Mechanism and Network Design with Private Negative Externalities

Alexandre Belloni, Changrong Deng, Saša Pekeč
The Fuqua School of Business, Duke University,
ab5@duke.edu, changrong.deng@duke.edu, pekec@duke.edu

A revenue-maximizing monopolist is selling a single indivisible good to buyers who face a loss if a rival buyer obtains it. The rivalry is modeled through a network, an arc between a pair of buyers indicates that a buyer considers another buyer its rival, and the magnitude of the loss is the private information of each buyer. First, given a network, we characterize the optimal mechanism. Second, we show that revenues depend on the network structure. Thus, in applications where it is possible, the monopolist might consider designing not only the mechanism but also the network (if not fully, at least partially). Third, we provide solutions to this joint network and mechanism design problem. Specifically, despite of the non-monotone impact of additional competition on the monopolist’s revenues, we determine revenue-maximizing rivalry networks (which in turn induce optimal mechanisms), and show that they are independent of distributional assumptions on buyers’ independent private loss values, provided virtual values are bounded from zero. We achieve these results under different restrictions on the network structure and formation. When rivalry is symmetric, matchings are optimal (with at most one path on three vertices). Thus, a market with a fragmented network structure yields higher revenues for the monopolist than a market with a fully connected network structure. However, asymmetric competitive relationships among buyers generate higher revenues than symmetric ones. The optimal asymmetric networks are characterized by (i) every buyer having at least one rival, and (ii) the existence of a buyer not considered a rival by anyone.

Key words: Mechanism Design, Network Design, Negative Externalities
1. Introduction

The potential for gains from winning in a competitive market often come hand in hand with the risk of losses if a rival wins instead. For example, a business seeking to carry a sought-after product or provide a specialized licensed service in a local area, could incur a loss if a rival secures the right to carry a product instead. Similarly, a company that is denied a regulatory approval (e.g., a drug developer seeking approval for a new drug, or a telecom company seeking a frequency spectrum license) is better off if all competitors also fail to obtain such approval, rather than if any of the competitors succeed and capture the market demand (e.g., for a drug or for bandwidth). Thus, comparisons with rivals’ successes or failures often matter as they are closely related to the market share, to capturing important customers or business opportunities, and to securing contracts, property rights or licenses.

This loss-exposure due to competitive considerations can be viewed as a negative externality: the value associated with a market outcome depends not only on one’s allocation but also on the allocation to one’s rivals. If market participants are exposed to such negative externalities, i.e., if they value their relative competitive position and relationships, this information could potentially play a role in market design.

In this paper, we show that the nature of competitive relationships dictates the format of the optimal mechanism, and thus, the underlying structure of competitive relationships among market participants is a relevant consideration in market design. Therefore, the monopolist should consider competitive relationships among market players when designing the market transaction rules.

Furthermore, once the optimal mechanism that maximizes the monopolist’s revenues is in place, impacting the structure of competitive relationships among market participants provides a new additional opportunity for improvement of the monopolist’s revenues. We show that the monopolist could be interested in investing in changes to these relationships, when possible. This is because different competitive structures not only have different optimal mechanisms, but might yield different revenues to the monopolist. We study this question of optimizing competitive relationships in the context of a revenue-maximizing monopolist who might have the ability to affect these competitive relationships among buyers. In particular, we characterize optimal structures from the monopolist’s perspective.

In many settings, it is possible for the monopolist to fully or partially impact the competitive relationships.\(^1\) For example, Facebook can choose their Preferred Marketing Developers (PMDs) in

\(^1\) Alternatively, even in settings in which the monopolist cannot change the network structure, the results of this paper also allow us to understand which network structures lead to higher revenues. Furthermore, such revenue information can be useful for designing future markets, or even to monopolists that are considering to potentially invest in changing the network (which could require substantial costs).
a given region. The competitive relationships among selected PMDs affect the best practices and choices for potential Facebook customers on that market. (Enterprise Resource Planing software providers have a similar ability in approving and licensing IT consulting companies for software implementation and integration.) When a manufacturer chooses a location for a production facility that depends on local suppliers (say, a food processing facility), it could take into account the competitive relationships among local suppliers for each of the locations considered and choose the location (and the optimal mechanism corresponding to the competitive structure of local suppliers) that maximizes its expected revenues. There are also many situations in which the regulatory environment impacts the competitive relationships. For example, operating licenses for pharmacies are regulated in many European countries in a way that limits the number of pharmacies in a geographic and demographic area (e.g., distance between two pharmacies should be at least 500 meters and there cannot be more than one pharmacy per 5,000 inhabitants of any municipality.) Similarly, the number of taxi licenses is often capped and, in addition, non-competition is mandated (e.g., while a Boston taxi can drive a customer from Boston to Cambridge, MA, it is not allowed to pick up a customer in Cambridge. The opposite restriction is in place for Cambridge taxis, thus eliminating any competition between Boston and Cambridge taxis.) These examples suggest that understanding the impact of the competitive relationships among market participants on market design is important for market-designer’s decision-making.

In this paper, we develop a theoretical model that focuses on the privately held negative externalities and their impact on the monopolist’s market design problem. Thus, we attempt to capture the value of competitive relationships among buyers with these externalities and abstract away from other potentially important aspects of realistic market design considerations. In particular, we study a market design problem for a risk-neutral revenue-maximizing monopolistic seller that has a single item that could be allocated to any of the \(n\) buyers.\(^2\) The value of the item is publicly known and is \(v\) to the buyer who obtains the item (e.g., price of the enterprise resource software suite, frequency spectrum license, contract to supply a commodity with a well-defined market price, etc.). However, each competitor \(i\) of the buyer who obtained the item, suffers a loss of \(\alpha_i\) in that case. The loss-value \(\alpha_i\) is private to each buyer \(i\) and is the negative externality realized due to the allocation to a rival. If neither the buyer nor any of its rivals obtains the item (for example, if the item does not get allocated), then the value for such a buyer is zero. We formally introduce the model in Section 2.

In Section 3, we solve the mechanism design problem for a generic structure of competitive relationships, i.e., in the case where the set of rivals for every buyer is predetermined. The revenue-maximizing mechanism considers the aggregate negative externality that a possible allocation to

\(^2\) The case of the risk-neutral cost-minimizing monopsonist buyer of a single item facing \(n\) competing sellers is analogous.
buyer $i$ would create. The mechanism allocates the item if the benefit of the allocation is larger than the aggregate negative externality induced by it. There are two interesting features of the optimal mechanism. First, the optimal mechanism might not allocate the item. Second, even buyers who do not get the item and do not experience any negative externalities from the allocation, might have to pay. A recent school naming-rights example illustrates this point: thirteen donors gave a combined $85$ million, with a minimum single donor gift of $5$ million, to the Wisconsin School of Business at the University of Wisconsin-Madison to “preserve the Wisconsin name for at least 20 years. During that time, the school will not be named for a single donor or entity.” (The Wisconsin School of Business, 2007). Hence, each of these donors paid millions of dollars for not obtaining the item (and for ensuring that no other rival obtains it either). Not only did the Wisconsin School of Business raise $85$ million for not naming the school (compared, e.g., with the $55$ million naming gift for the Tepper School of Business at Carnegie Mellon University, 2004), but they can also try to raise funds again by (not) selling the name again in 2027.

The fact that the optimal mechanism, and thus the monopolistic seller’s revenues, depend on the structure of competitive relationships among buyers, indicates that the monopolist could have preferences over these structures, and, consequently, a revenue-driven interest in designing, changing, or influencing competitive relationships among buyers. Thus, in addition to finding and implementing an optimal mechanism, the market-designer has an additional important tool for revenue optimization: optimizing over structures of market participants’ competitive relationships. Ignoring the network design part of revenue-optimization is a choice that could result in leaving money on the table, even if the optimal mechanism is implemented.

In Section 4, we focus on this important aspect of the market designer’s problem and analyze expected revenues from optimal mechanisms for different structures of competitive relationships. In particular, we characterize competitive structures that are revenue-maximizing for the market designer. This requires jointly analyzing competitive structures and mechanisms, and cannot be decoupled: each structure has its own optimal mechanism and these mechanisms do change as the competitive structures change. Furthermore, it is not a priori clear that such optimal structures exist independently of problem parameters such as the value of the item $v$ and distributional assumptions on privately held negative externalities $\alpha_i$: one competitive structure might be optimal for one distributional assumption on negative externalities, while another might be optimal for a different distributional assumption. We show that, somewhat surprisingly, the optimality of competitive structures does not depend on the underlying distributional assumptions for externalities, and characterize these revenue-maximizing competitive structures and mechanisms.

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3 This is in contrast to the optimal mechanism with a publicly known value $v$ but without any negative externalities, i.e., without considering competitive relationships among buyers. The revenue-maximizing mechanism in this case simply allocates to any buyer and charges $v$.

4 The numerical value of the expected revenues achieved by the optimal mechanism does depend on parameters.
A natural starting point for optimizing over competitive structures is the fully competitive setting in which every buyer experiences a negative externality whenever any other buyer obtains the item. In other words, every buyer considers all other buyers to be its rivals. In this setting, buyers are vulnerable and thus the revenue-maximizing mechanism should somehow exploit this by possibly capturing some of these negative externality valuations. The expected revenues in such fully competitive environment are higher than the expected revenues when there are no negative externalities. Thus, negative externalities can be exploited by the optimal mechanism in the fully competitive setting. We then investigate the extent to which negative externalities can be exploited in optimal mechanisms with limited competition. In particular, we analyze how the optimal mechanism performs when buyers are partitioned in disjoint competing blocks. Such fragmentation of competitive relationships is often due to geographical constraints. For example, as already noted, competition among taxis in the Boston area is fragmented into two competitive blocks on each side of the Charles River, while there is no competition among rivals from opposite sides of the river. Also, authorized retailers or licensed service providers (e.g., retailers of a luxury brand or authorized repair/service companies) compete for business with all rivals within the boundaries of a predefined region, such as a state or a country, and do not compete with rivals across these boundaries.\(^5\) In an attempt to capture the nature of competitive relationships in similar situations, we study settings in which buyers are fragmented into separate competing blocks, so that a buyer in a block considers all other buyers from that block only as its rivals (and thus it does not consider any buyer from a different block to be a rival). A fragmented competitive structure could have higher expected revenues than the fully competitive structure. Thus, the fully competitive setting is not revenue-maximizing among such market fragmentations. We show that the competitive structure fragmented into two-buyer competitive blocks is revenue-maximizing: each buyer considers only one other buyer as its competitor (when there is an odd number of buyers, there is either exactly one block of three buyers or a single buyer that does not have rivals).

The reason why buyer competition fragmented into blocks of size two is revenue-optimal, is two-fold. First, it is important in terms of revenue considerations for each buyer to face a possibility of experiencing negative externality (and thus the willingness to pay to avoid such an outcome), but this can be achieved already with blocks of size two. Second, if a buyer who gets the item imposes negative externality on many other buyers (because many consider her a rival), payments will not be collected from those buyers since they will experience their worst possible outcome.

\(^5\) Fragmentation of competitive relationships could be due to other reasons, such as business type. For example, advertisers competing for an advertising slot perceive similar businesses as their competitors. An online shoe retailer is competing for potential customers with other shoe retailers, but does not necessarily lose potential business if it loses an ad to a financial institution. Thus, while shoe retailers and financial institutions compete for customers within their line of business, they are not in competitive relationships across lines of business.
Thus, having large blocks of buyers limits optimal mechanism’s revenue-potential from exploiting negative externalities from buyers in that block. Another way to explain why revenues in a market fragmented into buyer matchings dominates revenues in fully competitive markets, is through the value of privately held information on negative externalities. In the fully competitive market, each buyer’s information is relevant for $n$ possible allocations ($n-1$ rivals that could get an item, and no-allocation), while in the market fragmented in blocks of two buyers, this information is relevant for only two possible allocations (that to the unique rival or no-allocation). Thus, in a fully competitive market, privately held information is more valuable than in a fragmented market, so buyers can command higher information rents, which consequently lowers the expected revenue. Hence, while competition among buyers can be and should be exploited in order to maximize the monopolistic seller’s revenues, its effect on revenues is not monotone: too much competition (e.g., fully competitive structure) or lack of competition (no competitive relationships) are both revenue inefficient.

The model of localized competition through partitioning buyers into blocks generalizes to graphical representations. The buyers are modeled as nodes in the graph and the edge in the graph indicates that two buyers (nodes) are rivals. Thus, the graph neighborhood of a buyer (node) corresponds to the set of rivals. The fully competitive setting corresponds to the complete graph, while the market fragmented in the blocks of size two corresponds to a matching. Graph representations allow for a much larger set of possible competitive structures, yet we establish that matchings remain optimal. (In the case of an odd number of users, the optimal graph is a matching with a path on three vertices, somewhat different than the case of partitioning in blocks.) In other words, the optimal mechanism on a matching has higher expected revenues than an optimal mechanism on any other graph. Note that the matchings are sparse among all graphs, i.e., a randomly selected or constructed graph representing competitive relationships is not likely to be revenue-maximizing.

