Chapter 5

REFLECTION AND TRANSMISSION

This chapter (Chapter 5 in textbook) discusses the reflection and transmission of plane waves at planar interfaces between different media.

We first consider the general case for lossy media by noting that all reflection and refraction angles can be complex. For the special case of lossless media, these angles become real except for the post-critical incidence where the angle of refraction is complex.

Furthermore, in contrast to the textbook, we consider the general oblique incidence. The normal incidence is just a trivial special case. We first define several terms in the following. For convenience, the normal of the planar interface between medium 1 and medium 2 is \( \hat{n} = \pm \hat{z} \).

**Plane of incidence (POI).** The plane formed by the direction of incidence \( \hat{k}_i \) and the normal direction of the interface \( \hat{n} \). We can always choose this to be the \( xz \) plane.

**Angles of incidence, reflection, and transmission.** The angles of incidence and reflection (\( \theta_i \) and \( \theta_r \)) are defined as the angles formed by the incidence direction \( \hat{k}_i \) and \( \hat{n}_{21} = -\hat{z} \), and by the reflection direction \( \hat{k}_r \) and \( \hat{n}_{21} = -\hat{z} \), respectively. Similarly, the angle of transmission (\( \theta_t \)) is that formed by the transmission direction \( \hat{k}_t \) and the normal \( \hat{n}_{12} = \hat{z} \). Written in terms of these angles, the unit vectors in the directions propagation of the incident, reflected, and refracted (transmitted) waves are

\[
\hat{k}_i = \hat{x} \sin \theta_i + \hat{z} \cos \theta_i, \quad \hat{k}_r = \hat{x} \sin \theta_r - \hat{z} \cos \theta_r, \quad \hat{k}_t = \hat{x} \sin \theta_t + \hat{z} \cos \theta_t. \quad (5.1a)
\]

If we define the wavenumber vector \( \mathbf{k}_i = k \hat{k} = \hat{x} k_{ix} + \hat{y} k_{iy} + \hat{z} k_{iz} \), then we have

\[
k_{ix} = k \sin \theta_i, \quad k_{iy} = 0, \quad k_{iz} = k \cos \theta_i, \quad (5.1b)
\]

and similarly for the reflected and transmitted waves.

**Polarization of the incident wave.** Two types of polarizations are possible with respect to the POI.
(a). **Perpendicular (horizontal, \( E \), or \( \text{TM}_y \)) polarization:** the \( E \) field of the incident wave is perpendicular to the POI.

(b). **Parallel (vertical, \( H \), or \( \text{TE}_y \)) polarization:** the \( E \) field of the incident wave is parallel to the POI.

Note that any plane wave can be written as the sum of these two polarizations: \( \mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel \), as shown in Chapter 4. Since there is no coupling between these two components at a planar interface, we can treat them separately.

## 5.1 Oblique Incidence at a Single Interface

We first consider a single interface at \( z = 0 \) separating two conductive media with \((\mu_1, \varepsilon_1)\) and \((\mu_2, \varepsilon_2)\). The complex wavenumber is

\[
k_m = \omega \sqrt{\mu_m \varepsilon_m}, \quad m = 1, 2
\]

where both \( \mu \) and \( \varepsilon \) can be complex.

### 5.1.1 Perpendicular (Horizontal, or \( E \), or \( \text{TM}_y \)) Polarization

For the perpendicular polarization, the incident electric and magnetic fields can be written as

\[
\mathbf{E}_i = \hat{y}E_{0i}e^{-jk_i \hat{k}_i \cdot \mathbf{r}} = \hat{y}E_{0i}e^{-jk_1(x \sin \theta_i + z \cos \theta_i)}
\]

\[
\mathbf{H}_i = \frac{1}{\eta_i} \hat{k}_i \times \mathbf{E}_i = (\hat{x} \cos \theta_i + \hat{z} \sin \theta_i) \frac{E_{0i}}{\eta_i} e^{-jk_1(x \sin \theta_i + z \cos \theta_i)}
\]

Similarly for the reflected and transmitted fields

\[
\mathbf{E}_r = \hat{y}E_{0r}e^{-jk_r \hat{k}_r \cdot \mathbf{r}} = \hat{y}E_{0r}e^{-jk_2(x \sin \theta_r - z \cos \theta_r)}
\]

\[
\mathbf{H}_r = \frac{1}{\eta_r} \hat{k}_r \times \mathbf{E}_r = (\hat{x} \cos \theta_r + \hat{z} \sin \theta_r) \frac{E_{0r}}{\eta_r} e^{-jk_2(x \sin \theta_r - z \cos \theta_r)}
\]

\[
\mathbf{E}_t = \hat{y}E_{0t}e^{-jk_t \hat{k}_t \cdot \mathbf{r}} = \hat{y}E_{0t}e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}
\]

\[
\mathbf{H}_t = \frac{1}{\eta_t} \hat{k}_t \times \mathbf{E}_t = (\hat{x} \cos \theta_t + \hat{z} \sin \theta_t) \frac{E_{0t}}{\eta_t} e^{-jk_2(x \sin \theta_t + z \cos \theta_t)}
\]

Again, in general these angles \((\theta_i, \theta_r, \theta_t)\) should be considered complex unless determined otherwise.

