Chapter 2

Wave Equation and Its Solutions

In this chapter (Chapter 3 in the textbook) we derive the wave equations for homogeneous, conductive medium, and find the general solutions of these equations for source-free regions. The formulation is derived for lossy medium as a general case, which can be simplified if the conductivity $\sigma = 0$.

2.1 Time-Varying Electromagnetic Fields

In this chapter we are concerned with homogeneous media. For a homogeneous medium, we have

$$\nabla \mu = 0, \quad \nabla \epsilon = 0, \quad \nabla \sigma = 0. \quad (2.1)$$

Maxwell’s equations for a homogeneous, conductive medium is

$$\nabla \times \mathbf{E} = -\mathbf{M}_i - \sigma_m \mathbf{H} - \mu \frac{\partial \mathbf{H}}{\partial t} \quad (2.2)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_i + \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (2.3)$$

$$\nabla \cdot \mathbf{E} = \rho_e \quad (2.4)$$

$$\nabla \cdot \mu \mathbf{H} = \rho_m \quad (2.5)$$

In the above, we assume nonzero electric conductivity $\sigma$ and magnetic conductivity $\sigma_m$. Note that the total free electric and magnetic charge densities are

$$\rho_e = \rho_{ei} + \rho_{ec}, \quad \rho_m = \rho_{mi} + \rho_{mc} \quad (2.6)$$

where $\rho_{ei}$ ($\rho_{mi}$) and $\rho_{ec}$ ($\rho_{mc}$) are the electric (magnetic) charge densities due to the imposed and conduction electric (magnetic) current, respectively. Therefore, for an inhomogeneous electrically (magnetically) conductive medium, normally
\( \rho_{ec} \neq 0 \) (\( \rho_{mc} \neq 0 \)). However, for a homogeneous electrically (magnetically) conductive medium, \( \rho_{ec} = 0 \) (\( \rho_{mc} = 0 \)).

Taking the curl of (2.2) and making use of (2.3) and (2.4), we arrive at

\[
\nabla^2 \mathbf{E} - \sigma_m \mathbf{E} - (\mu \sigma + \varepsilon \sigma_m) \frac{\partial \mathbf{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{S}_e (r, t) \tag{2.7}
\]

Similarly,

\[
\nabla^2 \mathbf{H} - \sigma_m \mathbf{E} - (\mu \sigma + \varepsilon \sigma_m) \frac{\partial \mathbf{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = \mathbf{S}_m (r, t) \tag{2.8}
\]

These are the uncoupled, second-order PDE’s for the electric and magnetic fields. They are the vector wave equations. In the above, the source terms are given by

\[
\begin{align*}
\mathbf{S}_e (r, t) &= \nabla \times \mathbf{M}_i + \sigma_m \mathbf{J}_i + \mu \varepsilon \nabla \rho_e - \frac{\nabla \rho_e}{\varepsilon}, \\
\mathbf{S}_m (r, t) &= -\nabla \times \mathbf{J}_i + \sigma_m \mathbf{M}_i + \mu \varepsilon \nabla \rho_m - \frac{\nabla \rho_m}{\mu}.
\end{align*}
\]

Note that these two wave equations (2.7) and (2.8) are symmetric, i.e., \( \mathbf{E} \) and \( \mathbf{H} \) satisfy the duality. This duality principle will be discussed later.

### 2.1.1 Special case: A source-free region

For a source-free region, the vector wave equations reduce to the homogeneous vector Helmholtz equations

\[
\begin{align*}
\nabla^2 \mathbf{E} - \sigma_m \mathbf{E} - (\mu \sigma + \varepsilon \sigma_m) \frac{\partial \mathbf{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \tag{2.9} \\
\nabla^2 \mathbf{H} - \sigma_m \mathbf{E} - (\mu \sigma + \varepsilon \sigma_m) \frac{\partial \mathbf{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} &= 0 \tag{2.10}
\end{align*}
\]