The rivalry need not be mutual. A small local store might consider a multinational giant retailer a competitor and could be affected by the assortment and pricing of products the giant retailer carries. On the other hand, the multinational might not consider some small local store a rival. Such situations are better modeled with a directed network, rather than with an undirected graph. We show that breaking up the requirement for a competitive relationship to be mutual, fundamentally changes the structure of optimal networks. In order to describe the structure of optimal directed networks, we note two types of sources of revenue (i.e., buyer payments) for the monopolistic seller. First, the seller will collect $v$ if they allocate the object. Second, the seller will collect some payment

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6 The distinction of mutual and one-directional relationships is important in social networks: Facebook friendships are bidirectional, while Twitter following is one-directional. This distinction does have potential implications for product placement and advertising decisions based on the underlying social network structure.
Table 1  Optimal Networks under Different Externality Structures and Objectives

<table>
<thead>
<tr>
<th>Positive Externalities</th>
<th>Negative Externalities</th>
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<tbody>
<tr>
<td>Revenue Maximization</td>
<td>Complete Graph</td>
</tr>
<tr>
<td>Social Surplus Maximization</td>
<td>Complete Graph</td>
</tr>
</tbody>
</table>

from all buyers who faced a possibility of experiencing negative externality, but the allocation did not impose it on them. We show that any directed network which ensures both sources of revenue is optimal. More precisely, the highest expected revenues are achieved with the optimal mechanism on a network in which (i) every buyer has at least one rival, and (ii) there exists a (benevolent) buyer that is not considered a rival by any other buyer. These two properties can be simultaneously achieved by a directed network, but cannot be achieved by an undirected graph. In another contrast to the undirected case, optimal directed networks are dense among all networks in the sense that any network can be transformed into an optimal network by a small number of arc additions/deletions. Thus, even if changing competitive relationships among pairs of buyers is costly, one needs to make a small number of changes to reach an optimal network.

In Section 5, we discuss several important aspects of the presented results. We show that optimizing over network is an important tool at the market-designer’s disposal that cannot be ignored in the sense that the gap between revenues on the optimal network and revenues on a randomly chosen network can be arbitrarily large. We also provide conditions for the emergence of no-allocation in the optimal mechanism, just as in the case of Wisconsin School of Business naming rights fundraising. We also discuss the robustness of our main findings to distributional assumptions on privately held information on negative externalities $\alpha_i$ as well as choice of the objective. Specifically, in Section 5.4 we show that the market design problem with the objective of maximizing social welfare (efficiency) is simpler than the revenue maximization problem. Indeed, any graph/network with a benevolent buyer (i.e., a buyer that does not impose negative externalities to any other buyer) is optimal and the mechanism allocates the item to this buyer. Thus, there are no benefits of exploiting competitive relationships if the objective is the welfare of all buyers. Moreover, the network optimization is straightforward in settings with positive network externalities, that is, in settings where buyers experience positive externality when a neighbor obtains the item. In such settings, whether the objective is maximizing revenues or efficiency, the value of objective function is increasing with respect to edge/arc addition and thus, a fully connected graph is optimal. We summarize the optimal networks under different externality structures and objectives in Table 1. Finally, in Section 5, we also discuss the simultaneous presence of positive and negative externalities (or their virtual valuations). In this case, the structure of optimal networks becomes dependent
on problem parameters such as $v$ and distributional assumptions about externalities. Hence, the case of revenue-maximizing on a network with negative externalities seems to be the only one in which there exists a robust characterization of non-trivial optimal structures.

Some concluding remarks are provided in Section 6. The proofs are relegated to the Appendix.

1.1. Related Literature

In this paper, we consider the optimal network design when buyers have localized allocation dependent valuations. One important element of our framework is that buyers not only have valuations for getting the item, but they also have (negative) externality valuations (when losing the item to direct competitors). Therefore, our modeling of negative externalities is related to the interdependent valuations models in economics literature (e.g., see survey Maskin 2003). In this literature, each buyer typically perceives all other buyers as direct competitors, which corresponds to a symmetric fully competitive setting that, in contrast to our setting, does not fully capture the underlying network structure. Models with externalities have been studied in, e.g., Jehiel et al. (1996), Jehiel et al. (1999), Jehiel and Moldovanu (2001), Aseff and Chade (2008), Figueroa and Skreta (2011), Deng and Pekeč (2011), and Brocas (2012). Furthermore, in some of the literature, such as Jehiel et al. (1996), externalities are modeled as private information of the rivals. Such an information structure is critical in reducing the multi-dimensional mechanism design problem to a 1-dimensional problem, since a buyer has no incentive to misreport rivals’ externalities. Our work, however, considers a different private information structure in which negative externalities are buyers’ own private information. Such a setting enables us to study how the private information and network structure of negative externalities affects the market design.³ The work of Jehiel et al. (1999) identifies the difficulty of the multi-dimensional mechanism design problem in the setting where both valuations and externalities are buyers’ own private information. Moreover, by considering the symmetric case, they are able to establish the optimality of second-price auction formats. The symmetry assumption is critical: a consequence of the results we establish is that second-price auctions are not optimal for a generic network structure of externalities (even with publicly known item valuations). The reason is that buyers may have different competitor relationships, and the symmetry may not hold inherently in our (network) setting. In particular, under optimal undirected graphs, second-price auctions cannot implement the optimal mechanism, since losing buyers may pay differently (When the winning buyer is a direct competitor of a losing buyer, the losing buyer pays zero; however, when the winning buyer is not a direct competitor of a losing buyer, the losing buyer has to pay for not suffering from negative externalities.)

³ Under the information structure of Jehiel et al. (1996) with negative externalities, Deng and Pekeč (2011) provide a rationale for the no-allocation equilibrium. We show that no-allocation is a property of suboptimal network structures in our model, and that it could also emerge in the fully competitive setting and other sub-optimal network structures.
Another important element of our framework is network design. Network design problem has been considered mostly in the context of network formation games among buyers who build or maintain the network, in both economics literature (e.g., see Jackson 2003, Epstein et al. 2009, Arcaute et al. 2013) and computer science literature (e.g., see Anshelevich et al. 2003, Chen et al. 2008, Marden and Wierman 2013). In these papers, the existence and quality of the equilibrium are studied from the perspective of independent buyers. There are also recent papers studying the impact of networks on economic systems. For example, Acemoglu et al. (2012), Acemoglu et al. (2013a), and Acemoglu et al. (2013b) analyze the role of networks as shock propagation and amplification mechanisms. Network design in our work has different motivation and goals: we design and evaluate network structures that maximize expected revenues for the associated optimal mechanism.

Furthermore, as externalities considered in our paper are localized, the network structure is crucial to understanding these localized externalities. Thus, our paper is also related to the economics literature of network externalities. In particular, buyers’ valuations are often assumed to depend on the (expected) size of their associated network (see, e.g., Katz and Shapiro 1985, Parker and Alstyne 2005) or the behavior of other buyers (see, e.g., Farrell and Saloner 1985, Johari and Kumar 2010), while valuations in our model depend on the allocation in the neighborhood. Several recent papers also study allocation and pricing procedures on networks with positive and negative externalities. In Candogan et al. (2012), a monopolistic seller’s pricing strategies for a divisible good are examined in a public information setting with a local positive network effect, i.e., a buyer’s utility increases with the usage level of its peers. In Haghpanah et al. (2013), positive externalities are modeled so that a buyer’s value is the product of a fixed private type and a known submodular function of the allocation of its peers. The work of Bhattacharya et al. (2011) examines algorithmic properties of some allocation and pricing procedures on a network in a setting where negative externalities are publicly known. Our paper focuses on the impact of privately held negative externalities on the structure of the optimal mechanism, and consequently, on the choice of the optimal network structure for the monopolistic seller (who is able to influence the structure of competitive relationships among buyers).

The network structure in our framework is publicly known. While there is work that studies mechanisms with private information on certain proximity networks (see, e.g., Schummer and Vohra 2002), allowing for private information on the perceived competitors would impose difficulties in finding the optimal mechanism as such private information on sets of competitors might not have sufficient structural properties for, e.g., establishing existence of belief-free optimal mechanisms. However, our results (with a publicly known network structure) provide a natural benchmark for any results in the setting with private information on the network structure.
The suboptimality of expected revenues in a fully competitive setting of our model (and optimality of fragmenting symmetric competitive relationships into pairwise matchings), could be viewed as an indication that some markets have the tendency to fragment. This has been argued in the context of some financial two-sided markets in a recent work of Peivandi and Vohra (2014), and has been observed in the context of labor markets (e.g., Roth and Xing 1994).

2. Model

A monopolistic seller has an indivisible item that can be allocated among \( N = \{1, \ldots, n\} \) buyers. Buyer \( j \)'s valuation for obtaining the item is \( v \), but she faces a negative externality \(-\alpha_j\) if a competing buyer obtains the item instead. The competitive relationships among buyers are represented by a network \( \mathcal{N} = (N, E) \), with \( E \subseteq N^2 \). The nodes \( j \in N \) correspond to buyers, and an arc \((i, j) \in E\) represents that buyer \( i \) is perceived to be a competitor to buyer \( j \), i.e., that buyer \( j \) experiences negative externality \(-\alpha_j\) if buyer \( i \) obtains the item. The neighborhoods in such network capture information on competitive relationships: the set of buyer \( j \)'s competitors, i.e., those that could impose negative externality \(-\alpha_j\) to buyer \( j \) if they obtain the item, is denoted by \( \mathcal{N}^-(j) = \{ i \in N : (i, j) \in E \} \), and the set of buyers \( i \) that perceive buyer \( j \) to be their competitor, i.e., those that would suffer negative externality \(-\alpha_i\) if buyer \( j \) obtains the item, is denoted by \( \mathcal{N}^+(j) = \{ i \in N : (j, i) \in E \} \). We also use notation \( d^+(j) = |\mathcal{N}^+(j)| \) and \( d^-(j) = |\mathcal{N}^-(j)| \), and \( \delta^+(\mathcal{N}) = \min_{j \in \mathcal{N}} d^+(j) \) and \( \delta^-(\mathcal{N}) = \min_{j \in \mathcal{N}} d^-(j) \).

A special case is that of symmetric competitive relationships, \((i, j) \in E\) if and only if \((j, i) \in E\). Then, we use common terminology: \( \mathcal{N} \) is a graph (i.e., an undirected network) and \((i, j)\) such that \((i, j), (j, i) \in E\) is an (undirected) edge. Note that, when \( \mathcal{N} \) is a graph, \( \mathcal{N}^+(j) = \mathcal{N}^-(j) \) and we denote it as \( \mathcal{N}(j) \); consequently, \( d^+(j) = d^-(j) \) and it is denoted by \( d(j) \), and \( \delta^+(\mathcal{N}) = \delta^-(\mathcal{N}) \) is denoted by \( \delta(\mathcal{N}) \).

The information structure is as follows. The item valuation \( v \) is publicly known and equal across buyers. The network \( \mathcal{N} \) is publicly known (and potentially directed). The magnitude of the negative externality \( \alpha_i \) is privately known by each buyer, and it is drawn independently from cumulative distribution function \( F_i \). Its support is given by \( \Omega = [\underline{\alpha}, \overline{\alpha}] \) and the corresponding density function is denoted by \( f_i \).

By the Revelation Principle (Myerson 1981), we consider direct mechanisms that allocate the item based on buyers’ reports. Reports from all the buyers are \( \hat{\alpha} = (\hat{\alpha}_j, \hat{\alpha}_{-i}) \in \Omega \). A direct mechanism specifies the allocation probabilities \( (p_i : \Omega^n \rightarrow [0, 1] \text{ is buyer } i \text{'s probability to get the item}) \) and payments \( (x_i : \Omega^n \rightarrow \mathbb{R} \text{ is the payment from buyer } i \text{ to the seller}) \) for each \( \hat{\alpha} \in \Omega^n \). If buyer \( i \)

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8 Note that buyers need not to be ex ante identical, and a negative externality value \( \alpha \in [\underline{\alpha}, \overline{\alpha}] \) can be held by any buyer. The buyer-specific support \( \Omega_i = [\underline{\alpha}_i, \overline{\alpha}_i] \) is discussed in Section 5.3.
does not participate, the trigger strategy is to allocate the item to one of buyer \( i \)'s competitors, i.e., \( j \in \mathcal{N}^-(i) \), (see, e.g., Jehiel et al. 1996).

Buyer \( i \)'s ex post utility when she reports her type as \( \hat{\alpha}_i \) while her true type is \( \alpha_i \) and other buyers truthfully report is

\[
U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i}) = v p_i(\hat{\alpha}_i, \alpha_{-i}) - \sum_{j \in \mathcal{N}^-(i)} \alpha_i p_j(\hat{\alpha}_i, \alpha_{-i}) - x_i(\hat{\alpha}_i, \alpha_{-i}),
\]

where we will write \( U_i(\alpha_i, \alpha_{-i}) \) as \( U_i(\hat{\alpha}_i, \alpha_{-i}) \) for simplicity.

Throughout the paper we will focus on ex-post constraints. Therefore, the seller’s revenue maximization problem is

\[
\max_{p, x} \sum_{i=1}^n \int x_i(\alpha_i, \alpha_{-i}) dF(\alpha)
\]

subject to

(EPIC) \( U_i(\alpha_i, \alpha_{-i}) \geq U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i}) \) for all \( i \) and all \( \alpha_i, \hat{\alpha}_i, \alpha_{-i} \),

(EPIR) \( U_i(\alpha_i, \alpha_{-i}) \geq -\alpha_i 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} \) for all \( i \) and all \( \alpha_i, \alpha_{-i} \),

(Feasibility) \( \sum_{i=1}^n p_i(\alpha_i, \alpha_{-i}) \leq 1 \) and \( p_i(\alpha_i, \alpha_{-i}) \geq 0 \), for all \( i \).