The boundary conditions at \( z = 0 \) require \( \hat{n}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \) and \( \hat{n}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0 \). Therefore,

\[
E_{0i} e^{-jk_1 x \sin \theta_i} + E_{0r} e^{-jk_2 x \sin \theta_r} = E_{0t} e^{-jk_2 x \sin \theta_t}
\]

(5.9)
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\[-\frac{E_{r0}}{\eta_1} \cos \theta_r e^{-jk_1 x \sin \theta_1} + \frac{E_{t0}}{\eta_1} \cos \theta_t e^{-jk_1 x \sin \theta_1} = -\frac{E_{t0}}{\eta_2} \cos \theta_t e^{-jk_2 x \sin \theta_2}\]

(5.10)

**Snell’s laws.** It is possible for (5.9) and (5.10) hold for all \(x\) if all the phase terms are equal. That is

\[k_1 \sin \theta_t = k_1 \sin \theta_r, \quad k_1 \sin \theta_t = k_2 \sin \theta_t\]

(5.11)

These are called the **phase matching conditions.** Equivalently, we have

\[\theta_r = \theta_i\]

(5.12)

and

\[\theta_t = \sin^{-1} \left( \frac{k_1}{k_2} \sin \theta_i \right)\]

(5.13)

Equations (5.12) and (5.13) are called the Snell’s laws of reflection and refraction, respectively. Note that for a real incidence angle \(\theta_i\), the angle of reflection is also real, but the angle of refraction \(\theta_t\) is in general complex if either medium 1 or medium 2 is conductive.

Using these Snell’s laws, (5.9) and (5.10) can be simplified as

\[E_{\theta 0} + E_{r0} = E_{t0}\]

\[-\left( \frac{E_{r0}}{\eta_1} + \frac{E_{t0}}{\eta_1} \right) \cos \theta_i = -\frac{E_{t0}}{\eta_2} \cos \theta_t\]

which leads to

\[R_\perp \equiv \frac{E_{r0}}{E_{\theta 0}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}\]

(5.14)

\[T_\perp \equiv \frac{E_{t0}}{E_{\theta 0}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}\]

(5.15)

The coefficients \(R_\perp\) and \(T_\perp\) are called the reflection and transmission coefficients for the perpendicular polarization. The total field in medium 1 and medium 2 are thus

\[E_1 = \hat{y} \hat{E}_{00} e^{-jk_1 (x \sin \theta_i + z \cos \theta_i)} [1 + R_\perp e^{i2k_1 z \cos \theta_i}]\]

(5.16)

\[E_2 = \hat{y} \hat{E}_{00} T_\perp e^{-jk_2 (x \sin \theta_i + z \cos \theta_i)}\]

(5.17)

Given the parameters for the incident field \((\theta_i, E_{\theta 0})\), the reflected and transmitted fields can be obtained from (5.16) and (5.17). The magnetic fields can also be obtained from (5.6) and (5.8).

For the convenience of the discussions on multiple interfaces, we now introduce the following components of \(k_m\) \((m = 1, 2)\) for the two media:

\[k_x = k_1 \sin \theta_i = k_2 \sin \theta_t, \quad (k_x = k_{1x} = k_{2x}),\]

(5.18)

\[k_{1z} = k_1 \cos \theta_i = \sqrt{k_1^2 - k_x^2}, \quad k_{2z} = k_2 \cos \theta_t = \sqrt{k_2^2 - k_x^2}\]

(5.19)
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Then the reflection and transmission coefficients in (5.14) and (5.15) can be rewritten as

\[
R_{\perp} = \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \tag{5.20}
\]

\[
T_{\perp} = \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \tag{5.21}
\]

And the electric fields in (5.16) and (5.17) can be written as

\[
E_1 = \hat{y} E_0 e^{-jk_{1}x} [e^{-jk_{1z}z} + R_{\perp} e^{jk_{1z}z}] \tag{5.22}
\]

\[
E_2 = \hat{y} E_0 T_{\perp} e^{jk_{1}x} e^{-jk_{1z}z} \tag{5.23}
\]

5.1.2 Parallel (Vertical, or H, or TE_y) Polarization

Similarly, for the parallel polarized waves, the incident, reflected, and transmitted fields can be written as

\[
E_i = (\hat{z} \cos \theta_i - \hat{z} \sin \theta_i) E_0 e^{-jk_{1}(x \sin \theta_i + z \cos \theta_i)} \tag{5.24}
\]

\[
H_i = \hat{y} \frac{E_0}{\eta_1} e^{-jk_{1}(x \sin \theta_i + z \cos \theta_i)} \tag{5.25}
\]

\[
E_r = (\hat{x} \cos \theta_r - \hat{z} \sin \theta_r) E_0 e^{-jk_{1}(x \sin \theta_r - z \cos \theta_r)} \tag{5.26}
\]

\[
H_r = \hat{y} \frac{E_0}{\eta_1} e^{jk_{1}(x \sin \theta_r + z \cos \theta_r)} \tag{5.27}
\]

\[
E_t = (\hat{z} \cos \theta_t - \hat{z} \sin \theta_t) E_0 e^{-jk_{1}(x \sin \theta_t + z \cos \theta_t)} \tag{5.28}
\]

\[
H_t = \hat{y} \frac{E_0}{\eta_2} e^{jk_{1}(x \sin \theta_t + z \cos \theta_t)} \tag{5.29}
\]

