least one \((n - 1)\)-member subelectorate. We note their necessity proofs next.

**Necessity proofs, P3 and P4.** Assume that \( C = C_t \) for an integral \( t \), and let

\[
\alpha^*_t(V) = \max_{s \in \xi} \alpha_t(S, V).
\]

**P3.** Suppose \( S \in C(V^-_p) \cap C(V^+_p) \). Then \( \alpha_t(S, V^-_p) = \alpha^*_t(V^-_p) \) and \( \alpha_t(S, V^+_p) = \alpha^*_t(V^+_p) \), so \( \alpha_t(S, V) = \alpha^*_t(V) \) and \( S \in C_t(V) \). Given \( T \not\in C_t(V^-_p) \cap C_t(V^+_p) \), we have \( \alpha_t(T, V^-_p) \leq \alpha^*_t(V^-_p) \) and \( \alpha_t(T, V^+_p) \leq \alpha^*_t(V^+_p) \), with < in one or both cases, so \( \alpha_t(T, V) < \alpha^*_t(V) \) and \( T \not\in C_t(V) \).

**P4.** Suppose \( S \in C_t(V^*_i) \), so \( \alpha^*_t(V^*_i) = \alpha_t(S, V^*_i) \geq \alpha_t(T, V^*_i) \) for all \( T \in \xi \). Given P4’s hypotheses, we have \( \alpha_t(S, V^o_i) = 0 \), \( \alpha_t(R, V^o_i) = 1 \) for some \( R \in \xi \) and for all such \( R \), \( \alpha_t(R, V^*_i) < \alpha_t(S, V^*_i) \). Then \( \alpha_t(S, V) \geq \alpha_t(T, V) \) for all \( T \in \xi \), so \( S \in C_t(V) \). It follows that \( C_t(V^*_i) \subseteq C_t(V) \), hence that \( \bigcup_i C_t(V^*_i) \subseteq C_t(V) \).
Suppose $S \notin C_t(V_i^*)$ for $i = 1, \ldots, n$, i.e., $S \notin \bigcup_i C_t(V_i^*)$, but contrary to P4, $S \in C_t(V)$. Then $\forall T_i \in C_t(V_i^*)$, we have

$$\alpha_t(S, V_i^*) = \alpha_t(T_i, V_i^*) - 1 = \alpha_t(V_i^*) - 1 \leq n - 2;$$

$$\alpha_t(S, V_i^o) = 1, \alpha_t(T_i, V_i^o) = 0.$$

But then $V_i S \geq t(S)$ for all $i$, so $\alpha_t(S, V) = n$, a contradiction to $\alpha_t(S, V_i^*) \leq n - 2$ and $\alpha_t(S, V_i^o) = 1$.

Axioms P3 and P4 provide routes to the extension of $C_t$ from the set of all constant profiles, i.e., from $\nu_C = \{A^n : A \subseteq 2^{|m|}\}$ to all of $\nu$. The following lemma will guide our construction of $C = C_t$ on $\nu_C$.

**Lemma 6.1** Suppose $n \geq 2$ and P2 and P3 hold. Then for every $A$ with $\emptyset \subset A \subseteq [m]$ and every profile $V$ with $V_i \in \{\emptyset, A\}$ for all $i$ with $V_i = A$ for some $i$, we have

$$C(V) = C(A, A, \ldots, A)$$

In particular, $C(A, \emptyset, \ldots, \emptyset) = C(A, A, \ldots, A)$.

**Proof.**

By P2, $C(A, \emptyset, \ldots, \emptyset) = C(\emptyset, A, \emptyset, \ldots, \emptyset) = C(\emptyset, \emptyset, \ldots, \emptyset, A)$.

By P3, $C(A, A, \emptyset, \ldots, \emptyset) = C(A, \emptyset, \ldots, \emptyset) \cap C(\emptyset, A, A, \ldots, \emptyset) = C(A, \emptyset, \ldots, \emptyset)$.

Further uses of P3 give $C(A, A, A, \emptyset, \ldots, \emptyset) = C(A, A, \emptyset, \ldots, \emptyset) \cap C(\emptyset, \emptyset, A, \ldots, \emptyset) = C(A, \emptyset, \ldots, \emptyset)$,
and consequently, \( C(A, A, \ldots, A) = C(A, \emptyset, \ldots, \emptyset) \).

P2 completes the proof.  

■

**Axioms for Constant Profiles and \( n = 1: \)**

The lemma’s final equation allows us to obtain \( C = C_t \) on constant profiles from \( C \) on \( \{(A, \emptyset, \ldots, \emptyset) : A \subseteq [m]\} \). Let

\[
C^*(A) = C(A, \emptyset, \ldots, \emptyset) \text{ for all } A \in \nu_1.
\]

This simplifies notation and has the added feature that it applies to \( n = 1 \). We work with \( C^* \) until later.