### 2.1.2 Special case: A source-free nonconductive medium

For a nonconductive medium, \( \sigma = \sigma_m = 0 \). The wave equations are

\[
\begin{align*}
\nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \tag{2.11} \\
\nabla^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} &= 0 \tag{2.12}
\end{align*}
\]

In principle, these equations can be solved by either analytical or numerical methods.
2.2 Time-Harmonic Electromagnetic Fields

For time-harmonic fields, we use the phasor notations. Then, by simply replacing $\partial/\partial t$ by $j\omega$, we can derive the wave equations in frequency domain. However, before doing that, let’s introduce the complex permittivity to simplify Maxwell’s equations. For a conductive medium, if we define a complex permittivity and a complex permeability

\[
\hat{\varepsilon} = \varepsilon + \frac{\sigma}{j\omega}, \quad \hat{\mu} = \mu + \frac{\sigma_m}{j\omega},
\]

(2.13)

and note that the charge density due to the conduction current can be obtained by the continuity condition

\[
\rho_{ec} = -\frac{1}{j\omega} \nabla \cdot \sigma \mathbf{E},
\]

(2.14)

we can rewrite Maxwell’s equations as

\[
\nabla \times \mathbf{E} = -\mathbf{M}_i - j\omega \hat{\mu} \mathbf{H}
\]

(2.15)

\[
\nabla \times \mathbf{H} = \mathbf{J}_i - j\omega \hat{\varepsilon} \mathbf{E}
\]

(2.16)

\[
\nabla \cdot \hat{\varepsilon} \mathbf{E} = \rho_{ei}
\]

(2.17)

\[
\nabla \cdot \hat{\mu} \mathbf{H} = \rho_{mi}
\]

(2.18)

From Maxwell’s equations, we can derive the wave equations

\[
\nabla^2 \mathbf{E} + k^2 \mathbf{E} = \mathbf{S}_e
\]

(2.19)

\[
\nabla^2 \mathbf{H} + k^2 \mathbf{H} = \mathbf{S}_m
\]

(2.20)

where the complex wavenumber $k$ is

\[
k^2 = \omega^2 \hat{\varepsilon} \hat{\mu} = \omega^2 (\mu + \frac{\sigma_m}{j\omega})(\varepsilon + \frac{\sigma}{j\omega})
\]

(2.21)

and the source terms

\[
\mathbf{S}_e = \nabla \times \mathbf{M}_i + j\omega \hat{\mu} \mathbf{J}_i + \frac{\nabla \rho_{ei}}{\varepsilon}
\]

(2.22)

\[
\mathbf{S}_m = -\nabla \times \mathbf{J}_i - j\omega \hat{\varepsilon} \mathbf{M}_i + \frac{\nabla \rho_{mi}}{\hat{\mu}}
\]

(2.23)

Note now the equations for $\mathbf{E}$ and for $\mathbf{H}$ are symmetric after the complex permittivity is introduced. In the future, the tilde on $\hat{\varepsilon}$ and $\hat{\mu}$ will be neglected, keeping in mind that $\varepsilon$ and $\mu$ are in general complex.

2.3 Solution to the Wave Equation

Solutions to wave equations in Cartesian, cylindrical, and spherical coordinate systems are given in this section.
2.3.1 Cartesian Coordinate System

In Cartesian coordinates, \( \mathbf{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z \), and the Laplacian \( \nabla^2 \) operator commutes with the unit vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \). Therefore, for a source-free region, equation (2.19) becomes

\[
\nabla^2 E_x + k^2 E_x = 0
\]

(2.24)

for \( E_x \), and similarly for other components of \( \mathbf{E} \) and \( \mathbf{H} \). We seek the separable form of the solution

\[
E_x = f(x)g(y)h(z)
\]

(2.25)

After substitution into (2.24) we obtain

\[
\frac{d^2 f}{dx^2} + k_x^2 f = 0
\]

(2.26)

\[
\frac{d^2 g}{dy^2} + k_y^2 g = 0
\]

