CLASSNOTES FOR EE 271:
ELECTROMAGNETICS THEORY

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Chapter 1

Review

In this chapter (also corresponding to Balanis’ Chapter 1) we review the fundamental concepts and equations in electromagnetics, including Maxwell’s equations, boundary conditions, and Poynting theorem.

1.1 Maxwell’s Equations

Maxwell’s equations describe the relation between electromagnetic sources and the fields. We are interested in finding the following electromagnetic fields:

- E Electric field intensity (V/m)
- H Magnetic field intensity (A/m)
- D Electric flux density (C/m²)
- B Magnetic flux density (W/m²)

These electromagnetic fields are generated by a variety of sources:

\[
\mathbf{J}_i, \mathbf{J}_c, \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}
\]

Impressed, conduction, and displacement electric current densities (A/m²)

\[
\mathbf{M}_i, \mathbf{M}_c, \mathbf{M}_d = \frac{\partial \mathbf{B}}{\partial t}
\]

Impressed, conduction, and displacement magnetic current densities (V/m²)

\[
\rho_{ei}, \rho_{ec}
\]

Impressed and conduction electric charge densities (C/m³)

\[
\rho_{mi}, \rho_{mc}
\]

Impressed and conduction magnetic charge densities (W/m³)
1.1.1 Maxwell’s Equations in Differential Form

Maxwell’s equations in differential form are written as

\[
\nabla \times \mathbf{E} = -\mathbf{M}_i - \mathbf{M}_c - \frac{\partial \mathbf{B}}{\partial t} \tag{1.1}
\]

\[
\nabla \times \mathbf{H} = \mathbf{J}_i + \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \tag{1.2}
\]

\[
\nabla \cdot \mathbf{D} = \rho_{ei} + \rho_{ec} \tag{1.3}
\]

\[
\nabla \cdot \mathbf{B} = \rho_{mi} + \rho_{mc} \tag{1.4}
\]

In addition, the continuity equations governing the conservation law of charges are

\[
\nabla \cdot \mathbf{J}_{i,c} = -\frac{\partial \rho_{ei,c}}{\partial t}, \tag{1.5}
\]

\[
\nabla \cdot \mathbf{M}_{i,c} = -\frac{\partial \rho_{mi,c}}{\partial t}, \tag{1.6}
\]

respectively for the impressed sources and the conduction sources. Note that the total free electric and magnetic charge densities are

\[
\rho_e = \rho_{ei} + \rho_{ec}, \quad \rho_m = \rho_{mi} + \rho_{mc}
\]

where \(\rho_{ei}\) (\(\rho_{mi}\)) and \(\rho_{ec}\) (\(\rho_{mc}\)) are the electric (magnetic) charge densities due to the imposed and conduction electric (magnetic) current, respectively.

Note that in this course, we introduce the magnetic sources even though none isolated magnetic charges have been found. The reason for the introduction of these sources is for symmetry and for the convenience in future discussions on reciprocity.

1.1.2 Maxwell’s Equations in Integral Form

Two fundamental theorems are used to rewrite Maxwell’s equations into an integral form. Stokes’ theorem relates the line integral of an arbitrary vector \(\mathbf{v}\mathbf{A}\) with the surface integral of the curl of this vector:

\[
\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} \tag{1.7}
\]

where the direction of the closed line integral \(C\) follows the right-hand rule of the normal direction of the surface \(S\). Similarly, by Gauss theorem, the close surface integral of \(\mathbf{A}\) over a surface \(S\) is equal to the volume integral of the divergence of \(\mathbf{A}\) over the volume \(V\) enclosed by \(S\):

\[
\iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V \nabla \cdot \mathbf{A} dv \tag{1.8}
\]

where the normal direction of \(S\) is outward.
1.2. **Constitutive Relations**

The integral form of Maxwell’s equations are listed below. Faraday’s law is

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S (\mathbf{M}_i + \mathbf{M}_c) \cdot d\mathbf{s} - \frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{s}. \]  

(1.9)

The generalized Ampere’s circuitual law is

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\mathbf{J}_i + \mathbf{J}_c) \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} d\mathbf{s}. \]  

(1.10)