### 3. Optimal Mechanism for a Given Network

This section solves the optimal mechanism design problem under the negative externality structure discussed in the previous section when the network \( \mathcal{N} \) is fixed. This characterization is of interest on its own, but it also plays a crucial role in the following sections in which we consider the designer of the mechanism who can further optimize over the network itself.

To describe the optimal mechanism, define the virtual negative externality

\[
\pi_i(\alpha_i) := \alpha_i - \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)}
\]

and for a set of nodes \( A \subseteq N \) we let \( \pi_A(\alpha) = \sum_{j \in A} \pi_j(\alpha_j) \) and \( p_A(\alpha) = \sum_{j \in A} p_j(\alpha) \). By the Envelope Theorem, the seller’s expected revenue can be expressed through the virtual negative externalities as

\[
\sum_{i=1}^n \int x_i(\alpha_i, \alpha_{-i}) dF(\alpha) = \sum_{i=1}^n \alpha_i 1_{\{\mathcal{N}^-(i) \neq \emptyset\}} + \int \sum_{i=1}^n \{ p_i(\alpha) (v - \pi_{\mathcal{N}^-(i)}(\alpha)) \} dF(\alpha).
\]

The first term in (2) represents the revenue from individual negative externalities, while the second term is the revenue from payments for selling the good, discounted by the information rent for the externalities a buyer imposes on the system. The proof of the equality in (2) is in the Appendix.
This allows us to restate the seller’s revenue maximization problem as a function of the allocation variable

\[
\Pi(N) := \max_p \left\{ \Pi(p, N) := \sum_{i=1}^{n} \alpha_i 1\{N^{-}(i) \neq \emptyset\} + \int \sum_{i=1}^{n} p_i(\alpha_i) \left( v - \pi_{N^{+}(i)}(\alpha) \right) dF(\alpha) \right\}
\]

subject to

(Feasibility) \[ \sum_{i=1}^{n} p_i(\alpha_i, \alpha_{-i}) \leq 1 \text{ and } p_i(\alpha_i, \alpha_{-i}) \geq 0, \text{ for all } i, \]

(Monotonicity) \[ \sum_{j \in N^{-}(i)} p_j(\alpha_i, \alpha_{-i}) \] is decreasing in \( \alpha_i \).

Note that the last constraint is part of the (EPIC). To solve the problem above, we apply the standard argument of ignoring monotonicity, maximizing point-wise for each \( \alpha_i \), and then verifying that the solution satisfies monotonicity under regularity conditions. The following result summarizes the optimal allocation and payment rules under standard monotonicity assumption on the virtual negative externalities (i.e., \( \pi_i \) is non-decreasing in \( \alpha_i \)).

**Theorem 1.** Suppose \( \pi_i(\cdot) \) is non-decreasing for each \( i \in N \) and the network \( N \) is given. Then it is optimal to allocate to buyer \( i \) at evaluations \( \alpha \) if and only if

\[
\pi_{N^{+}(i)}(\alpha) = \min_{j \in N} \pi_{N^{+}(j)}(\alpha) \text{ and } \pi_{N^{+}(i)}(\alpha) \leq v, \]

in which case \( p^*_i(\alpha) = 1 \), and \( p^*_j(\alpha) = 0 \) for \( j \in N \setminus \{i\} \), is optimal. Moreover, the optimal revenue equals to \(^9\)

\[
\Pi(N) = \sum_{i=1}^{n} \alpha_i 1\{N^{-}(i) \neq \emptyset\} + \int \max_{i \in N} \left( v - \pi_{N^{+}(i)}(\alpha) \right)_+ dF(\alpha), \tag{3}
\]

and for each \( i \in N \) the associated optimal payment rule is given by

\[
x^*_i(\alpha) = v p^*_i(\alpha) + \alpha_i \left\{1\{N^{-}(i) \neq \emptyset\} - p^*_N^{-}(i) - \int^{\alpha_i} \left\{1\{N^{-}(i) \neq \emptyset\} - p^*_N^{-}(i, \alpha_{-i}) \right\} dF(\alpha) \right\}. \tag{4}
\]

Theorem 1 characterizes the optimal mechanism for any given network \( N \). The allocation is determined by the impact of a buyer on others, \( N^{+} \), while the payment of a buyer is determined by how the buyer is affected, \( N^{-} \). Therefore, the network structure \( N \) plays a critical role in the optimal mechanism.

Theorem 1 also establishes that the payments of the optimal mechanism are between \( v p^*_i(\alpha) \) and \( v p^*_i(\alpha) + \alpha_i \). The first two terms in \( x^*_i(\alpha) \) can be seen as the “cost” for the agent of the allocation and the last term in \( x^*_i(\alpha) \) is negative and is the information rent which can be interpreted as a discount for the buyer.

\(^9\) For a scalar \( t \in \mathbb{R} \) we use the notation \( (t)_+ = \max\{0, t\} \).
Remark 1 (Inefficiency of Second-Price Auctions). The optimal payment (4) indicates that, unlike the symmetric setting of Jehiel et al. (1999), second-price auctions may not be able to implement the optimal mechanism for a generic network \( \mathcal{N} \), since losing buyers’ payments may differ. In particular, if the winning buyer is a direct competitor of a losing buyer, the losing buyer pays zero; but if the winning buyer is not a direct competitor of a losing buyer, such losing buyer has to pay for not being exposed to negative externalities.

Remark 2 (Non-monotonicity of Monopolist’s Revenues). The expression (3) implies that the revenue from the optimal mechanism is not monotone with respect to changes in competition (adding/deleting arcs), i.e. changes in underlying network structure. In the next section, we examine which network structures yield higher revenues for the monopolist.

4. Optimal Network Design

The previous section characterizes the revenue maximizing mechanism for a given network. In order to further improve revenues, the seller needs to resort to changing the network itself. In this section we study the problem of the revenue-maximizing seller who does not only design the optimal mechanism but also optimizes over the set of feasible networks. We will consider three different cases: (i) a feasible network must have a group structure; (ii) a feasible network is symmetric, i.e., an undirected graph; and (iii) any network is feasible.

The characterization of the optimal mechanism under a fixed network \( \mathcal{N} \) established in Section 3 points to two potential difficulties with this joint network and mechanism optimization problem. First, the optimal mechanism could change as the underlying network \( \mathcal{N} \) changes. Second, there is no monotonicity of expected revenues with respect to network arc additions or deletions.

Formally, in this section, we are addressing the following problem:

\[
\max_{\mathcal{N} \in \mathcal{R}} \Pi(\mathcal{N}) = \max_{\mathcal{N} \in \mathcal{R}} \max_{p \in \mathcal{P}} \Pi(p; \mathcal{N})
\]

where \( \mathcal{R} \) denotes the set of feasible network structures (e.g., group structure, undirected graph, or a generic network), and where \( \mathcal{P} \) denotes the set of feasible allocation rules. More precisely, \( \mathcal{P} \) is the set of allocation rules satisfying (EPIC), (EPIR), and (Feasibility) constraints, and, thus, \( \Pi(\mathcal{N}) \) is given by equation (3) in Theorem 1. We call the optimal solution to problem (5) an optimal network (e.g., optimal group structure, optimal undirected graph, or optimal directed network).\(^{10}\)

The focus of our analysis is on the impact of negative externalities on the solution to (5). In order to isolate the impact of negative externalities, throughout this section, the virtual negative externalities are assumed to be non-negative, namely we assume that \( \pi_i(\alpha_i) \geq 0 \) for all \( \alpha_i \in \Omega \).\(^{10}\)

\(^{10}\)Note that optimal network \( \mathcal{N}^* \) defines the optimal mechanism (by Theorem 1) and, consequently, the allocation rule \( p^* \) that jointly with \( \mathcal{N} \) maximizes the righthand side of (5).
This condition implies that negative externalities are non-zero, i.e. $\alpha > 0$. Several commonly used distributions imply this condition. For example, $\alpha_i \sim U[\alpha, \overline{\alpha}]$ for $0 < \alpha \leq \overline{\alpha} \leq 2\alpha$; $\alpha_i - \alpha \sim \exp(\lambda)$ where $\lambda \alpha \geq 1$. (In Section 5.5, we briefly discuss the consequences of relaxing this condition.)

We first consider problem (5) with restrictive choice of $\mathcal{N}$: we only consider network structures corresponding to partitioning players into groups, and describe the structure of optimal group structures. We show in Section 4.1 that the optimal group structures are largely independent of the distributional assumption on negative externalities. However, the distributional assumptions do matter for a complete description of optimal group structures when the number of players is odd. Interestingly, once the set of feasible structures $\mathcal{N}$ is extended to include all undirected graphs, in Section 4.2 we characterize optimal graphs and establish that their optimality is independent of distributional assumptions. The optimal undirected graphs are unique (up to a graph homomorphism). In contrast, in Section 4.3 we demonstrate that there is a large number of optimal directed networks, all independent of distributional assumptions.

4.1. Optimal Group Design

A group structure is defined by a partition of buyers, i.e., $\mathcal{G} = \left\{ \{N_h\}_h \in \mathcal{H} : \bigcup_{h=1}^{H} N_h = A \right\}$, where $H$ is the number of sets in the partition. Buyers in the set $N_h$ are said to be in the same group $h$, and have mutual competition relationships, i.e., buyer $i \in N_h$ when getting the item imposes negative externalities on all other buyers $j \in N_h \setminus \{i\}$. However, a buyer does not perceive any other buyers in a different group as competitors and vice versa. Therefore, under a group structure $\mathcal{G}$, we have $\mathcal{G}(i) = N_h \setminus \{i\}$ for every buyer $i \in N_h$.

Group structures that partition buyers in as equal-sized sets $N_h$ as possible are important in our analysis. A partition $\{N_h : h \in L\}$ of $A$ is said to be $k$-equipartition if $|N_h| = k$ for all $h \in L = \{1, \ldots, l\}$; to be an almost $k$-equipartition if $||N_h| - k| \leq 1$ with equality holding for at most one $h$.

In order to optimize over group structures we need to account that the optimal mechanism will also change as we search over different groups as characterized in Theorem 1. With ex ante identical buyers (who have the same externality distribution and item value), there are optimal allocation rules that are symmetric among buyers in the same group. This means that any permutation on valuations of buyers in $\mathcal{G}(i)$ will not change the optimal allocation rule. Therefore we can focus on the set of symmetric allocation rules to search the optimal allocation rule.

**Theorem 2.** Consider the optimization problem (5) where $\mathcal{N}$ is the set of group structures and externality valuations are identically distributed. Almost 2-equipartitions are optimal solutions to problem (5) for any non-negative and non-decreasing $\pi_i$. Furthermore, if $v > \overline{\alpha} - 1/f(\alpha) > 0$, every optimal solution to problem (5) is an almost 2-equipartition.
When the number of buyers \( n \) is even, Theorem 2 implies that the optimal group design is 2-equipartition, i.e., the revenues are maximized when buyers are fragmented and partitioned into competitive pairs (\( |G(i)| = 1 \) for all \( i \in N \)). Note that this optimal group structure is independent of distributional assumptions on \( \alpha_i \) (as long as \( \pi_i(\alpha_i) > 0 \)).

When \( n \) is an odd number, the optimal group structure is still fragmented and consists of at least \((n - 3)/2\) pairs of competing buyers. The remaining three buyers are either grouped in a clique of three competing buyers, or are fragmented into a pair of competing buyers with the last buyer not having any competitors. The competitive relationship among these three buyers in the optimal group design depends on the item value and the externality distribution. If the externality is sufficiently large, it is optimal to have these three buyers in a clique. However, if the item value is sufficiently large, it is optimal to have a single buyer without any competitors. (A numerical example is provided in Example 1 of the Appendix.)

The following corollary of Theorem 2 demonstrates that, under mild assumptions on the value \( v \), the optimal group structure induces allocation.

**Corollary 1.** Under the conditions of Theorem 2 and provided \( v > \bar{\alpha} \), the optimal mechanism induced by an optimal group structure with \( n > 3 \) always allocates the item.

Note that optimal mechanisms for arbitrary group structures do not necessarily allocate the item. In fact, there are group structures for which no-allocation is optimal for a range of \( v \) that intersects with \( v > \bar{\alpha} \), see Section 5.1.

### 4.2. Optimal Undirected Graphs

In this section we consider undirected graphs. Using \( \mathcal{N}(i) = \mathcal{N}^{-}(i) = \mathcal{N}^{+}(i) \) to denote the neighbors of buyer \( i \), the optimal mechanisms under a network design \( \mathcal{N} \) are described in Theorem 1. Furthermore, the intuition behind Theorem 2 extends to the case of undirected graphs, with calling a graph \( k \)-regular if \( d_i = k \) for all \( i \), and almost \( k \)-regular if \( |d_i - k| \leq 1 \) with equality holding for at most one \( i \in N \).

**Theorem 3.** Consider the optimization problem (5) where \( \mathcal{N} \) is the set of undirected graphs. Almost 1-regular graphs with \( \delta = 1 \) are optimal solutions to problem (5) for any non-negative and non-decreasing \( \pi_i \). Furthermore, if \( v > \alpha - 1/f_i(\alpha) > 0 \) for all \( i \in N \), every optimal solution to problem (5) is an almost 1-regular graph with \( \delta = 1 \).