(2.27)

\[
\frac{d^2 h}{dz^2} + k_z^2 h = 0
\]

(2.28)

and

\[
k_x^2 + k_y^2 + k_z^2 = k^2.
\]

(2.29)

The general solutions for (2.26)–(2.28) can be written in the forms of traveling waves

\[
f(x) = A_1 e^{-j k_x x} + B_1 e^{j k_x x}
\]

(2.30)

\[
g(y) = A_2 e^{-j k_y y} + B_2 e^{j k_y y}
\]

(2.31)

\[
h(z) = A_3 e^{-j k_z z} + B_3 e^{j k_z z}
\]

(2.32)

or in the forms of standing waves

\[
f(x) = C_1 \cos(k_x x) + D_1 \sin(k_x x)
\]

(2.33)

\[
g(y) = C_2 \cos(k_y y) + D_2 \sin(k_y y)
\]

(2.34)

\[
h(z) = C_3 \cos(k_z z) + D_3 \sin(k_z z)
\]

(2.35)

The constants \( A_i \) and \( B_i \) are determined by the appropriate boundary conditions.

Note that for a general lossy medium, \( \sigma \neq 0 \) or \( \sigma_m \neq 0 \), and hence \( k, k_x, k_y, \) and \( k_z \) are complex in general

\[
k \equiv -j \gamma = \beta - j \alpha, \quad k_i \equiv -j \gamma_i = \beta_i - j \alpha_i, \quad i = (x, y, z)
\]

(2.36)

where both \( \alpha \) and \( \beta \) are real. In the above, \( k \) is known as the wavenumber, and \( \gamma \) as the propagation constant, both being complex in general. As the solution consists of two parts as in (2.30), we can always choose real parts of \( k_i \) (i.e., \( \beta_i \)) to be positive and determine imaginary parts (i.e., \( \alpha_i \)) from the physical condition
of the solution. We note that there are two possibilities after $\Re(k_i) = \beta_i \geq 0$
is fixed:
\[
\begin{align*}
\alpha_i & \geq 0 \\
+\alpha_i & \leq 0
\end{align*}
\]  
(2.37)
To determine the appropriate sign, we examine the first term of (2.30)
\[
f^+(x) = A_1 e^{-\alpha x} e^{-j\beta x} = A_1 e^{-\alpha x} e^{-j\beta x}.
\]  
(2.38)
Assuming $A_1$ is real, the instantaneous expression of (2.38) is
\[
f^+(x, t) = A_1 e^{-\alpha x} \cos(\omega t - \beta x),
\]  
(2.39)
which represents a wave traveling in $+x$ direction because $\beta_i \geq 0$. If the medium
is lossy ($\sigma \geq 0$), we expect the solution to decay in the $+x$ direction; and if the
medium is active ($\sigma < 0$), we expect the solution to grow in the $+x$ direction.
Therefore,
\[
\Re(k_i) = \beta_i \geq 0, \quad \exists \arg \{k_i\} = \begin{cases} 
-\alpha_i & \leq 0 \text{ for } \sigma \geq 0 \\
+\alpha_i & \geq 0 \text{ for } \sigma > 0
\end{cases}
\]  
(2.40)
We can also introduce the propagation constants
\[
\gamma = jk = \begin{cases} 
+\left(\alpha + j\beta\right) & \text{for } \sigma \geq 0 \\
-\left(\alpha + j\beta\right) & \text{for } \sigma < 0
\end{cases}, \quad \gamma_i = jk_i = \begin{cases} 
+\left(\alpha_i + j\beta_i\right), & \text{for } \sigma \geq 0 \\
-\left(\alpha_i + j\beta_i\right), & \text{for } \sigma < 0
\end{cases}
\]  
(2.41)
In the above, $\beta_i$ is called the phase constant, and $\alpha_i$ the attenuation constant.
Then the general solutions can be written as
\[
f(x) = A_1 e^{-\gamma x} + B_1 e^{\gamma x}
\]  
(2.42)
\[
g(y) = A_2 e^{-\gamma y} + B_2 e^{\gamma y}
\]  
(2.43)
\[
h(z) = A_3 e^{-\gamma z} + B_3 e^{\gamma z}
\]  
(2.44)
or in the forms of standing waves
\[
f(x) = C_1 \cosh(\gamma x) + D_1 \sinh(\gamma x)
\]  
(2.45)
\[
g(y) = C_2 \cosh(\gamma y) + D_2 \sinh(\gamma y)
\]  
(2.46)
\[
h(z) = C_3 \cosh(\gamma z) + D_3 \sinh(\gamma z)
\]  
(2.47)
The general solution for $E_x$, $E_y$, and $E_z$ can be written in the form of (2.25).