We adopt two structural restrictions on \( \xi \) with respect to \( C^* \). They forbid two kinds of anomalous committees. The first type applies to \( S \) if, whenever \( S \) is a choice for \( A \in \nu_1 \) then all committees are choices for \( A \)[i.e., \( C^*(A) = \xi \)], and this is true for every \( A \in \nu_1 \). The second type applies to \( S \) if \( S \) is a choice regardless of \( A \), or \( S \in C^*(A) \) for all \( A \in \nu_1 \). The following restrictions rule out these anomalies.

**R1.** \( \forall S \in \xi, \exists A \in \nu_1 \text{ such that } S \in C^*(A) \text{ and } T \not\in C^*(A) \text{ for some } T \in \xi; \)

**R2.** \( \forall S \in \xi, \exists A \in \nu_1 \text{ such that } S \not\in C^*(A). \)

It is still possible to have \( C^*(A) = \xi \) for some \( A \in \nu_1 \), and in fact P1 requires this for \( A = \emptyset \).

We partition \( \nu_1 \) into two parts, \( \eta \) and \( \zeta \):

\[
\eta = \{ A \in \nu_1 : C^*(A) = \xi \} \\
\zeta = \{ A \in \nu_1 : S \not\in C^*(A) \text{ for some } S \in \xi \}.
\]
A 0/1 matrix for $C^*$ in which 0 denotes $S \notin C^*(A)$ and 1 denotes $S \in C^*(A)$ has the following form with rows in $\nu_1$ and columns in $\xi$:

<table>
<thead>
<tr>
<th></th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1 0</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

[Note that every row in $\zeta$ has a 0 (by definition of $\zeta$) and a 1 (when $C^*(A) \neq \emptyset$). Also, every column in $\zeta$ has a 0 and a 1 by R1 and R2: all 0’s violate R1 and all 1’s violate R2].

We use three more axioms for $C^*$. The first is a monotonicity axiom that is easily seen to be necessary for $C^* = C_t$ (on $\nu_1$) for some $t$.

**P5 (Monotonicity).** For all $A, B \in \nu_1$ and all $S \in \xi$: if $A \in \zeta$, $S \in C^*(A)$ and $BS \geq AS$ then $S \in C^*(B)$.

This has the following implications in conjunction with other conditions.

**Lemma 6.2** For all $A \in \zeta$ and all $S \in C^*(A)$, $A \cap S \neq \emptyset$, i.e., $AS > 0$.

**Lemma 6.3** $C^*([m]) = \xi$.

**Proof Comments.** If $AS = 0$ with $A \in \zeta$ and $S \in C^*(A)$, then P5 implies a 1’s column for $S$ in the preceding matrix, and this violates R2. For Lemma 6.3, every column $S$ in the matrix has a 1 in the $\zeta$ part, so, because $[m]S \geq AS$, P5 gives $S \in C^*([m])$ for every $S \in \xi$.  

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Our other two conditions for \( C^* \) use the following definition of an integral function \( \tau \) on \( \xi \):

\[
\tau(S) = \min\{BS : B \in \zeta, S \in C^*(B)\}
\]

for all \( S \in \xi \).

Our structural restrictions assume that \( \tau \) is unambiguously defined, Lemma 6.2 implies that \( \tau(S) \geq 1 \) for every \( S \) and P5 says that \( S \in C^*(A) \) whenever \( AS \geq \tau(S) \). It may be anticipated that \( t \) for \( C^* = C_t \) is identical to \( \tau \).

The first of our two \( \tau \)-based conditions is

**P6 (Existential).** \( \forall S \in \xi, \exists A \in \zeta \) such that \( AS = \tau(S) - 1 \).

**Necessity Proof for P6.** We presume that \( C^* = C_t \) for a positive integral \( t \) on \( \xi \), and refer to this as “the model”.

Given \( S \in \xi \), let \( B \in \zeta \) satisfy \( BS = \tau(S) \) with \( S \in C^*(B) \). Also, let

\[
\mathcal{F} = \{F \in \nu_1 : FS = \tau(S) - 1\}.
\]

Clearly, \( \mathcal{F} \neq \emptyset \). We suppose to the contrary of P6 that \( \mathcal{F} \subseteq \eta \) and will derive a contradiction.

By the model, \( t(S) \leq \tau(S) \).

Suppose \( t(S) = \tau(S) \). By P5 and \( \xi \)’s structure, some \( G \in \xi \) has \( GS \leq \tau(S) - 2 \). Let \( F \in \mathcal{F} \) be such that \( G \subset F \). Because \( FS < t(S) \) and \( F \in \eta \) by supposition, the model implies \( FT < t(T) \) for all \( T \in \xi \). Then \( GT \leq FT < t(T) \) for all \( T \in \xi \), so we obtain the contradiction that, according to the model, \( G \in \eta \).