Gauss’ law for electric flux density is

\[ \int_S \mathbf{D} \cdot d\mathbf{s} = Q_{ei} + Q_{ec}. \]  

(1.11)

Gauss’ law for magnetic flux density is

\[ \int_S \mathbf{B} \cdot d\mathbf{s} = Q_{mi} + Q_{mc}. \]  

(1.12)

Finally, the continuity equations are

\[ \int_S \mathbf{J}_{i,c} \cdot d\mathbf{s} = - \frac{\partial Q_{ei,c}}{\partial t} \]  

(1.13)

\[ \int_S \mathbf{M}_{i,c} \cdot d\mathbf{s} = - \frac{\partial Q_{mi,c}}{\partial t} \]  

(1.14)

where \( Q_{ei,c} \) and \( Q_{mi,c} \) are the electric and magnetic charges inside the volume \( V \), respectively.

### 1.2 Constitutive Relations

Given the impressed electric and magnetic sources \( \mathbf{J}_i, \rho_{ei}, \mathbf{M}_i, \rho_{mi} \), the unknowns quantities to be solved are \( \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}_c, \mathbf{M}_c, \rho_{ec}, \) and \( \rho_{mc} \). Therefore there are a total of 20 scalar unknowns. However, there only 10 scalar equations in (1.1)–(1.6), excluding the scalar equations in (1.5) and (1.6) for the impressed sources. Furthermore, equations (1.3) and (1.4) are not independent equations from the rest, as it can be proven that (1.3) and (1.4) can be derived from the rest equations. Therefore, there are only 8 scalar equations, not enough to solve all 20 scalar unknowns.

Additional equations are provided by constitutive relations. For simple isotropic media, we have

\[ \mathbf{D} = \varepsilon \mathbf{E}, \]  

(1.15)

\[ \mathbf{B} = \mu \mathbf{H}. \]  

(1.16)
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Furthermore, for linear media, \( \varepsilon \) and \( \mu \) are independent of the field strength. Usually the permittivity \( \varepsilon \) and permeability \( \mu \) are also written in terms of the relative permittivity (i.e., dielectric constant) \( \varepsilon_r \) and relative permeability \( \mu_r \),

\[
\varepsilon = \varepsilon_r \varepsilon_0, \quad \varepsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi} \text{ (F/m)}
\]

\[
\mu = \mu_r \mu_0, \quad \mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}
\]

where \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of the vacuum. The conduction current densities are related to the field intensities by the Ohm’s law:

\[
J_c = \sigma E,
\]

\[
M_c = \sigma H,
\]

where \( \sigma \) (Semiens/meter) and \( \sigma_m \) (Ohms/m) are called the electric and magnetic conductivities, respectively. In general, electromagnetic media can be inhomogeneous, that is, all parameters \( \varepsilon, \mu, \sigma, \sigma_m \) may be function of position. Furthermore, some media can be dispersive, i.e., the parameters are functions of frequency.

Equations (1.15)–(1.18) together with (1.1), (1.2), (1.5), and (1.6) provide 20 scalar equations, from which all 20 unknowns can be solved. With the above constitutive relations, Maxwell’s equation can now be written as

\[
\nabla \times \mathbf{E} = -\mathbf{M}_i - \sigma_m \mathbf{H} - \frac{\mu}{\mu_0} \frac{\partial \mathbf{H}}{\partial t} \quad (1.19)
\]

\[
\nabla \times \mathbf{H} = \mathbf{J}_i + \sigma E + \frac{\varepsilon}{\varepsilon_0} \frac{\partial \mathbf{E}}{\partial t} \quad (1.20)
\]

\[
\nabla \cdot \varepsilon \mathbf{E} = \rho_e + \rho_{ec} \quad (1.21)
\]

\[
\nabla \cdot \mu \mathbf{H} = \rho_m + \rho_{mc} \quad (1.22)
\]

1.3 Boundary Conditions

Behaviors of electromagnetic fields across the boundary between two different media are important in the solution of Maxwell’s equations for discontinuous media. We consider an interface \( S \) between two different media with material properties \((\varepsilon_1, \mu_1, \sigma_1, \sigma_{m1})\) and \((\varepsilon_2, \mu_2, \sigma_2, \sigma_{m2})\) respectively. Let the electromagnetic fields in these two media be denoted as \((\mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \mathbf{B}_1)\) and \((\mathbf{E}_2, \mathbf{H}_2, \mathbf{D}_2, \mathbf{B}_2)\) respectively.