2.3.2 Cylindrical Coordinate System

A. Derivation of coupled equations in cylindrical coordinates
We consider a conductive, homogeneous, source-free region. In the cylindrical
coordinate system,
\[
\mathbf{E}(\rho, \phi, z) = \hat{\rho} E_{\rho} + \hat{\phi} E_{\phi} + \hat{z} E_z
\]  
(2.48)
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The basic operations in cylindrical coordinates can be summarized as

\[
\nabla \times \mathbf{E} = \hat{\rho} \left( \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right) + \hat{\phi} \left( \frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z} \right) + \hat{z} \left( \frac{1}{\rho} \frac{\partial (\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right)
\]

(2.49)

\[
\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}
\]

(2.50)

\[
\nabla \cdot \mathbf{E} = \frac{1}{\rho} \frac{\partial (\rho E_\rho)}{\partial \rho} + \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}
\]

(2.51)

The vector wave equation

\[
\nabla^2 \mathbf{E} = -k^2 \mathbf{E}
\]

(2.52)

applies to cylindrical coordinates. However, the Laplacian operator \( \nabla^2 \) does not commute with unit vectors \( \hat{\rho} \) and \( \hat{\phi} \), as it does with \( \hat{z} \). Therefore, except for the \( z \) components \( E_z \) and \( H_z \), all the field components are coupled.

For the \( z \) components, since \( \nabla^2 \hat{z} = \hat{z} \nabla^2 \), we obtain

\[
\nabla^2 E_z + k^2 E_z = 0
\]

(2.53)

\[
\nabla^2 H_z + k^2 H_z = 0.
\]

(2.54)

Or more specifically

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = 0
\]

(2.55)

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial H_z}{\partial \rho} \right) + \frac{\partial^2 H_z}{\partial \phi^2} + k^2 H_z = 0
\]

(2.56)

The equations for \( \rho \) and \( \phi \) components can be derived in two ways. The first way is to rewrite, for a homogeneous medium,

\[
\nabla^2 \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} = -\nabla \times \nabla \times \mathbf{E}
\]

(2.57)

and make use of \( \nabla \cdot \mathbf{E} = 0 \), or

\[
\frac{1}{\rho} \frac{\partial (\rho E_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = 0
\]

(2.58)

We first find the \( \rho \) component of (2.57)

\[
\left( \nabla \times \nabla \times \mathbf{E} \right)_\rho = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \frac{\partial (\rho E_\rho)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right] - \frac{1}{\rho} \frac{\partial (\rho E_\phi)}{\partial z} E_\rho - \frac{1}{\rho} \frac{\partial (\rho E_z)}{\partial \rho} E_\rho
\]

(2.59)

\[
= \frac{1}{\rho} \frac{\partial^2 E_\rho}{\partial \rho^2} - \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial}{\partial \rho} \left[ \frac{\partial E_\phi}{\partial \rho} + \frac{\partial E_\phi}{\partial \phi} \right]
\]

(2.60)

\[
= \frac{1}{\rho} \frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial E_\phi}{\partial \rho^2} + \frac{\partial E_\phi}{\partial \phi^2} - \frac{\partial^2 E_\phi}{\partial \rho \partial \phi}
\]