Theorem 3 demonstrates that optimal undirected graphs are unique up to a graph homomorphism (since buyers are ex ante symmetric): for even \( n \), the optimal graph is a matching, and for odd \( n \) the optimal graph is a matching on \( n - 3 \) vertices and a path on the remaining three vertices.\(^{11}\)

\(^{11}\)This is the unique almost 1-regular graph with \( \delta = 1 \) when \( n \) is odd, up to a graph homomorphism.
The almost 1-regularity of optimal undirected graphs corresponds to the almost 2-equipartition structure of optimal group structures. However, unlike optimal group structures and their natural graph representations, optimal graphs do not have a clique of size three or three independent vertices of degree zero. Moreover, unlike optimal group structures, the optimality of the described graphs is independent of the distributional assumption (provided virtual valuations are positive, as stated in the theorem).

It also follows from Theorem 3 that the optimal network among all undirected graphs always allocates under mild conditions on the value $v$.

**Corollary 2.** Under the conditions of Theorem 3 and provided $v > \bar{v}$, the optimal mechanism induced by an optimal undirected graph always allocates the item.

### 4.3. Optimal Directed Networks

Next we fully extend the set of feasible structures and optimize (5) over all (directed) networks. As in preceding subsections, we build upon the characterization of Theorem 1. The flexibility of asymmetric relations allows for emergence of new optimal structures (and additional revenue gains), as characterized by the following theorem.

**Theorem 4.** Consider the optimization problem (5) where $\mathcal{N}$ is the set of directed graphs. Let $\pi_i$ be non-decreasing and non-negative for $i \in N$. A network $\mathcal{N}$ is optimal if and only if (i) $\delta^- \geq 1$, and (ii) $\delta^+ = 0$. Moreover, the optimal revenue associated with those networks is $v + \alpha n$. Furthermore, any directed network can be transformed into an optimal network by $O(n)$ arc additions/deletions.

Results in Theorem 4 describe two characterizing properties of the optimal networks: (i) each buyer has loss-exposure, i.e., could experience negative externalities, and (ii) there exists a buyer who does not impose negative externalities on any other buyers. These two conditions describe the best-case scenario from revenue-maximization perspective: (i) there is a potential to exploit loss-exposure of each buyer, and (ii) the revenues from allocating the item do not need to be offset by negative externalities imposed by such allocation. Note that these two conditions cannot be simultaneously achieved by an undirected graph (even with ex-ante symmetric buyers), hence any optimal network must have asymmetric competitive relationships among buyers. These conditions do not depend on distributional assumptions, and consequently, optimality is independent of these assumptions (analogous to the undirected graph case). Furthermore, the implementation of the optimal mechanism on any optimal network is straightforward: the item is allocated to one of the “benevolent” buyers (i.e., to any buyer $i$ with $d^+(i) = 0$) who is charged $v + \alpha$, while all other buyers are charged $\alpha$. 
Corollary 3. Under the conditions of Theorem 3 and provided \( v > 0 \), the optimal mechanism induced by an optimal directed network always allocates the item.

We illustrate optimal networks by presenting two extremal structures: the \( S_{ij} \)-network which has the minimum number of arcs among all optimal networks (there needs to be at least \( n \) arcs due to condition (i)), and \( K_{\{i\}} \)-network which has the maximum number of arcs among all optimal networks (there has to be at least \( n - 1 \) arcs missing due to condition (ii)). The \( S_{ij} \)-network is a variant of a star network defined by \( n \) arcs: \( n - 1 \) arcs \((i, k), k \in N \setminus \{i\}\) and the arc \((j, i)\). (Figure 1a shows \( S_{12} \)-network for four buyers.) \( K_{\{i\}} \)-network has all possible arcs except for \( n - 1 \) arcs \((k, i), k \in N \setminus \{i\}\). (Figure 1b shows \( K_{\{3\}} \)-network for four buyers.) There are many more optimal directed networks: Theorem 4 also establishes that, in sharp contrast to undirected graphs, optimal (directed) network structures are not unique and are “near” any network. More precisely, an optimal network can be obtained from any given network with at most \( 2(n - 1) \) arc additions/deletions. Such number of additions/deletions is very small compared with the average distance between two (uniformly drawn) random networks which is \( n(n - 1)/2 = O(n^2) \). Thus, if influencing competitive relationships among pairs of buyers is not prohibitively costly, the mechanism designer can improve expected revenues by investing in influencing a small number of competitive relationships among pairs of buyers.

5. Additional Results and Extensions
In this section, we examine the robustness of our findings in several ways. We first investigate the property that the item is always allocated by the optimal mechanism on the optimal structures (Corollaries 1, 2 and 3). In Section 5.1, we show that such property does not hold for many suboptimal network structures: no-allocation equilibrium can emerge, i.e., it may be optimal for the seller not to allocate the item. Even when the item is always allocated by the optimal mechanism for a suboptimal network structure, in Section 5.2, we indicate that the revenue loss could be
substantial. More precisely we compare revenues between random and optimal undirected graphs when allocation always occurs.

In Section 5.3, we discuss the impact of the assumption that buyers are ex ante symmetric and point to some results that generalize to settings with heterogeneous buyers.

Section 5.4 focuses on efficiency (instead of revenue-maximization). We provide a characterization for the joint mechanism and network design problem (5). It turns out that finding an optimal network structure under the efficiency (social welfare maximization) objective is simpler than under the revenue-maximization objective studied in Section 4.

Finally, in Section 5.5, we relax the requirement for virtual valuations of negative externalities to be bounded from zero. When negative externalities are non-substantial in the sense that the virtual values can be both positive and negative, the solution to the joint mechanism and network design problem (5) depends on the distributional details of negative externalities and is sensitive to other problem parameters, such as the item value v. Hence, with such relaxation, there are no belief-free optimal solutions to problem (5).

5.1. No-Allocation Equilibrium

Corollaries 1, 2 and 3 provide sufficient conditions under which the optimal network structure induces an optimal mechanism which always allocates the item. For non-optimal network structures, however, the corresponding optimal mechanism might not allocate. The purpose of this subsection is to investigate when no-allocation equilibrium arises from the optimal mechanism for a given network. We next formally state conditions on the value v for no-allocation equilibrium to exist.

**Proposition 1.** Given a network \( N \), let \( \delta^+ := \min_{i \in N} |N^+(i)| \). Suppose \( \pi_i \) is non-decreasing for \( i \in N \). Provided that \( \delta^+ \min_{i \in N} (\frac{a}{1/f_i(a)}) > v \), the optimal mechanism induced by \( N \) has \( p^*_i(\alpha) = 0 \) for all \( i \in N \).

The proof of Proposition 1 is a direct consequence of Theorem 1. If the value \( v \) does not grow with the number of buyers and virtual negative externality is bounded away from zero, \( \pi_i(\alpha) = \frac{a}{1/f_i(a)} > 0 \) for all \( i \), networks in which every node is highly connected are prone to no-allocation. Therefore, even small negative externalities may still be worth exploiting with the no-allocation equilibrium. Under relaxed assumptions, the following related proposition provides a lower bound for the probability of no allocation.

**Proposition 2.** Given a network \( N \), let \( \delta^+ := \min_{i \in N} |N^+(i)| \). Suppose \( \pi_i \) is non-decreasing for \( i \in N \). Then, the probability of no-allocation arising as the outcome of the optimal mechanism induced by \( N \) is at least \( 1 - n \exp(-\frac{(\alpha \delta^+ - v)^2}{2\alpha^2}) \).
The intuition is similar to one behind Proposition 1. If the value of allocating the item is not too large, it is optimal not to allocate if all agents are connected and subject to negative externality valuations. Finally, a necessary condition for the no-allocation equilibrium is that isolated nodes cannot exist.

**Proposition 3.** Given a network \( N \), let \( \delta^+ := \min_{i \in N} |N^+(i)| \). Suppose \( \pi_i \) is non-decreasing for \( i \in N \) and \( v > 0 \). If no allocation is optimal for some realization of the types, there are no isolated nodes, i.e. \( \delta^+ = \min_{i \in N} |N^+(i)| \geq 1 \).

Results from this subsection suggest that no allocation emerges in large networks where every buyer has a large number of perceived competitors. Thus, large highly competitive network structures cannot be optimal due to results of Corollaries 1, 2 and 3.

### 5.2. Revenue Comparison between Random and Optimal Undirected Graphs

Not allocating the item forfeits an opportunity to get \( v \) from allocating the item and, as shown in Corollaries 1, 2 and 3, the optimal mechanism allocates the item on the optimal structure. However, there are many other structures whose corresponding optimal mechanisms always allocate the item. In this subsection, we briefly investigate the difference between revenues from the optimal structure and a random structure which always allocates the item (and hence capitalizes on the item value \( v \)).

Specifically, we focus on undirected graphs. As shown in Section 4.2, the almost1-regular graph is optimal and yields \( \alpha(n - 1) + v \) in revenue. (Note that by ignoring negative externalities, a suboptimal mechanism can easily achieve \( v \) by allocating the item to any buyer.)

In order to simplify analysis, in this subsection, we further assume \( v \geq n\overline{\alpha} \) in addition to \( \pi_i(\alpha_i) \geq 0 \).\(^{12}\) To compare revenues between optimal undirected graphs (almost 1-regular graph) and random graphs, we consider a random graph \( R \) with \( n \) nodes where each edge is added independently with probability \( q \). Therefore, with high probability \( 1 - \gamma \), we have

\[
\left| |R^+(i)| - (n - 1)q \right| \leq \sqrt{nq(1 - q) \log(2/\gamma)}.
\]  

In turn this allows us to bound the revenue of the optimal mechanism given the network \( R \) with high probability as

\[
\Pi(p^*; R) = \sum_{i=1}^{n} \alpha 1\{r^-(i) = 0\} + \int \sum_{i=1}^{n} \{p_i^*(\alpha) (v - \pi_{R^+(i)})\} \ dF(\alpha)
= (1) \quad \overline{n} + v - E[\sum_{i=1}^{n} p_i^*(\alpha) \pi_{R^+(i)}] 
\leq (2) \quad \overline{n} + v - E[\min_{i \in N} \pi_{R^+(i)}] 
\leq \overline{n} + v - \overline{\alpha} \min_{i \in N} |R^+(i)| - E \left[ \min_{i \in N} \pi_{R^+(i)} - \overline{\alpha} |R^+(i)| \right] 
\leq \overline{n} + v - \overline{\alpha} \min_{i \in N} |R^+(i)| + E \left[ \max_{i \in N} |\pi_{R^+(i)} - \overline{\alpha} |R^+(i)| \right]
\]

\(^{12}\) The same conclusion would hold provided \( \max_{\xi \leq 1} E[\alpha^2_\xi] < \infty \) and \( v \geq v\bar{\xi} = \min\{v \in E[\pi_N(\alpha) \wedge \bar{\xi}] \geq \xi E[\pi_N(\alpha)] \text{ for } \xi \in (0, 1] \). Note that if \( v\bar{\xi} \leq n\overline{\alpha} \) but \( v\bar{\xi} \) allows for unbounded support of \( \alpha_i \).
where (1) holds by $v \geq n\alpha \geq n \max_{i \in N} \pi_i(\alpha_i) \geq \pi_{\mathcal{R}^+(i)}(\alpha)$, (2) holds by allocations adding up to one pointwise. By (6), $\min_{i \in N} |\mathcal{R}^+(i)| \geq nq - \sqrt{nq(1-q)\log(2/\gamma)}$ with probability $1 - \gamma$. Next, because $\pi_j(\alpha_j) - \underline{\alpha}$ is independent across $j$ and has zero mean, we have

$$E[\max_{i \in N} |\sum_{j \in N(i)} \{\pi_j(\alpha_j) - \underline{\alpha}\}|] \leq \{E[\max_{i \in N} |\sum_{j \in N(i)} \{\pi_j(\alpha_j) - \underline{\alpha}\}|^2]\}^{1/2}$$

$$\leq (1) \sqrt{2\{1 + 4\log(2n)\}}^{1/2}\{E[\max_{i \in N} \sum_{j \in N(i)} \pi_j^2(\alpha_j)]\}^{1/2}$$

$$\leq (2) \sqrt{2\{1 + 4\log(2n)\}}^{1/2}\{E[\sum_{i \in N} \pi_i^2(\alpha_i)]\}^{1/2}$$

$$\leq (3) \sqrt{2\{1 + 4\log(2n)\}}^{1/2}\{n \max_{i \in N} E[\alpha_i^2]\}^{1/2}$$

where (1) follows from Nemirovski’s inequality (e.g., see Exercise 11.8 in Boucheron et al. 2013), (2) from $\mathcal{N}(i) \subset N$, and (3) from $0 \leq \pi_i(\alpha) \leq \alpha$ by assumption and its definition. Then, collecting terms we have

$$\Pi(p^*; \mathcal{R}) \leq n\underline{\alpha}(1 - q) + v + \alpha\sqrt{n}\{q(1-q)\log(2/\gamma)\}^{1/2} + \sqrt{n}\{2 + 8\log(2n)\}^{1/2}\alpha$$

where the last two terms are $o(n)$. Since the optimal matching yields $\underline{\alpha}(n - 1) + v$, we have effective gains of the order of $n\underline{\alpha}q$.