The boundary conditions between these two sets of fields at the interface \( S \) can be derived either from the integral form or the differential form of Maxwell’s equations. Denoting the normal direction \( \mathbf{n} \) of \( S \) as the normal pointing from medium 1 to medium 2, we can derive the general boundary conditions as

\[
\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\mathbf{M}_s \quad (1.23)
\]

\[
\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \quad (1.24)
\]
1.3. BOUNDARY CONDITIONS

\[ \hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_{es} \quad (1.25) \]
\[ \hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = \rho_{ms} \quad (1.26) \]

where \( \mathbf{J}_s \) and \( \mathbf{M}_s \) are the surface electric and magnetic current densities, and \( \rho_{es} \) and \( \rho_{ms} \) are the surface electric and magnetic charge densities, respectively. These surface sources are defined as, for example,

\[ \mathbf{J}_s = \lim_{h \to 0} \frac{1}{h} \mathbf{J} \quad (1.27) \]

where \( h \) is a small thickness along the normal (\( \hat{n} \)) direction on the interface. In the above, the surface sources contain the impressed sources and the conduction sources.

1.3.1 Special Case I: Source-Free Media with a Finite Conductivity

If there are no impressed sources, and the conductivities are finite in both media, then we have

\[ \hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad (1.28) \]
\[ \hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0 \quad (1.29) \]
\[ \hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0 \quad (1.30) \]
\[ \hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \quad (1.31) \]

since the surface conduction sources are zero as \( h \to 0 \).

1.3.2 Special Case II: Medium 1 is a Perfect Electric Conductor (PEC)

If medium 1 is a perfect electric conductor, \( \sigma_1 \to \infty \), surface conduction electric current and charge densities may exist. Then the boundary conditions are

\[ \hat{n} \times \mathbf{E}_2 = 0 \quad (1.32) \]
\[ \hat{n} \times \mathbf{H}_2 = \mathbf{J}_s \quad (1.33) \]
\[ \hat{n} \cdot \mathbf{D}_2 = \rho_{es} \quad (1.34) \]
\[ \hat{n} \cdot \mathbf{B}_2 = 0 \quad (1.35) \]

Note that the surface electric charge density and electric current density are also governed by the continuity equation.
1.3.3 Special Case III: Medium 1 is a Perfect Magnetic Conductor (PMC)

If medium 1 is a perfect magnetic conductor, \( \sigma_{m1} \to \infty \), surface conduction magnetic current and charge densities may exist. Then the boundary conditions are

\[
\hat{n} \times \mathbf{E}_2 = -\mathbf{M}_s \quad (1.36)
\]
\[
\hat{n} \times \mathbf{H}_2 = 0 \quad (1.37)
\]
\[
\hat{n} \cdot \mathbf{D}_2 = 0 \quad (1.38)
\]
\[
\hat{n} \cdot \mathbf{B}_2 = \rho_{ms} \quad (1.39)
\]

Similarly, the surface magnetic charge density and magnetic current density are also governed by the continuity equation.

1.4 Power and Energy

From Maxwell’s equations we can derive the conservation of energy in an electromagnetic system. We start from the vector identity

\[
\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})
\]

and make use of Maxwell’s equations (1.19) and (1.20). Then by performing a volume integral over \( V \) and using Gauss theorem, we arrive at

\[
- \int_V (\nabla \cdot (\mathbf{E} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i)) dv = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot ds + \int_V (\sigma E^2 + \sigma_m H^2) dv + \frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv
\]

(1.40)

This equation can also be rewritten as

\[
P_s = P_e + P_d + \frac{\partial}{\partial t} (W_e + W_m)
\]

(1.41)

where

\[
P_s = - \int_V (\nabla \cdot (\mathbf{E} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i)) dv \quad \text{Supplied power (W)}
\]

\[
P_e = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot ds \quad \text{Exiting power (W)}
\]

\[
P_d = \int_V (\sigma E^2 + \sigma_m H^2) dv \quad \text{Dissipated power (W)}
\]

\[
W_e = \int_V \frac{\varepsilon}{2} E^2 dv \quad \text{Stored electric energy (J)}
\]

\[
W_m = \int_V \frac{\mu}{2} H^2 dv \quad \text{Stored magnetic energy (J)}
\]
Therefore, this equation is nothing but the conservation of energy within a volume $V$. It applies even if the medium is inhomogeneous.