(2.61)

\[
= -\nabla^2 E_\rho + \frac{E_\phi}{\rho^2} + \frac{E_\phi}{\rho^2}
\]
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Then, the ρ component of (2.52) can be derived as

\[
\nabla^2 E_\rho - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} = -k^2 E_\rho
\]

(2.59)

Similarly,

\[
\nabla^2 E_\phi - \frac{E_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} = -k^2 E_\phi
\]

(2.60)

\[
\nabla^2 H_\rho - \frac{H_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial H_\phi}{\partial \phi} = -k^2 H_\rho
\]

(2.61)

\[
\nabla^2 H_\phi - \frac{H_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial H_\rho}{\partial \phi} = -k^2 H_\phi
\]

(2.62)

The other way of deriving these equations is to recognize that

\[
\frac{\partial \rho}{\partial \rho} = \frac{\partial \phi}{\partial \phi} = \frac{\partial \hat{z}}{\partial \rho} = 0 = \frac{\partial \hat{z}}{\partial \phi} = \frac{\partial \hat{\rho}}{\partial z} = \frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{z}}{\partial z}
\]

(2.63)

\[
\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}
\]

(2.64)

These equations can be substituted into \( \nabla^2 \mathbf{E} = -k^2 \mathbf{E} \) to obtain the above partial differential equations.

In short, equations for the ρ and φ components are coupled, while those for the z components are uncoupled.

**B. Solutions of scalar Helmholtz equation in cylindrical coordinates**

The solutions to the coupled PDE’s for the ρ and φ field components will be postponed. Here we search the solution to the scalar Helmholtz equation

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0
\]

(2.65)

Assuming a separable solution for \( \psi \) of the form

\[
\psi(\rho, \phi, z) = f(\rho)g(\phi)h(z),
\]

(2.66)

we obtain the three differential equations

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{df}{d\rho} \right) + \left( k^2 - \frac{m^2}{\rho^2} \right) f = 0
\]

(2.67)

\[
\frac{d^2 g}{d\phi^2} + m^2 g = 0
\]

(2.68)

\[
\frac{d^2 h}{dz^2} + k_z^2 h = 0
\]

(2.69)

and the dispersion relation

\[
k_z^2 + k_\rho^2 = k^2
\]

(2.70)
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Equation (2.67) is the Bessel differential equation,

\[ \frac{1}{x} \frac{d}{dx} \left( x \frac{df}{dx} \right) + \left( 1 - \frac{m^2}{x^2} \right) f = 0 \]  

(2.67a)

The general solutions of (2.67)–(2.69) are written in the forms of traveling waves

\[ f(\rho) = A_1 H_m^{(2)}(k_\rho \rho) + B_1 H_m^{(1)}(k_\rho \rho) \]  

(2.71)
\[ g(\phi) = A_2 e^{-jm\phi} + B_2 e^{jm\phi} \]  

(2.72)
\[ h(z) = A_3 e^{-jk_z z} + B_3 e^{jk_z z} \]  

(2.73)

or in the forms of standing waves

\[ f(\rho) = C_1 J_m(k_\rho \rho) + D_1 Y_m(k_\rho \rho) \]  

(2.74)
\[ g(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi) \]  

(2.75)
\[ h(z) = C_3 \csc(k_z z) + D_3 \sin(k_z z) \]  

(2.76)

The signs of the real and imaginary parts of \( k_\rho = k_\rho' + jk_\rho'' \) and \( k_z = k_z' + jk_z'' \) are similar to (2.40) and (2.41)

\[ k_\rho' = +\beta_i \geq 0, \quad k_\rho'' = \begin{cases} -\alpha_i \leq 0 & \text{for } \sigma \geq 0 \\ +\alpha_i \geq 0 & \text{for } \sigma > 0 \end{cases} \]  

(2.77)

\[ \gamma = jk = \begin{cases} + (\alpha + j\beta) & \text{for } \sigma \geq 0 \\ - (\alpha + j\beta) & \text{for } \sigma < 0 \end{cases} \quad , \quad \gamma_i = jk_i = \begin{cases} + (\alpha_i + j\beta_i), & \text{for } \sigma \geq 0 \\ - (\alpha_i + j\beta_i), & \text{for } \sigma < 0 \end{cases} \]  

(2.78)

where \( i = (\rho, z) \).