Therefore, as $n$ grows large, the revenue gains from choosing an optimal structure rather than a random one that guarantees allocation, are unbounded. This motivates a revenue-maximizing seller to consider the aspect of network optimization.

### 5.3. Heterogenous Buyers

Throughout the analysis, we allowed for heterogeneous distributions $F_i$ describing privately held negative externalities $\alpha_i$. However, we assumed that support $\Omega = [\underline{\alpha}, \overline{\alpha}]$ is identical for all $F_i$. Our methods and most results extend naturally when allowing for heterogeneous support $\Omega_i = [\underline{\alpha}_i, \overline{\alpha}_i]$.

For instance, the characterization of the optimal mechanism for a given network shown in Theorem 1 readily extends to the heterogeneous support setting (with $\underline{\alpha}$ replaced by $\underline{\alpha}_i$ in (4) and (3).

Similarly, Theorem 4 directly generalizes to the setting with heterogeneous support $\Omega_i = [\underline{\alpha}_i, \overline{\alpha}_i]$ (now with the expected revenue from an optimal directed network expressed as $v + \sum_i \underline{\alpha}_i$).

Allowing for heterogenous support does not fully generalize Theorem 3.

**Theorem 5.** Assume $\Omega_i = [\underline{\alpha}_i, \overline{\alpha}_i]$ and consider the optimization problem (5) where $\mathcal{N}$ is the set of undirected graphs. An optimal undirected graph is 1-regular. Furthermore, if there exists $\underline{\alpha}$ such that $\underline{\alpha}_i = \underline{\alpha}$ for all $i \in N$ and if $v > \underline{\alpha} - 1/f_i(\underline{\alpha}) > 0$ for all $i \in N$, then the undirected graph is optimal if and only if it is almost 1-regular with $\delta = 1$. 
Thus, results and the proof arguments from Theorem 3 directly extend only to the setting with buyers that have heterogenous support for externality valuations, provided that the lowest possible externality is the same for all buyers. (With private information, $\alpha_i$ are critical parameters for determining the seller’s expected revenues.) When $\alpha_i$ are different, the seller could exploit these differences and focus on some particular buyers. The optimal way to exploit such differences is distribution-dependent, and in the Appendix we provide an example (see Example 2) with three buyers where the optimal graph could either have one or two edges, depending on the problem parameters. Also, with heterogenous $\alpha_i$, while almost 1-regular graphs are optimal, there could be other optimal graphs.\(^{13}\)

5.4. Efficiency Objective

Competitive relationships among buyers have potential to negatively impact their values, and the optimal network and mechanism design is exploiting these negative externalities. However, if the designer’s objective is efficiency, i.e., maximizing social welfare (surplus), then the negative externalities could only have a negative impact on the objective. Thus, efficient mechanisms will avoid negatively impacting buyers’ values, whenever possible.

More formally, the efficiency objective is to maximize the sum of buyers’ values\(^ {14}\)

$$T(N) = \max_{p,x} \sum_{i=1}^{n} \left( v p_i - \sum_{j \in N^{-}(i)} \alpha_i p_j \right)$$  \hspace{1cm} (7)

subject to

(EPIC) $U_i(\alpha_i, \alpha_{-i}) \geq U_i(\tilde{\alpha}_i, \alpha_i, \alpha_{-i})$ for all $i$ and all $\alpha_i, \tilde{\alpha}_i, \alpha_{-i}$,

(EPIR) $U_i(\alpha_i, \alpha_{-i}) \geq -\alpha_i 1_{\{N^{-}(i) \neq \emptyset\}}$ for all $i$ and all $\alpha_i, \alpha_{-i}$,

(Feasibility) $\sum_{i=1}^{n} p_i(\alpha_i, \alpha_{-i}) \leq 1$ and $p_i(\alpha_i, \alpha_{-i}) \geq 0$, for all $i$.

Thus, given the feasibility constraint and negative externalities, $v$ is an upper bound on the value of the objective function (7) for any network structure $N$. However, for a network structure with $\delta^+ = 0$ (i.e., with a buyer that is not perceived a competitor by anyone), a mechanism that allocates to a buyer $i$ with $d^+(i) = 0$ achieves the total surplus of $v$.

**Proposition 4.** A network structure $N$ has $T(N) = v$ if and only if $\delta^+(N) = 0$.

\(^{13}\)The optimal mechanism will avoid imposing negative externalities to any node with a sufficiently large value of $\alpha_i$ (and “sufficiently large” is distribution-dependent, as it depends on the relative comparisons of $v$ and virtual valuations). Thus, an optimality is not affected by any edge additions/deletions among such nodes with “sufficiently large” $\alpha_i$ (provided that these nodes are not isolated).

\(^{14}\)The payments $x_i$ cancel out in the objective as they are transfers from buyers’ to the seller.
Thus, the total welfare is maximized for network structures that have at least one “benevolent” buyer who will get the item. This observation aligns with simple intuition that eliminating competitive relationships should limit the impact of negative externalities and thus weakly improve total welfare. However, as demonstrated in the paper, when the objective is revenue-maximization, managing the structure of competitive relationships has a delicate impact on revenues: one one hand, eliminating competitive relationships eliminates possibilities for revenue generation, while on the other hand, introducing too many competitive relationships among buyers also negatively impacts revenues.

5.5. Generalized Externality Values

The main focus of this paper are negative externalities and their impact on mechanism and network design. In this subsection, however, we discuss the implications of possibly relaxing the condition $\pi_i(\alpha) \geq 0$ for all $\alpha \in \Omega$. 

A natural setting contrasting negative externalities is that of positive externalities. Specifically, requiring that in our setup virtual valuations always be negative, $\pi_i(\alpha) \leq 0$ for all $\alpha \in \Omega_i$. (Since the impact on buyer’s utility is $-\alpha_i$, buyers are experiencing positive externalities.)

With positive externalities, it is intuitive and straightforward to establish that adding pairwise relationships among buyers can only increase revenues (as all neighbors of the buyer who obtains the item benefit from such allocation and there are no negative consequences of any allocation), and consequently, the revenue-maximizing network structure is a complete graph.

Thus, in both cases with the uniform impact of externalities, corresponding to all positive externalities or all sufficiently large negative externalities (i.e., independently drawn externalities with $\max_i \pi_i(\alpha_i) \leq 0$ or $\min_i \pi_i(\alpha_i) > 0$, respectively), the optimal network structure does not depend on any further distributional details. In the case of positive externalities, the revenues are maximized on a complete graph, while determining the existence and form of the optimal structures in presence of negative externalities has been the main focus of this paper.

However, when the impact of externalities is not uniform, (i.e., if $\min_i \pi_i(\alpha_i) < 0 < \max_i \pi_i(\alpha_i)$), the optimal network structure does depend on distributional details and problem parameters. We illustrate this with a concrete example of four buyers. We present optimal networks for two different item values, $v = 0$ and $v = 10$, and for externalities independently and uniformly distributed on $[-1, 0]$ (positive externalities), on $[0, 1]$ (both positive and negative impact possible since $\pi_i(\alpha) = -1 < 0 < 1 = \pi_i(\pi)$), and on $[1, 2]$ (negative externalities). The optimal network structure with the smallest number of arcs for each case is presented in Table 2.

Specifically, the optimal network structure changes with changes in externality valuations. As externalities change from uniformly positive to uniformly significantly negative, the optimal network changes from the complete graph to optimal networks identified in this paper. While the
Table 2  Optimal Networks with \( \min_{i \in N} (|N^+(i)| + |N^-(i)|) \) (Different Distributions of \( \alpha_i \) and Different values of \( v \)).

<table>
<thead>
<tr>
<th>( v = 0 )</th>
<th>( \alpha_i \sim U[-1,0] )</th>
<th>( \alpha_i \sim U[0,1] )</th>
<th>( \alpha_i \sim U[1,2] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Network Diagram]</td>
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<table>
<thead>
<tr>
<th>( v = 10 )</th>
<th>( \alpha_i \sim U[-1,0] )</th>
<th>( \alpha_i \sim U[0,1] )</th>
<th>( \alpha_i \sim U[1,2] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Network Diagram]</td>
<td>![Network Diagram]</td>
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</table>

optimal network structures in these uniform cases are belief-free (i.e., they do not depend on details of distributional assumptions and as such can provide general insights in optimal competitive market structures, the optimal networks in intermediate cases that simultaneously allow for both positive and negative externalities depend on the problem parameters (e.g., the case of optimal networks with \( \alpha_i \sim U[0,1] \) for different values of \( v \) in Table 2) and consequently a general parameter-free guidance on optimal market structure cannot be provided.

### 6. Conclusions

We study the impact of rivalry among competing buyers on the revenues of a monopolist seller of a single indivisible object. We show that the structure of competitive relationships among buyers dictates the design of the optimal mechanism and, consequently, the seller’s expected revenues. Furthermore, we establish the existence of optimal rivalry structures for the revenue-maximizing seller. While the seller can exploit and benefit from buyers’ loss exposure due to rivalry, we show that the seller’s expected revenue is maximized for structures that limit the rivalry. Thus, fierce competitive relationships among buyers undermine revenue potential of the monopolist seller.

Competitive relationships among buyers are modeled as negative externalities: a buyer suffers a loss (which is private information, independent across buyers) if any of its rivals obtains the item. We show that the revenue-maximizing mechanism for the seller depends on the network describing the structure of the competitive relationships (that are common knowledge). In the optimal mechanism for a given network, buyers might pay the seller even if they don’t obtain the item: they pay to avoid the loss that would be induced by an allocation to a rival. When such payments are large in aggregate, it might be optimal for the seller not to allocate the item.

Expected revenues depend both on the distributional assumptions the seller uses to model buyers’ independent private information on their losses, and on the network structure describing
competitive relationships. However, we show that the optimal network structure is independent of distributional assumptions (provided that anticipated losses are sufficiently large). Thus, the revenue-maximizing seller who has ability to shape the network structure of competitive relationships among buyers, would choose the same optimal network structure regardless of its beliefs on buyers’ loss values.

Specifically, we show that for symmetric relationships, matchings are optimal structures (with a single path on three vertices in the case of an odd number of buyers). Thus, the seller’s revenues are maximized when facing maximally fragmented rival buyers grouped in pairs of mutually perceived competitors. This result indicates that the effect of negative externalities could be (at least in our setting) one of the drivers for market fragmentation to emerge naturally.

Symmetry of competitive relationships limits revenue potential for the seller. We show that there are many other asymmetric structures that revenue-dominate matchings. There are many optimal networks, none of which are symmetric, and are characterized by (i) every buyer having a possibility of experiencing a loss, and (ii) the existence of a buyer that is not perceived as a rival by anyone.

It is interesting to note that, unlike efficiency objective and/or positive externalities settings, non-trivial network structures emerge as optimal ones, yet the optimality does not depend on the distributional assumptions (provided sufficiently large negative externalities). If some buyers could experience a gain rather than a loss when a rival obtains the item (positive externality), or if the virtual valuation of a loss changes sign, we show that the structure of the optimal competitive structure depends on the distributional assumption.

Our results are established in a model that deliberately focuses on the effects of negative externalities, and could be viewed as a starting point of the analysis in richer settings. For example, even if the item value is buyers’ private information, it is likely that the optimal structures are not far from those we identify in this paper (e.g., if uncertainty about this privately held information is not large). Similarly, if multiple items are to be allocated to unit-demand buyers, it is straightforward to extend some of our results to that setting (as one needs the existence of multiple buyers not considered rivals by anyone). The main insight from our work, that the monopolist’s revenues depend on the network structure and that non-trivial networks are optimal in our model, indicates that potentially influencing competitive relationships among buyers is a relevant strategy that a revenue-maximizing monopolist should consider in richer and potentially more realistic settings.
Appendix.

A. Proofs of Section 3.

Proof of Equation (2).