We can define the Poynting (power density) vector

$$\mathbf{P}(r, t) = \mathbf{E} \times \mathbf{H} \quad (1.42)$$

and the electric and magnetic energy densities

$$w_e(r, t) = \frac{\varepsilon}{2} E^2 \quad (1.43)$$
$$w_m(r, t) = \frac{\mu}{2} H^2 \quad (1.44)$$

Note that these quantities are in general functions of both position and time.

### 1.5 Time-Harmonic Electromagnetic Fields

Time-harmonic EM fields are those with a sinusoidal variation in time, for example,

$$\mathbf{E}(r, t) = E_0(r) \cos[\omega t + \phi(r)] \quad (1.45)$$

where the constant $\omega$ is the angular frequency. To simplify the treatment of time-harmonic fields, we can introduce the phasor notation $\mathbf{E}(\mathbf{r})$ of the corresponding instantaneous field $\mathbf{E}(\mathbf{r}, t)$. The relation between the instantaneous and phasor expressions is

$$\mathbf{E}(\mathbf{r}, t) = \Re\{\mathbf{E}(\mathbf{r})e^{j\omega t}\} \quad (1.46)$$

In general, the phasor expression $\mathbf{E}(\mathbf{r})$ is complex, even though the instantaneous expression $\mathbf{E}(\mathbf{r}, t)$ is always real.

#### 1.5.1 Maxwell’s Equations for Phasor EM Fields

With this relation between the instantaneous and phasor expressions of the field, the time derivative $\frac{\partial}{\partial t}$ on the instantaneous field corresponds to $j\omega$ times the phasor expression of the field. Therefore, one can derive Maxwell’s equations for the phasor form of electromagnetic fields as

$$\nabla \times \mathbf{E} = -\mathbf{M}_i - \sigma_m \mathbf{H} - j\omega \mu \mathbf{H} \quad (1.47)$$
$$\nabla \times \mathbf{H} = \mathbf{J}_i + \sigma \mathbf{E} + j\omega \varepsilon \mathbf{E} \quad (1.48)$$
$$\nabla \cdot \mathbf{E} = \rho_{ei} + \rho_{cc} \quad (1.49)$$
$$\nabla \cdot \mathbf{H} = \rho_{mi} + \rho_{me} \quad (1.50)$$

The continuity equations are

$$\nabla \cdot \sigma \mathbf{E} = -j\omega \rho_{ec}, \quad (1.51)$$
$$\nabla \cdot \sigma_m \mathbf{H} = -j\omega \rho_{mc}. \quad (1.52)$$
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The boundary conditions for the phasor form of electromagnetic fields remain the same as for the instantaneous form of EM fields. An additional approximate boundary condition which is often used for good (but not perfect) conductors is the impedance boundary condition. For example, when medium 1 is a good conductor (i.e., \( \sigma_1 \gg \omega \varepsilon_1 \)), then we can define the surface impedance

\[
Z_s = R_s + jX_s \approx (1 + j) \sqrt{\frac{\omega \mu_1}{2\sigma_1}}
\]  

(1.53)

The conduction current in medium 1 concentrates on a very thin layer (within the skin depth) on the surface. Therefore we can approximate the volume electric current as a surface electric current as

\[
J_s \approx \hat{n} \times \mathbf{H}_2
\]

(1.54)

Then the boundary condition for the electric field intensity is

\[
\mathbf{E}_{1t} = \mathbf{E}_{2t} \approx Z_s \mathbf{J}_s = Z_s \hat{n} \times \mathbf{H}_2 = \hat{n} \times \mathbf{H}_2 \sqrt{\frac{\omega \mu_1}{2\sigma_1}}(1 + j)
\]  

(1.55)

where the subscript t denotes the tangential component.