The general solution for \( \psi(\rho, \phi, z) \), therefore, is

\[ \psi(\rho, \phi, z) = [C_1 J_m(k_\rho \rho) + D_1 Y_m(k_\rho \rho)][C_2 \cos(m\phi) + D_2 \sin(m\phi)] \left[ A_3 e^{-jk_z z} + B_3 e^{jk_z z} \right] \]  

(2.79)

Note that \( J_m \) and \( Y_m \) represent standing waves, while \( H_m^{(1)}(k_\rho \rho) \) and \( H_m^{(2)}(k_\rho \rho) \) represent the traveling waves in \( \rho \) direction. \( m \) is usually integer so that the solution satisfy the periodicity in the \( \phi \) direction. The constants \( A_i, B_i, C_i \), and \( D_i \) are determined by the appropriate boundary conditions.

C. Asymptotic behaviors of Bessel and Hankel functions

The relations between Bessel functions and Hankel functions are

\[ H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + jY_m(k_\rho \rho) \]  

(2.80)
\[ H_m^{(2)}(k_\rho \rho) = J_m(k_\rho \rho) - jY_m(k_\rho \rho) \]  

(2.81)
\[ J_m(k_\rho \rho) = \frac{1}{2} [H_m^{(1)}(k_\rho \rho) + H_m^{(2)}(k_\rho \rho)] \]  

(2.82)
\[ Y_m(k_\rho \rho) = \frac{1}{2j} [H_m^{(1)}(k_\rho \rho) - H_m^{(2)}(k_\rho \rho)] \]  

(2.83)
2.3. SOLUTION TO THE WAVE EQUATION

As $\rho$ approaches infinity, one would expect that the cylindrical waves approach the waves in Cartesian coordinates. Indeed, the asymptotic behaviors of the cylindrical wave functions for $k_\rho \rho \to \infty$ are:

$$J_m(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \cos(k_\rho \rho - \frac{m\pi}{2} - \frac{\pi}{4})$$

$$Y_m(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \sin(k_\rho \rho - \frac{m\pi}{2} - \frac{\pi}{4})$$

$$H_m^{(1)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} e^{j(k_\rho \rho - \frac{m\pi}{2} - \frac{\pi}{4})}$$

$$H_m^{(2)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} e^{-j(k_\rho \rho - \frac{m\pi}{2} - \frac{\pi}{4})}$$

Therefore, $H_m^{(2)}(k_\rho \rho)$ and $H_m^{(1)}(k_\rho \rho)$ represent waves traveling in $+\rho$ and $-\rho$ directions, respectively; while $J_m(k_\rho \rho)$ and $Y_m(k_\rho \rho)$ represent standing waves.

The asymptotic behaviors of cylindrical waves as $k_\rho \rho \to 0$ are different for $m = 0$ and for $m > 0$. For $m = 0$ as $k_\rho \rho \to 0$,

$$J_0(k_\rho \rho) \sim 1$$

$$Y_0(k_\rho \rho) \sim \frac{2}{\pi} \ln(\gamma k_\rho \rho / 2)$$

$$H_0^{(1)} \sim \frac{2j}{\pi} \ln(\gamma k_\rho \rho / 2)$$

$$H_0^{(2)} \sim -\frac{2j}{\pi} \ln(\gamma k_\rho \rho / 2)$$

However, for $m > 0$ as $k_\rho \rho \to 0$,

$$J_m(k_\rho \rho) \sim \frac{(k_\rho \rho / 2)^m}{m!}$$

$$Y_m(k_\rho \rho) \sim -\frac{(m - 1)!}{\pi} \left(\frac{2}{k_\rho \rho}\right)^m$$