We rewrite the (EPIC) as follows. By the Envelope Theorem,

$$\frac{dU_i(\alpha_i, \alpha_{-i})}{d\alpha_i} = \frac{\partial U_i(\hat{\alpha}_i, \alpha_i, \alpha_{-i})}{\partial \alpha_i} \bigg|_{\hat{\alpha}_i=\alpha_i} = - \sum_{j \in N^{-}(i)} p_j(\alpha_i, \alpha_{-i}).$$

(8)

By (1), (8), and (EPIR), we rewrite the ex post payment as

$$x_i(\alpha_i, \alpha_{-i}) = vp_i(\alpha_i, \alpha_{-i}) + \alpha_1\{N^{-}(i)\neq \emptyset\} - \sum_{j \in N^{-}(i)} \alpha_i p_j(\alpha_i, \alpha_{-i}) + \int_{\alpha_i}^{\alpha_i} \sum_{j \in N^{-}(i)} p_j(t, \alpha_{-i}) dt.$$  

(9)

The seller’s expected revenue is

$$\sum_{i \in N} \int x_i(\alpha_i, \alpha_{-i}) dF(\alpha)$$

$$= \sum_{i \in N} \left\{ \alpha_1\{N^{-}(i)\neq \emptyset\} + \int \left( v p_i(\alpha) - \sum_{j \in N^{-}(i)} \alpha_i p_j(\alpha) \right) dF(\alpha) + \int_{\alpha_i}^{\alpha_i} \sum_{j \in N^{-}(i)} p_j(t, \alpha_{-i}) dt dF(\alpha) \right\}$$

$$= \sum_{i \in N} \left( \alpha_1\{N^{-}(i)\neq \emptyset\} + \int v p_i(\alpha) dF(\alpha) - \sum_{i \in N^{-}(i)} \int \sum_{j \in N^{-}(i)} \alpha_i p_j(\alpha) dF(\alpha) \right. \right.$$  

$$\left. \left. + \sum_{i \in N^{-}(i)} \int_{\alpha_i}^{\alpha_i} \sum_{j \in N^{-}(i)} p_j(t, \alpha_{-i}) dF_i(\alpha) dt dF_{-i}(\alpha_{-i}) \right\}$$

$$= \sum_{i \in N} \left( \alpha_1\{N^{-}(i)\neq \emptyset\} + \int v p_i(\alpha) dF(\alpha) - \sum_{i \in N^{-}(i)} \int \sum_{j \in N^{-}(i)} \alpha_i p_j(\alpha) dF(\alpha) \right. \right.$$  

$$\left. \left. + \sum_{i \in N^{-}(i)} \int_{\alpha_i}^{\alpha_i} \sum_{j \in N^{-}(i)} p_j(t, \alpha_{-i}) \left( 1 - F_i(t) \right) dt dF_{-i}(\alpha_{-i}) \right\}$$

$$= \sum_{i \in N} \left( \alpha_1\{N^{-}(i)\neq \emptyset\} + \int v p_i(\alpha) dF(\alpha) - \sum_{i \in N^{-}(i)} \int \sum_{j \in N^{-}(i)} \alpha_i p_j(\alpha) dF(\alpha) \right. \right.$$  

$$\left. \left. + \sum_{i \in N^{-}(i)} \int_{\alpha_i}^{\alpha_i} \sum_{j \in N^{-}(i)} p_j(t, \alpha_{-i}) \frac{1 - F_i(t)}{f_i(t)} dF_i(t) dt dF_{-i}(\alpha_{-i}) \right\}$$

$$= \sum_{i \in N} \left( \alpha_1\{N^{-}(i)\neq \emptyset\} + \int v p_i(\alpha) dF(\alpha) + \sum_{i \in N^{-}(i)} \int \left( \frac{1 - F_i(\alpha_i)}{f_i(\alpha_i)} - \alpha_i \right) \left( \sum_{j \in N^{-}(i)} p_j(\alpha) \right) dF(\alpha). \right.$$

Note that the second equality comes from changing the order of integration.

We can further rewrite the revenue as

$$\sum_{i \in N} \int x_i(\alpha_i, \alpha_{-i}) dF(\alpha)$$
Let \( \alpha_1 \in \mathcal{N}^{-}(i \neq \emptyset) \) be a function with \( \alpha_1(N^{-}(i \neq \emptyset)) = 1 \) for any index \( i \). Then we can rewrite the seller’s expected revenue (2) for a given allocation rule \( p \) as

\[
\Pi(p;G) := \sum_{i \in N} \alpha_1(N^{-}(i \neq \emptyset)) + \int \sum_{i \in N} p_i(\alpha) dF(\alpha) - \sum_{i \in N} \sum_{j \in \mathcal{N}^+(i)} \int p_i(\alpha) \pi_j(\alpha) dF(\alpha).
\]  

(10)

Note that the third equality comes from changing the order of summation. The fourth equality is obtained by changing the notation. \( \square \)

**Proof of Theorem 1.**

Ignoring (Monotonicity) the seller’s revenue maximization problem decouples for each \( \alpha \in \Omega^n \). Since the first term is constant in \( p \), the solution of this relaxed problem is setting \( p_i^{*}(\alpha) = 1 \) for any index \( i^{*} \in \mathop{\arg \max}_{i \in N} v - \pi^{-1}(i, \alpha) \), and \( p_j(\alpha) = 0 \) for \( j \in N \setminus \{i^{*}\} \). This choice satisfies (Feasibility). Next we show this also satisfies (Monotonicity).

Fix \( k \in N \) and we will show that \( \sum_{j \in \mathcal{N}^{-}(k)} p_j(\alpha_k, \alpha_{-k}) \) is decreasing in \( \alpha_k \). By increasing \( \alpha_k \) we have that \( \pi_k(\alpha_k) \) is non-decreasing. Since \( j \in \mathcal{N}^{-}(k) \) implies that \( k \in \mathcal{N}^+(j) \) we have that \( \pi_{\mathcal{N}^+(j)}(\alpha) \) is non-decreasing in \( \alpha_k \). Then the choice of \( p \) above yields that \( \sum_{j \in \mathcal{N}^{-}(k)} p_j(\alpha_k, \alpha_{-k}) \) is decreasing in \( \alpha_k \). \( \square \)

**B. Proofs of Section 4.**

**Proof of Theorem 2.**

First rewrite the seller’s expected revenue (2) for a given allocation rule \( p \) and group structure \( G \) as

\[
\Pi(p;G) := \sum_{i \in N} \alpha_1(G(i) \neq \emptyset) + \int \sum_{i \in N} p_i(\alpha) dF(\alpha) - \sum_{i \in N} \sum_{j \in G(i)} \int p_i(\alpha) \pi_j(\alpha) dF(\alpha).
\]

(10)

Let \( \Gamma \) denote all possible group structures and let \( \mathcal{P}(G) \) denote all feasible allocation rules. The seller’s group design problem is stated as follows

\[
\max_{\mathcal{G} \in \Gamma} \max_{p \in \mathcal{P}(G)} \Pi(p;G).
\]

Next, for any \( j \in N \) define the linear functional \( H_j \) as

\[
H_j(p) \triangleq \int p_i(\alpha;G) \pi_j(\alpha) dF(\alpha).
\]
Note that, since $\pi_i \geq 0$ (indicating $\omega > 0$) for $i \in N$ and $0 \leq p_A(\alpha) \leq 1$ for $A \subseteq N$, we have $H_j(p_i) \geq 0$ and
\[
\sum_{i \in A} H_j(p_i) = H_j(p_A) = \int p_A(\alpha) \left( \alpha_j - \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} \right) dF(\alpha) \leq \int \left( \alpha_j - \frac{1 - F_j(\alpha_j)}{f_j(\alpha_j)} \right) dF(\alpha)
\]
\[
= \int \alpha_j F_j(\alpha_j) - \int (1 - F_j(\alpha_j)) d\alpha_j = \int \alpha_j F_j(\alpha_j) + \alpha - \int \alpha_j dF_j(\alpha_j) = \alpha.
\]

Because of the group structure and ex-ante symmetry, note that for $p \in P(G)$, $H_j(p_i) = \overline{H}(p_i)$ for any $j \in G(i)$. The third term of equation (10) can be restated as
\[
\sum_{i \in N} \sum_{j \in G(i)} \int p_i(\alpha) \pi_j(\alpha_j) dF(\alpha) = \sum_{i \in N} \sum_{j \in G(i)} H_j(p_i) = \sum_{i \in N} \overline{H}(p_i) d(i),
\]
where $d(i) := |G(i)|$ is the degree of buyer $i$, and $F_i = F_j$ for all indices. With these notations, $\Pi(p; G)$ can be restated as
\[
\sum_{i \in N} \omega^1_{G(i) \neq \emptyset} + v \int \sum_{i \in N} p_i dF(\alpha) - \sum_{i \in N} \overline{H}(p_i) d(i).
\]

Given an arbitrary group design $G$ and the associated optimal allocation rule $p^*(\cdot; G)$, recall that
\[
\Pi(p^*(\alpha; G); G) = \sum_{i \in N} \omega^1_{d(i) \geq 1} + v \int \sum_{i \in N} p_i^* (\alpha; G) dF(\alpha) - \sum_{i \in N} \overline{H}(p_i^* (\alpha; G)) d(i).
\]

Let $M$ denote an almost 2-equipartition, $d(i) = 1$ for all buyers with the exception of one group (let buyer $h$ in this group) with $d(h) = 2$.

We will first prove that
\[
\Pi(p^*(\alpha; G); M) \geq \Pi(p^*(\alpha; G); G).
\]

Step 1. Consider the group structure with $d(i) < 1$ for some $i$. Suppose there are two isolated nodes, $i$ and $j$. Connecting $i$ and $j$ yields an additional revenue of
\[
2\alpha - H(p_j) - H(p_i) \geq 0.
\]

Therefore, the seller can increase its revenue by setting the optimal network to contain at most one isolated node.

Step 2. Consider the group structure with $d(i) > 1$ for some $i$. Since $\overline{H}(p_i) \geq 0$, the seller’s revenue can be increased by decreasing the degree of buyer $i$ while maintaining $d(i) \geq 1$ for buyer $i$. Also note that when connecting an isolated buyer with one of other buyers, the seller’s expected revenue from this isolated buyer will not decrease since $\overline{H}(p_i) \leq \alpha$. These operations will result in the group structure of almost 2-equipartition. When $n$ is an even number, it is possible to set $d(i) = 1$ for all $i$. When $n$ is an odd number, the group structure is set to be either size-two groups plus an isolated node or size-two groups plus a size-three group.
Hence, $\Pi (p^* (\alpha; \mathcal{G}); \mathcal{M}) \geq \Pi (p^* (\alpha; \mathcal{G}); \mathcal{G})$.

Note that $p^* (\cdot; \mathcal{G})$ may not be in the set of $\mathcal{P}(\mathcal{M})$, which includes all symmetric allocation rules among buyers in the same group of $\mathcal{M}$. However, with ex ante identical buyers, there are optimal allocation rules that are symmetric among buyers in the same group. Hence, there exists $p^* (\cdot; \mathcal{M}) \in \mathcal{P}(\mathcal{M})$ such that

$$\Pi (p^* (\alpha; \mathcal{M}); \mathcal{M}) \geq \Pi (p^* (\alpha; \mathcal{G}); \mathcal{M}) .$$

Therefore, when $n$ is an even number, 2-equipartitions are optimal. When $n$ is an odd number, the seller should set as many size-two groups as possible until there are three buyers left. Analogous to Step 1, isolating each of the three buyers is dominated by grouping two of the buyers and leaving the third buyer independent. Therefore, the seller faces two possible options: either fully connecting all three buyers (option 1) or grouping two of the buyers and leaving the third buyer independent (option 2). Which one is optimal depends on the parameters.

To demonstrate this, let’s consider an example of three buyers. With option 1, the seller’s optimal revenue from the three buyers $i = 1, 2, 3$, is

$$3\alpha + v \int \sum_{i=1}^{3} p^*_i dF(\alpha) - 2 \sum_{i=1}^{3} \Pi (p^*_i) ,$$

where $p^*$ is the optimal allocation rule under option 1. With option 2, the seller’s optimal revenue is

$$2\alpha + v .$$

Suppose the item value $v$ is sufficiently large such that allocating the item is always optimal in option 1. Then the seller’s optimal revenue in option 1 is $3\alpha + v - 2\alpha$, which is smaller than the seller’s optimal revenue in option 2, i.e., $2\alpha + v$. Hence, option 2 dominates option 1.

Suppose the item value $v$ is so small such that $\int p^*_i dF(\alpha) = \varepsilon$ for some $\varepsilon > 0$. Then there exists $\tilde{\alpha}_i \in [\alpha, \bar{\alpha}]$ for buyer $i$ such that the seller’s optimal revenue in option 1 is

$$3\alpha + 3\varepsilon v - 2\varepsilon \sum_{i=1}^{3} \pi_i (\tilde{\alpha}_i) .$$

Thus, for small enough $\varepsilon > 0$, option 1 dominates option 2 when

$$v < \frac{\alpha}{1 - 3\varepsilon} - \frac{2\varepsilon \sum_{i=1}^{3} \pi_i (\tilde{\alpha}_i)}{1 - 3\varepsilon} .$$

This condition is feasible if

$$\frac{\alpha}{1 - 3\varepsilon} - \frac{2\varepsilon \sum_{i=1}^{3} \pi_i (\tilde{\alpha}_i)}{1 - 3\varepsilon} > 2 (\alpha - 1/f(\alpha)) .$$

There exist parameter values such that this condition holds.
We next prove that almost 2-equipartitions are uniquely optimal if \( v > \underline{\alpha} - 1/f(\underline{\alpha}) > 0 \).

Note that in Step 1, \( p_i^* + p_j^* \leq 1 \). By definition of \( \overline{\Pi}(p_i^*) \) and \( \overline{\Pi}(p_j^*) \), at least one of them is strictly smaller than \( \underline{\alpha} \). Hence, connecting \( i \) and \( j \) yields an additional revenue of

\[
2\underline{\alpha} - H(p_i^*) - H(p_j^*) > 0,
\]

and, the seller can strictly increase its revenue by setting the optimal network to contain at most one isolated node. Thus, we will start from the group structure with at most one isolated node and consider two cases, following the argument in Step 2 earlier in the proof.

By assumption, we have \( \pi_i > 0 \) for all \( i \) and \( v > \underline{\alpha} - 1/f(\underline{\alpha}) \). The latter condition indicates that buyer \( i \) with \( d(i) = 1 \) gets the item with positive probability for any \( G \) with \( G(j) \neq \emptyset \) for all \( j \).

Case 1: \( n \) is an even number.