Therefore \( t(S) < \tau(S) \). Take \( F \subset B \) with \( F \in \mathcal{F} \). Then \( FS \geq t(S) \) so, because \( F \in \eta \), the model requires \( FT \geq t(T) \) for all \( T \in \xi \). But then \( BT \geq FT \geq t(T) \) for all \( T \in \xi \), so
we obtain the contradiction that \( B \in \eta \).

We conclude, for each \( S \in \xi \), that some member of \( \mathfrak{F} \) for \( S \) is in fact in \( \zeta \).

Axiom P6 forbids a cardinality gap in \( \zeta \) under \( S \) (see \( \mathfrak{F} \subseteq \eta \) in the preceding proof) between the \( AS \) with \( S \in C^*(A) \) and those with \( S \notin C^*(A) \). It completes our conditions for \( C^* = C_\tau \) except that the conditions thus far seem not to imply

\[
A \in \eta \text{ if and only if } \quad AS < \tau(S) \text{ for all } S \in \xi \text{ or } \quad AS \geq \tau(S) \text{ for all } S \in \xi
\]

which is a consequence of the threshold model for \( C^* \). Our other \( \tau \)-based condition addresses this concern. It is clearly necessary for the threshold model.

**P7.** \( \forall A \in \eta, \forall B_1, B_2 \in \zeta \) and \( \forall S, T \in \xi \): if \( B_1 S = \tau(S) \), \( B_2 T = \tau(T) \) and \( AS \geq B_1 S \), then \( AT \geq B_2 T \).

When joined to prior conditions, P7 implies

**Lemma 6.4** For all \( A \in \nu_1 \), if there are \( S, T \in \xi \) such that \( AS \geq \tau(S) \) and \( AT < \tau(T) \) then \( A \in \zeta \).

**Proof.** Suppose the lemma’s hypothesis hold but \( A \in \eta \). There exists \( B_1, B_2 \in \zeta \) with \( B_1 S = \tau(S) \) and \( B_2 T = \tau(T) \). We then have \( AS \geq B_1 S \) so, by P7, \( AT \geq B_2 T \), a contradiction to \( AT < \tau(T) \).

**Theorems:**

We assume P1-P7 along with R1 and R2.

**Theorem 6.5** \( \tau \) is the unique integral \( t \in \Upsilon \) for which \( C^* = C_t \).
Proof. For all $A \in \zeta$ and $S \in \xi$, the definition of $\tau$ and P5 imply $S \in C^*(A)$ iff $AS \geq \tau(S)$. By P6, $\tau$ is the unique integral $t$ for which this is true, and we have $C^* = C_\tau$ on $\zeta$. □

When $A \in \eta$, Lemma 6.4 implies that either $AS \geq \tau(S)$ for all $S \in \xi$, or $AS < \tau(S)$ for all $S \in \xi$. It follows that $C^* = C_\tau$ on $\eta$. □

Theorem 6.6 $C = C_t$ on $\nu$.

Proof. Because Theorem 6.5 covers $n = 1$, assume henceforth that $n \geq 2$. The general step in the proof of Theorem 6.6 shows how we go from $C = C_\tau$ on profiles with at most $k \geq 1$ non-empty $V_i$, $k < n$, to profiles with at most $k + 1$ non-empty $V_i$. This can be done under Lemma 6.1 by fixing the last $n - (k + 1)V_i$ at $\emptyset$ and then using P2 to get $C = C_\tau$ for all $V$ with at most $k + 1$ non-empty $V_i$. No real generality is lost if we take $k = n - 1$, so we do this in what follows.

Assume that $C = C_\tau$ on profiles with at least one $V_i = \emptyset$. We extend this to $C = C_t$ on $\nu$.

(P3) Suppose $(I, J)$ is a non-trivial two-part partition of $[n]$, suppose $V_I, V_J$ and $V$ are profiles such that $V_I$ equals $V$ for $i \in I$ and is $\emptyset$ otherwise, and $V_J$ equals $V$ for $i \in J$ and is $\emptyset$ otherwise, and suppose that $C(V_I) \cap C(V_J) \neq \emptyset$. By P3, $C(V) = C(V_I) \cap C(V_J)$. By the result for $n - 1$, $C(V_I) = C_\tau(V_I)$ and $C(V_J) = C_\tau(V_J)$, and by the necessity proof for P3, $C_\tau(V) = C_\tau(V_I) \cap C_\tau(V_J)$. Therefore, $C(V) = C_\tau(V)$.

(P4) With the notation for P4, suppose $C(V_i^*) \cap C(V_i^*) = \emptyset$ for $i = 1, \ldots, n$. By P4, $C(V) = \bigcup_i C(V_i^*)$. By the result for $n - 1$, $C(V_i^*) = C_\tau(V_i^*)$ for $i = 1, \ldots, n$, and by the necessity proof for P4, $C_\tau(V) = \bigcup_i C_\tau(V_i^*)$. Therefore, $C(V) = C_\tau(V)$.

We conclude that $C(V) = C_\tau(V)$ for all $V \in \nu$. □