$$H_m^{(1)} \sim \frac{j(m - 1)!}{\pi} e^{\frac{2}{k_\rho \rho}}$$

$$H_m^{(1)} \sim \frac{j(m - 1)!}{\pi} \left(\frac{2}{k_\rho \rho}\right)^m$$

These asymptotic behaviors are important to determine the constants $C_1$ and $D_1$. For example, if $\rho = 0$ is in the domain of interest, $D_1$ has to be zero because $Y_m(k_\rho \rho)$ is singular at $\rho = 0$. 
Note that the above expressions are the asymptotic behaviors of the cylindrical harmonics. For some applications, the forms including the additive constants are needed. For example, as \( k \rho \to 0 \),

\[
Y_0(k \rho) \approx \frac{2}{\pi} \ln(k \rho + \ln(\gamma/2)) = \frac{2}{\pi} \ln \left( \frac{\gamma k \rho}{2} \right) 
\] (2.89a)

where \( \gamma \approx 1.781 \) is the Euler constant. Equation (2.89a) is often used to evaluate the singular integral in the method of moment.

### 2.3.3 Spherical Coordinate System

#### A. Coupled Equations in Spherical Coordinates

In spherical coordinates \((r, \theta, \phi)\),

\[
\mathbf{E}(r, \theta, \phi) = \hat{r}E_r + \hat{\theta}E_\theta + \hat{\phi}E_\phi
\]

The basic operations are

\[
\nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 E_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \sin \theta E_\theta + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi}
\] (2.96)

\[
\nabla \times \mathbf{E} = - \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \sin \theta E_\phi - \frac{1}{r} \frac{\partial E_\theta}{\partial \phi} \right] \hat{\varphi} + \frac{1}{r} \left[ \frac{\partial r E_\phi}{\partial \theta} - \frac{\partial E_\phi}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \sin \theta E_r \right) - \frac{\partial E_r}{\partial \phi} \right] \hat{\varphi}
\] (2.97)

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\] (2.98)

By using the fact that

\[
\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\theta}}{\partial \theta} = \frac{\partial \hat{\phi}}{\partial \phi} = 0
\] (2.99)

\[
\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = 0
\] (2.100)

\[
\frac{\partial \hat{r}}{\partial \phi} = \sin \theta \hat{\phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \csc \theta \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\sin \theta \hat{r} - \cos \theta \hat{\theta}
\] (2.101)

one can derive from the vector wave equation \( \nabla^2 \mathbf{E} = -k^2 \mathbf{E} \) the following scalar equations

\[
\nabla^2 E_r - \frac{2}{r^2} \left( E_r + E_\theta \cot \theta + \csc \theta \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_\theta}{\partial \theta} \right) = -k^2 E_r
\] (2.102)

\[
\nabla^2 E_\theta - \frac{1}{r^2} \left( E_\theta \csc^2 \theta + 2 \frac{\partial E_r}{\partial \theta} + 2 \cot \theta \csc \theta \frac{\partial E_\phi}{\partial \phi} \right) = -k^2 E_\theta
\] (2.103)
2.3. **SOLUTION TO THE WAVE EQUATION**

\[
\nabla^2 E_\phi - \frac{1}{r^2} \left( E_\phi \csc^2 \theta - 2 \csc \theta \frac{\partial E_\phi}{\partial \phi} - 2 \cot \theta \csc \theta \frac{\partial^2 E_\phi}{\partial \phi^2} \right) = -k^2 E_\phi
\]  

(2.104)

Note all three components are coupled, and equations (2.102)–(2.104) have to be solved simultaneously. This is not an easy task.