If there exists \( i \) such that \( G(i) = \emptyset \), then \( p_i^*(\alpha;G) = 1 \) for all \( \alpha \). Note that \( p_i^*(\alpha;G) = 1 \) for all \( \alpha \) is not the optimal allocation rule under \( \mathcal{M} \). Hence, \( \Pi(p^*(\alpha;M);M) > \Pi(p^*(\alpha;G);M) \geq \Pi(p^*(\alpha;G);G) \).

If \( G(i) \neq \emptyset \) for all \( i \), there are two possibilities.

(i) There exists some \( i \) with \( p_i^*(\alpha;G) = 0 \) for all \( \alpha \), such \( p^*(\alpha;G) \) is not the optimal allocation rule under \( \mathcal{M} \). Thus, \( \Pi(p^*(\alpha;M);M) > \Pi(p^*(\alpha;G);M) \geq \Pi(p^*(\alpha;G);G) \).

(ii) \( p_i^*(\alpha;G) > 0 \) for some \( \alpha \) and all \( i \), \( H(p_i^*) > 0 \) by the definition of \( H \). Thus, the seller’s revenue can be strictly increased by decreasing \( d(i) \) while maintaining \( d(i) \geq 1 \) for buyer \( i \).

Hence, setting \( d(i) = 1 \) for all \( i \) is strictly optimal.

Case 2: \( n \) is an odd number.

We focus on the externality distribution such that size-two groups plus a size-three group (also denoted by \( \mathcal{M} \)) maximize the seller’s revenue. Using argument analogous to Case 1, it can be shown that the group structure with size-two groups plus a size-three group is strictly optimal when there exists \( i \) such that \( G(i) = \emptyset \).

When \( G(i) \neq \emptyset \) for all \( i \), \( n = 3 \) is straightforward, and thus we consider \( n \geq 5 \). There are also two possibilities.

(i) There exists at least four buyers with \( p_i^*(\alpha;G) = 0 \) for all \( \alpha \), such \( p^*(\alpha;G) \) is not an optimal allocation rule under \( \mathcal{M} \). Thus, \( \Pi(p^*(\alpha;M);M) > \Pi(p^*(\alpha;G);M) \geq \Pi(p^*(\alpha;G);G) \).

(ii) There exist at most three buyers with \( p_i^*(\alpha;G) = 0 \) for all \( \alpha \) (and for the rest of other buyers \( p_j^*(\alpha;G) > 0 \) for some \( \alpha \)), \( H(p_i^*) > 0 \). Thus, the seller’s revenue can be strictly increased by decreasing the degree of buyer \( j \) while maintaining \( d(j) \geq 1 \) for buyer \( j \).

Therefore, if \( v > \underline{\alpha} - 1/f(\underline{\alpha}) > 0 \), every optimal solution to problem (5) is an almost 2-equipartition. □
Remark 3. Note that if the value of the item is smaller than the smallest virtual externalities, \( v \leq \alpha - 1/f(\alpha) \), then the item is not allocated in the optimal mechanism, \( p_i^* (\alpha; G) = 0 \) with \( G(i) \neq \emptyset \) for all \( i \), and, thus, \( H(p_i^*) = 0 \). Hence, in this case, almost 2-equipartitions cannot make the seller strictly better off in terms of the expected revenue. In particular, when \( n \) is an even number, any group structure with \( d(i) \geq 1 \) achieves the same expected revenue as almost 2-equipartitions. When \( n \) is an odd number, an isolated node plus groups of any size (strictly greater than two) are always optimal.

Proof of Corollary 1.

Let \( d(i) \) be the degree of buyer \( i \) in the optimal group structure. Then by Theorem 1, the seller always allocates the item in the optimal mechanism if \( v > \pi \min_{i \in N} d(i) \). Note that when \( n \) is an odd number, \( n = 3 \), and the optimal group structure is a size-3 group, the condition is \( v > 2\pi \). In all other cases, \( v > \pi \) is sufficient for the seller’s always allocating the item. \( \square \)

Proof of Theorem 3.

Let \( \mathcal{N} \) denote an arbitrary (undirected) graph design so that \( \mathcal{N}(i) = \mathcal{N}^-(i) = \mathcal{N}^+(i) \). By Theorem 1 we have

\[
\max_p \Pi(p; \mathcal{N}) = \sum_{i=1}^{n} \alpha 1_{\mathcal{N}^+(i) \neq \emptyset} + \int \max_{i \in N} (v - \pi_{\mathcal{N}^+(i)}(\alpha))_+ dF(\alpha)
\]

First we show that the optimal structure is connected, namely \( \delta := \min_{i \in N} |\mathcal{N}(i)| \geq 1 \). Suppose \( d(i^*) = 0 \) for some \( i^* \). Let \( \widetilde{\mathcal{N}} = \mathcal{N} \cup \{(i^*, j^*)\} \) denote the graph with an additional edge connecting \( i^* \) to \( j^* \in N \). Then, since \( \max_{i \in N} (v - \pi_{\mathcal{N}^+(i)}(\alpha))_+ = v \) by \( d(i^*) = 0 \), we have

\[
\max_p \Pi(p; \mathcal{N}) = \sum_{i=1}^{n} \alpha 1_{\mathcal{N}^+(i) \neq \emptyset} + v = \sum_{i=1}^{n} \alpha 1_{\mathcal{N}^+(i) \neq \emptyset} + \alpha + \int (v - \alpha) dF(\alpha)
\]

\[
= \sum_{i=1}^{n} \alpha 1_{\widetilde{\mathcal{N}}(i) \neq \emptyset} + \int \{v - \pi_{\mathcal{N}^+(i)}(\alpha)\} dF(\alpha)
\]

\[
\leq \sum_{i=1}^{n} \alpha 1_{\widetilde{\mathcal{N}}(i) \neq \emptyset} + \int \max_{i \in N} (v - \pi_{\widetilde{\mathcal{N}^+(i)}(\alpha)})_+ dF(\alpha) = \max_p \Pi(p; \widetilde{\mathcal{N}}),
\]

where we used that \( \int \pi_i(\alpha) dF(\alpha) = \alpha \). The inequality comes from the assumption that \( \pi_i \geq 0 \) is for all \( i \in N \). Thus an optimal network is connected, i.e., \( \delta \geq 1 \). We divide the rest of the proof in two cases.

Case 1: \( n \) is an even number. Let \( \mathcal{M} \) denote a matching and again by Theorem 1

\[
\max_p \Pi(p; \mathcal{M}) = n\alpha + \int \sum_{i=1}^{n} \max_{i \in N} (v - \pi_{\mathcal{M}(i)}(\alpha))_+ dF(\alpha).
\]

Since \( \pi_i \) is non-negative for all \( i \in N \) and each realization of \( \alpha \in \Omega \),

\[
\max_{i \in N} (v - \pi_{\mathcal{N}(i)}(\alpha))_+ \leq \max_{i \in N} (v - \pi_1(\alpha))_+ = \max_{i \in N} (v - \pi_{\mathcal{M}(i)}(\alpha))_+.
\]
The equality comes from the fact that each buyer is connected to another buyer once and only once in $\mathcal{M}$. Together with $n\underline{\alpha} = \sum_{i=1}^{n} \mathbb{1}_{\{M(i) \neq \emptyset\}} = \sum_{i=1}^{n} \mathbb{1}_{\{N(i) \neq \emptyset\}}$, we have that $\mathcal{M}$ dominates $\mathcal{N}$, i.e.,

$$\max_{p} \Pi(p; \mathcal{M}) \geq \max_{p} \Pi(p; \mathcal{N}).$$

Case 2: $n$ is an odd number. Note that $n = 1$ is straightforward. Assume $n \geq 3$.

(Step 1. Reduction to union of disjoint Star Graphs.) Let $\bar{\mathcal{N}}$ be the graph obtained by removing an edge between buyer $i$ and $j \in \mathcal{N}(i)$ where $|\mathcal{N}(i)| \geq 2$ and $|\mathcal{N}(j)| \geq 2$. Since $\pi_i(\alpha_i) \geq 0$ for all $i \in N$, and $\bar{\mathcal{N}}(i) \neq \emptyset$ and $\bar{\mathcal{N}}(j) \neq \emptyset$, we have $\pi_{\bar{\mathcal{N}}(i)}(\alpha) \leq \pi_{\mathcal{N}(i)}(\alpha)$. It follows that $\max_{p} \Pi(p; \bar{\mathcal{N}}) \geq \max_{p} \Pi(p; \mathcal{N})$. Repeating this operation, the final network (denoted as $\bar{\mathcal{N}}$) is a union of disjoint star graphs.

(Step 2. Reduction to union of disjoint edges and paths over three vertices.) Let the degree of a central buyer $i^c$ be $d(i^c)$. We next prove that, if $d(i^c) \geq 3$, it is possible to improve revenues by the following operation: For $\bar{j}, \bar{k} \in \bar{\mathcal{N}}(i^c)$ remove the edges $\{(\bar{j},i^c)\}$ and $\{(\bar{k},i^c)\}$, and add the edge $\{(\bar{j},\bar{k})\}$. Let $\bar{\mathcal{N}}$ denote the newly constructed network. The seller’s optimal revenue under network $\bar{\mathcal{N}}$ is

$$n\underline{\alpha} + \int \max \left\{ 0, v - \pi_{\bar{\mathcal{N}}(i^c)}(\alpha), v - \pi_{i^c}(\alpha), \max_{i \in \mathcal{N}(i^c) \cup \{i^c\}} v - \pi_{\mathcal{N}(i)}(\alpha) \right\} dF(\alpha).$$

However, the seller’s optimal revenue under network $\bar{\mathcal{N}}$ is

$$n\underline{\alpha} + \int \max \left\{ 0, v - \pi_{\bar{\mathcal{N}}(i^c)}(\alpha), v - \pi_{\bar{j}}(\alpha_{j}), v - \pi_{\bar{k}}(\alpha_{k}), v - \pi_{i^c}(\alpha), \max_{i \in \mathcal{N}(i^c) \cup \{i^c\}} v - \pi_{\mathcal{N}(i)}(\alpha) \right\} dF(\alpha).$$

Since $\pi_i(\alpha_i) \geq 0$ for all $i \in N$ and $\bar{\mathcal{N}}(i^c) \subset \bar{\mathcal{N}}(i^c)$, $v - \pi_{\bar{\mathcal{N}}(i)}(\alpha) \leq v - \pi_{\bar{\mathcal{N}}(i)}(\alpha)$. Hence,

$$\max_{p} \Pi(p; \bar{\mathcal{N}}) \leq \max_{p} \Pi(p; \bar{\mathcal{N}}).$$

For a star graph with buyer $i^c$, this operation can be applied on buyers in $\bar{\mathcal{N}}(i^c)$ until $d(i^c) = 1$ or $d(i^c) = 2$. The above argument works for all star graphs in the network, and, thus, the network consisting of a union of disjoint edges and paths over three vertices improves the seller’s expected revenue.

(Step 3. Reduction to at most one path over three vertices.) Finally, we show that the seller’s expected revenue improves by by converting two paths over three vertices into three disjoint edges, while keeping the rest of the graph unchanged. Label buyer 1, 2, and 3 in the first path over three vertices with edges $\{(1,2)\}$ and $\{(1,3)\}$, and label buyer 4, 5, and 6 in the second path over three vertices with edges $\{(4,5)\}$ and $\{(4,6)\}$. The seller’s optimal revenue under a network consisting of two paths over three vertices, i.e., $\mathcal{N}_{2V}$, is

$$\max_{p} \Pi(p; \mathcal{N}_{2V}) = n\underline{\alpha} + \int \max \left\{ 0, v - \pi_2(\alpha_2) - \pi_3(\alpha_3), v - \pi_5(\alpha_5) - \pi_6(\alpha_6), \max_{i \in \mathcal{N}(\{1,4\})} (v - \pi_{\mathcal{N}_{2V}(i)}(\alpha)) \right\} dF(\alpha).$$
Note that $v - \pi_2(\alpha_2) - \pi_3(\alpha_3) \leq \max\{v - \pi_2(\alpha_2), v - \pi_3(\alpha_3)\}$ and $v - \pi_5(\alpha_5) - \pi_6(\alpha_6) \leq \max\{v - \pi_5(\alpha_5), v - \pi_6(\alpha_6)\}$. If the seller converts the two paths over three vertices into three disjoint edges, the seller’s optimal revenue under this converted network, i.e., $N_{3M}$ is

$$\max_p \Pi(p; N_{3M}) = n\alpha + \int \max \left\{0, \max_{i \in \{1, \ldots, 6\}} (v - \pi_i(\alpha_i)), \max_{i \in N_{i} \{1, \ldots, 6\}} (v - \pi_{N_{2V}(i)}(\alpha)) \right\} dF(\alpha).$$

Hence, $\max_p \Pi(p; N_{2V}) \leq \max_p \Pi(p; N_{3M})$. Therefore, the graph with matchings and only one path over three vertices is optimal.

We next prove that almost 1-regular graphs with $\delta = 1$ are uniquely optimal if $v > \alpha - 1/f_i(\alpha) > 0$ for all $i \in N$.

Note that $\pi_i > 0$ and $v > \alpha - 1/f_i(\alpha)$ for all $i \in N$. The latter condition indicates that buyer $i$ with $d(i) = 1$ gets the item with positive probability for any $N$ with $N'(j) \neq \emptyset$ for all $j$.