1.5.2 Power and Energy for Phasor EM Fields

Noting that the instantaneous expression for the time-harmonic field can be written as

\[
\mathbf{E}(r, t) = \Re \{ \mathbf{E}(r) e^{j\omega t} \} = \frac{1}{2} [\mathbf{E}(r) e^{j\omega t} + \mathbf{E}^*(r) e^{-j\omega t}]
\]

(1.56)

we can derive the Poynting vector as

\[
\mathbf{P}(r, t) = \frac{1}{2} [\Re \{ \mathbf{E}(r) \times \mathbf{H}^*(r) \} + \Re \{ \mathbf{E}(r) \times \mathbf{H}(r) e^{j2\omega t} \}]
\]

(1.57)

Therefore, the time-averaged Poynting vector is

\[
\mathbf{P}_{av}(r) = \mathbf{S} = \int_0^T \mathbf{P}(r, t) dt = \frac{1}{2} \Re \{ \mathbf{E}(r) \times \mathbf{H}^*(r) \}
\]

(1.58)

Similar to the derivation of Poynting’s theorem for instantaneous EM fields, starting from \( \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) \) we can derive the conservation of energy for phasor form of time-harmonic fields,

\[
-\frac{1}{2} \mathbf{H}^* \cdot \mathbf{M}_i - \frac{1}{2} \mathbf{E} \cdot \mathbf{J}_i^* = \nabla \cdot (\frac{1}{2} \mathbf{E} \times \mathbf{H}^*) + \frac{1}{2} \sigma |\mathbf{E}|^2 + \frac{1}{2} \sigma_m |\mathbf{H}|^2 + j2\omega (\frac{1}{4} \mu |\mathbf{H}|^2 - \frac{1}{4} \sigma |\mathbf{E}|^2)
\]

(1.59)

This is the conservation of energy in the differential form. In the integral form, it can be written as

\[
P_s = P_e + P_t + j2\omega (\mathbf{W}_m - \mathbf{W}_c)
\]

(1.60)
1.5. **TIME-HARMONIC ELECTROMAGNETIC FIELDS**

where

\[
P_s = -\frac{1}{2} \int_V (\mathbf{H}^* \cdot \mathbf{M}_t + \mathbf{E} \cdot \mathbf{J}_t^*) \, dV
\]

Supplied complex power (W)

\[
P_e = \oint_S \left( \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \right) \cdot ds
\]

Exiting complex power (W)

\[
P_d = \frac{1}{2} \int_V (|\mathbf{E}|^2 + \sigma_m |\mathbf{H}|^2) \, dV
\]

Dissipated real power (W)

\[
\overline{W}_e = \int_V \frac{\varepsilon}{2} E^2 \, dV
\]

Time-averaged stored electric energy (J)

\[
\overline{W}_m = \int_V \frac{\mu}{2} H^2 \, dV
\]

Time-averaged stored magnetic energy (J)

### 1.5.3 Complex Permittivity and Complex Permeability

If we define complex permittivity and complex permeability as

\[
\tilde{\varepsilon} = \varepsilon + \frac{\sigma}{j\omega},
\]

(1.61)

\[
\tilde{\mu} = \mu + \frac{\sigma_m}{j\omega},
\]

(1.62)

then by using (1.51) and (1.52), Maxwell’s equations (1.47)–(1.50) can be written as

\[
\nabla \times \mathbf{E} = -\mathbf{M}_t - j \omega \tilde{\mu} \mathbf{H}
\]

(1.63)

\[
\nabla \times \mathbf{H} = \mathbf{J}_t + j \omega \varepsilon \mathbf{E}
\]

(1.64)

\[
\nabla \cdot \varepsilon \mathbf{E} = \rho_{ei}
\]

(1.65)

\[
\nabla \cdot \tilde{\mu} \mathbf{H} = \rho_{mi}
\]

(1.66)

Then Maxwell’s equations are in the same form regardless the medium is conductive or nonconductive. This is convenient for many applications. In the remaining parts of this course, we will drop the tilde on \( \varepsilon \) and \( \mu \), keeping in mind that they can be in general complex.