**B. Solution of Scalar Wave Equation in Spherical Coordinates**

Alternatively, one can decomposed the fields into TE and TM components as in Chapter 10. This decomposition will result in the following scalar wave equation

\[
\nabla^2 \psi(r, \theta, \phi) = -k^2 \psi(r, \theta, \phi)
\]  

(2.105)

which will be solved here. Using the separation of variables,

\[
\psi(r, \theta, \phi) = f(r)g(\theta)h(\phi)
\]  

(2.106)

we can derive the following three ordinary differential equations

\[
\frac{d}{dr} r^2 \frac{df}{dr} + [(kr)^2 - n(n + 1)] f = 0
\]  

(2.107)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{dg}{d\theta} + [n(n + 1) - \left( \frac{m}{\sin \theta} \right)^2] g = 0
\]  

(2.108)

\[
\frac{dh}{d\phi} + m^2 h = 0
\]  

(2.109)

Equation (2.107) is called the spherical Bessel differential equation, and has the general solution of

\[
f(r) = A_i h_n^{[2]}(kr) + B_i h_n^{[1]}(kr)
\]  

(2.110)

or

\[
f(r) = C_i j_n(kr) + D_i y_n(kr)
\]  

(2.111)

where \( j_n(kr) \), \( y_n(kr) \) are the spherical Bessel functions of the first and second kind, and \( h_n^{[1]}(kr) \) and \( h_n^{[2]}(kr) \) are the spherical Hankel functions of the first and second kind. The spherical harmonic \( b_n(kr) \) is related to the corresponding cylindrical harmonic \( B_{n+1/2}(kr) \) by

\[
b_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+1/2}(kr)
\]  

(2.112)

Another form of the spherical harmonics is the Schelkunoff type, defined as

\[
\hat{B}_n(kr) = kr b_n(kr) = \sqrt{\frac{\pi kr}{2}} B_{n+1/2}(kr)
\]  

(2.113)

and satisfies the differential equation

\[
\frac{d^2 \hat{B}_n(kr)}{dr^2} + \left[ k^2 - \frac{n(n + 1)}{r^2} \right] \hat{B}_n(kr) = 0
\]  

(2.114)
CHAPTER 2. WAVE EQUATION AND ITS SOLUTIONS

Equation (2.108) is known as the Legendre differential equation and has the general solution

\[ g(\theta) = A_2 P_n^m(\cos \theta) + B_2 P_n^m(-\cos \theta), \quad n \neq \text{integer} \quad (2.115) \]

or

\[ g(\theta) = C_2 P_n^m(\cos \theta) + B_2 Q_n^m(\cos \theta), \quad n = \text{integer} \quad (2.116) \]

where \( P_n^m(\cos \theta) \) and \( Q_n^m(\cos \theta) \) are the associated Legendre functions of the first and second kind. Note that \( Q_n^m(\cos \theta) \) is singular at \( \theta = 0 \) and \( \theta = \pi \).

C. Asymptotic behaviors of spherical Bessel and Hankel functions

The asymptotic behaviors of the spherical harmonics can be easily derived from those of the cylindrical harmonics, and thus will not be given here.

D. Some examples of spherical harmonics

We list a few spherical harmonics here for convenience:

\[ h_0^{(1)}(kr) = \frac{e^{jkr}}{jkr}, \quad h_1^{(1)}(kr) = -(1 + \frac{j}{kr}) \frac{e^{jkr}}{k^r} \quad (2.117) \]

\[ h_0^{(2)}(kr) = -\frac{e^{-jkr}}{jkr}, \quad h_1^{(2)}(kr) = -(1 - \frac{j}{kr}) \frac{e^{-jkr}}{k^r} \quad (2.118) \]

\[ j_0(kr) = \frac{\sin kr}{kr}, \quad j_1(kr) = -\frac{\cos kr}{kr} + \frac{\sin kr}{(kr)^2} \quad (2.119) \]

\[ y_0(kr) = -\frac{\cos kr}{kr}, \quad y_1(kr) = -\frac{\sin kr}{kr} - \frac{\cos kr}{(kr)^2} \quad (2.120) \]

Note that although cylindrical harmonics are special functions, the spherical harmonics are some elementary functions for integer \( n \).