Since buyers have the same item value and the same support for externality valuation, together with the fact that $\pi_i(\overline{\pi}) = \overline{\pi} > \overline{\alpha}$, there always exists some realization of $\alpha$ such that for every $i$ and all $j \neq i$

$$v - \pi_i(\alpha_i) > \max(0, v - \pi_j(\alpha_j)).$$

Hence, with $\pi_i > 0$ for all $i$, there exist some realization of $\alpha$ such that

$$\max_{i \in N} (v - \pi_i(\alpha_i)) > \max_{i \in N} (v - \pi_{N'(i)}(\alpha)),$$

where $N \neq N^*$ with $N'(i) \neq \emptyset$ for all $i$ and $N^*$ denotes almost 1-regular graphs.

Thus, almost 1-regular graphs strictly dominate any $N \neq N^*$ with $N'(i) \neq \emptyset$ for all $i$. In particular, when $n$ is an even number, for some realization of $\alpha$,

$$\max_{i \in N} (v - \pi_{N'(i)}(\alpha)) < \max_{i \in N} (v - \pi_i(\alpha_i)),$$

and it follows that

$$\max_p \Pi(p; M) > \max_p \Pi(p; N).$$

Similarly, when $n$ is an odd number, this strict inequality on the expected revenues holds, i.e., $\max_p \Pi(p; \overline{N}) > \max_p \Pi(p; N)$, $\max_p \Pi(p; \overline{N}) > \max_p \Pi(p; \overline{N'})$, and $\max_p \Pi(p; N_{3M}) > \max_p \Pi(p; N_{2V}).$

We next prove that expected revenues for almost 1-regular graphs strictly dominate that from $N'(i)$ with $N'(i^*) = \emptyset$ for some $i^*$. As in the first part of the proof, let $\overline{N} = N \cup \{(i^*, j^*)\}$ denote the graph with an additional edge from $i^*$ to $j^* \in N$. (Without loss of generality, we consider that $\overline{N}(i) \neq \emptyset$ for all $i$.) There are two possibilities.

(i) $v < \overline{\pi}$. There exist some realization of $\alpha$ such that

$$v - \pi_{j^*}(\alpha_{j^*}) < 0 \leq \max_{i \in \overline{N}} (v - \pi_{\overline{N}(i)}(\alpha))_+.$$


(ii) $v \geq \overline{\alpha}$. Here, $v - \pi_{j^*}(\alpha_{j^*}) \geq 0$ for any $\alpha$. There are two subcases. When $N = N^*$, there exist some realization of $\alpha$ such that $v - \pi_{j^*}(\alpha_{j^*}) < \max_{i \in N} \left( v - \pi_{N(i)}(\alpha) \right)_+$ still holds (since buyers have the same item value and the same support for externality distribution). When $N \neq N^*$, we have shown above, as in (12), that there exists some realization of $\alpha$ such that $\max_{i \in N} \left( v - \pi_{N(i)}(\alpha) \right)_+ < \max_{i \in N} \left( v - \pi_{N^*(i)}(\alpha) \right)_+$.

Hence, we have

$$\int \{v - \pi_{j^*}(\alpha_{j^*})\} dF(\alpha) < \int \max_{i \in N} \left( v - \pi_{N^*(i)}(\alpha) \right)_+ dF(\alpha).$$

Therefore, if $v > \alpha - 1/f_i(\alpha) > 0$ for all $i \in N$, every optimal solution to problem (5) is an almost 1-regular graph with $\delta = 1$.

This proof approach could have also been used to establish the distribution-independent part of the optimality of group structures, i.e., for the case of even $n$. We still include a separate proof for optimal group structures as it exploits a natural symmetry in that setting and also exposes why, in general, asymmetric structures need to be considered. □

**Remark 4.** Note that if $v \leq \alpha - 1/f_i(\alpha)$ for all $i$, there is no allocation on almost 1-regular graphs for any realization of $\alpha$. Hence, there exist other undirected graphs that achieve the same expected revenue as almost 1-regular graphs. In particular, any undirected graph $N$ with $N(i) \neq \emptyset$ is optimal.

**Proof of Theorem 4.**

Recall that the seller’s expected revenue under any network $N$ is

$$\Pi(p; N) = \sum_{i=1}^n \alpha_1 \{N(i) \neq \emptyset\} + \int \sum_{i=1}^n \left\{ p_i(\alpha) \left( v - \pi_{N^*(i)} \right) \right\} dF(\alpha).$$

Note that since $\pi_i \geq 0$ (which indicates that $\overline{\alpha} > 0$), $\max_{i \in N} \left( v - \pi_{N^+(i)} \right) \leq v$. Consider any network with the following two properties: (i) each buyer experiences externalities, i.e., $\delta^- \geq 1$, and (ii) there exists at least one buyer who does not impose any externality on any other buyers, i.e., $\delta^+ = 0$.

Thus, the seller’s optimal revenue under any of these networks is

$$n\overline{\alpha} + v,$$

which is an upper bound on $\Pi(p; N)$. Therefore, the constructed network is optimal for the seller among all networks.

We next prove that for any network $N$, at most $2(n-1) = O(n)$ arc additions/deletions are needed to transform $N$ to an optimal network.

If $\delta^+ \geq 1$, each buyer if getting the item imposes negative externalities on at least one of other buyers. Since $\sum_{i=1}^n d(i)^+ = \sum_{i=1}^n d(i)^-$, there exists a buyer $j$ who experiences negative externalities,
i.e., \( d(j)^- \geq 1 \). We first remove all edges starting from buyer \( j \). This can be achieved by at most \( n - 1 \) arc deletions. We then add arcs (not starting from buyer \( j \)) pointing to buyers who do not experience negative externalities. This can be achieved by at most \( n - 1 \) arc additions, since \( d(j)^- \geq 1 \). Thus, at most \( 2(n - 1) \) arc additions/deletions are needed.

If \( \delta^+ = 0 \), there exists a buyer \( h \) who does not impose negative externalities on any buyer if getting the item. In order to complete the transformation to an optimal network, we add edges (not starting from buyer \( h \)) pointing to buyers who do not experience negative externalities. This can be achieved by at most \( n \) arc additions. Since \( n \geq 2 \), then \( 2(n - 1) \geq n \).

Hence, at most \( 2(n - 1) \) arc additions/deletions are needed in order to achieve the optimal network. □

C. Proofs of Section 5.

Proof of Proposition 1.

By Theorem 1, for any \( \alpha \in \Omega^n \), no allocation occurs if \( \min_{i \in N} \pi_{N^+}(\alpha) > v \). Since \( \pi_i \) is non-decreasing we have \( \pi_i(\alpha_i) \geq \pi_i(\alpha) = \alpha - 1/f_i(\alpha) \). Therefore \( \pi_{N^+}(\alpha) \geq \delta^+ \min_{i \in N} (\alpha - 1/f_i(\alpha)) > v \) by the assumed condition. □

Proof of Proposition 2.

No allocation is optimal in the event \( \min_{i \in N} \pi_{N^+}(\alpha) > v \). Recall that \( (\alpha_i)_{i=1}^n \) are independent across \( i \) and \( E[\pi_i(\alpha_i)] = \bar{\alpha} \). Thus,

\[
\Pr \left( \min_{i \in N} \pi_{N^+}(\alpha) \leq v \right) \leq n \max_{i \in N} \Pr \left( \pi_{N^+}(\alpha) \leq v \right) \\
= n \max_{i \in N} \Pr \left( \pi_{N^+}(\alpha) - \bar{\alpha}|N^+(i)| \leq v - \bar{\alpha}|N^+(i)| \right) \\
\leq n \exp(-\{\bar{\alpha}\delta^+ - v\}^2/(2\bar{\alpha}^2)).
\]

where we used that \( 0 \leq \pi_i(\alpha_i) \leq \alpha_i \) and Hoeffding inequality. □

Proof of Proposition 3.

If there exists a buyer \( i \) such that \( N^+(i) = \emptyset \), then \( \pi_{N^+}(\alpha) = 0 < v \). Thus, it is always profitable for the seller to allocate the item to buyer \( i \). This is a contradiction to the optimality of no allocation. □

Proof of Theorem 5.

Since buyers have different externality distributions with \( \bar{\alpha} = \bar{\alpha} \) for all \( i \), we need \( \pi_i > 0 \) for all \( i \) and \( v > \max_{i \in N} (\alpha - 1/f_i(\alpha)) > 0 \) to guarantee that almost 1-regular graphs are uniquely optimal.
With these conditions and the fact that \( \pi_i(\alpha_i) = \alpha_i > \alpha \), there always exists some realization of \( \alpha \) such that for every \( i \) and all \( j \neq i \)

\[
v - \pi_i(\alpha_i) > \max(0, v - \pi_j(\alpha_j)).
\]

The rest of the proof is similar to Theorem 3. □

**Remark 5.** Note that if \( v \leq \max_{i \in N} (\alpha - 1/f_i(\alpha)) \), there may exist other networks that achieve the same expected revenue as almost 1-regular graphs. We illustrate this with the following numerical example.

Consider four buyers with buyer \( i \)'s externality uniformly distributed on \([5, 5 + i] \), for \( i = 1, \ldots, 4 \). Thus, \( \max_{i \in N} (\alpha - 1/f_i(\alpha)) = 4 \). Buyers have the item valuation \( v = 3 \).

Almost 1-regular graphs are also optimal. Paths over four vertices, in particular, \( 1 - 4 - 3 - 2 \) and \( 1 - 3 - 4 - 2 \), achieve the same expected revenue as almost 1-regular graphs. The seller’s expected revenue is \( 20.31 \).

**Proof of Proposition 4.**

The objective function (7) can be expressed as

\[
\sum_{i=1}^{n} \left[ vp_i - \sum_{j \in N^{-}(i)} \alpha_i p_j \right] = \sum_{i=1}^{n} vp_i - \sum_{i=1}^{n} \sum_{j \in N^{+}(i)} \alpha_i p_j
\]

\[
= \sum_{i=1}^{n} vp_i - \sum_{i=1}^{n} \sum_{j \in N^{+}(i)} \alpha_j p_i
\]

\[
= \sum_{i=1}^{n} vp_i - \sum_{\{i:d^{+}(i) \geq 1\}} \sum_{j \in N^{+}(i)} \alpha_j p_i
\]

\[
= \sum_{\{i:d^{+}(i) = 0\}} vp_i + \sum_{\{i:d^{+}(i) \geq 1\}} \left( v - \sum_{j \in N^{+}(i)} \alpha_j \right) p_i \leq v, \quad (14)
\]

where the first equality is due to changing the order of summation and the second equality is due to changing the notation of \( i \) and \( j \). The third and fourth equality separate contribution of nodes with \( d^{+}(i) = 0 \) and those with \( d^{+}(i) \geq 1 \). Finally, the inequality follows from (Feasibility) and from \( \alpha_j > 0 \) for all \( j \). The equality is achieved throughout (14) if and only if

\[
\sum_{\{i:d^{+}(i) = 0\}} p_i = 1
\]

(and, consequently, \( p_i = 0 \) for all \( i \) with \( d^{+}(i) \geq 1 \)). Note that such allocation also satisfies (EPIC) and (EPIR) constraints. Thus, \( T(N) = v \) if and only if \( \{i:d^{+}(i) = 0\} \) is non-empty, i.e. \( \delta^{+}(N) = 0 \). □
D. Illustrative Theoretical Examples.

Example 1: Two Cases for Optimal Groups Design with Odd \( n \).

Consider \( n = 3 \) buyers. Each buyer’s externality \( \alpha_i \) is identically, independently, and uniformly distributed on \([1.55, 3]\). If \( v = 1.6 \), the seller’s expected revenue by putting all three buyers in a single group is 4.8, while the revenue when leaving a single buyer out of the group is 4.7. Thus, putting all three buyers in a group dominates leaving a single buyer out. However, when \( v = 2 \), the seller’s expected revenue by putting all three buyers in a group is 4.9, while the one by leaving a single buyer out is 5.1. In fact, if \( v \leq 1.72 \), putting all three buyers in a group dominates leaving a single buyer out. Otherwise, leaving a single buyer out dominates putting all three buyers in a group. □

Example 2: Non-Identical Buyers.

Consider \( n = 3 \) buyers. There are four possible graph structures. We know the isolated graph and the complete graph are not optimal. Therefore, without loss of generality, it is sufficient to only consider \( \mathcal{N}_{1-2,3} = (1, 2, 3, \{(1, 2), \{3\}) \) and \( \mathcal{N}_{2-1-3} = (1, 2, 3, \{(1, 2), \{(1, 3)\}) \). With \( \mathcal{N}_{1-2,3} \), the expected revenue for the seller is

\[
\alpha_1 + \alpha_2 + v,
\]

while, with \( \mathcal{N}_{2-1-3} \), the expected revenue is

\[
\alpha_1 + \alpha_2 + \alpha_3 + \int \max\{0, v - \pi_2 - \pi_3, v - \pi_1\} dF(\alpha).
\]

Note that if \( v \leq \min\{\pi_2 + \pi_3, \pi_1, \pi_3\} \) for all \( \alpha \), then \( \mathcal{N}_{2-1-3} \) is optimal. Similarly, if \( v \geq \max\{\pi_1, \pi_2 + \pi_3\} \) for all \( \alpha \) and \( \alpha_1 \geq \alpha_3 \), then \( \mathcal{N}_{1-2,3} \) is optimal. □

References