The boundary conditions are \(E_{1x} = E_{2x}\) and \(H_{1y} = H_{2y}\) at \(z = 0\). These two equations demand the phase matching conditions (5.11), or the Snell’s laws in (5.12) and (5.13). Furthermore, from these boundary conditions we can obtain

\[
R_{\parallel} \equiv \frac{H_{20}}{H_{10}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{\epsilon_2 k_{1z} - \epsilon_1 k_{2z}}{\epsilon_2 k_{1z} + \epsilon_1 k_{2z}} \tag{5.30}
\]

\[
T_{\parallel} \equiv \frac{H_{10}}{H_{20}} = \frac{2\eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i} = \frac{2\epsilon_2 k_{1z}}{\epsilon_2 k_{1z} + \epsilon_1 k_{2z}} \tag{5.31}
\]

These are called the reflection and transmission coefficients for parallel polarization. Note that, in contrast to the textbook, the reflection and transmission coefficients defined here are the ratio of the magnetic fields for this polarization. The reason for using this definition is that their expressions are exactly dual to the expressions for the perpendicular polarization.

Again, note that in general \(\eta_m\) and \(k_m\) \((m = 1, 2)\) are complex, and so are \(R_{\parallel}, T_{\parallel}\), and \(\theta_i\). Therefore, as a caution, the instantaneous expressions of fields have to been written with consideration of these complex quantities.
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Given the parameters for the incident field (θi, Ei), the total fields in medium 1 and medium 2 can be obtained from (5.24)–(5.29). In particular,

\[
\mathbf{H}_1 = \hat{y} \frac{E_{in}}{\eta_1} \left[ e^{-j k_1 x} e^{-j k_1 z \sin \theta_i \cos \theta_i} + R_\parallel e^{-j k_1 x} e^{-j k_1 z \sin \theta_i \cos \theta_i} \right] = \hat{y} H_{i0} e^{-j k_1 x} \left[ e^{-j k_{11} z} + R_\parallel e^{j k_{11} z} \right]
\]

\[
\mathbf{H}_2 = \hat{y} \frac{E_{in}}{\eta_2} T_\parallel e^{-j k_1 x} e^{-j k_1 z \sin \theta_i \cos \theta_i} = \hat{y} H_{i0} T_\parallel e^{-j (k_2 x + k_{12} z)}.
\]

where \( H_{i0} = E_{i0}/\eta_i \).

Finally, the relations between the reflection and transmission coefficients defined here and those in the textbook are

\[
R_\parallel = \Gamma_\parallel',
\]

\[
T_\parallel = \frac{\eta_2}{\eta_1} T_\parallel' \tag{5.35}
\]

where \( \Gamma_\parallel' \) and \( T_\parallel' \) refer to those in the textbook.

5.1.3 Total Transmission—Brewster Angle

Here we derive the conditions for zero reflection, i.e., the total transmission. The incidence angle at which there is total transmission is called the Brewster angle \( \theta_{iB} \).

A. Perpendicular polarization

The condition for total transmission is

\[
R_\perp = 0 = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \tag{5.36}
\]

However, since \( \cos \theta_i = \sqrt{1 - \sin^2 \theta_i} = \sqrt{1 - \frac{k_1^2}{k_2^2} \sin^2 \theta_{iB}} \), we have from (5.28) that

\[
\eta_2^2 \sin^2 \theta_{iB} = \eta_1^2 (1 - \frac{k_2^2}{k_1^2} \sin^2 \theta_{iB})
\]

or

\[
\sin \theta_{iB \perp} = \left[ \frac{\epsilon_2}{\epsilon_1} - \frac{\mu_2}{\mu_1} \right]^{1/2} \tag{5.37}
\]

It is seen that in order for \( \theta_{iB \perp} \) to be real \( (0 \leq \sin \theta_{iB} \leq 1) \), \( \epsilon_2/\epsilon_1 \) has to be real, and one of the following two conditions has to be satisfied:

i) \( \mu_2/\mu_1 \leq \epsilon_2/\epsilon_1 \leq \mu_1/\mu_2 \),

ii) \( \mu_1/\mu_2 \leq \epsilon_2/\epsilon_1 \leq \mu_2/\mu_1 \).
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For non-magnetic materials (most materials except for ferromagnetic materials), \( \mu_1 \approx \mu_2 \), no \( \theta_i, \theta_r, \theta_t \) exists because \( \sin \theta_i \theta_t \rightarrow \infty \).

One should note that in general, two conductive media can also have a Brewster angle if \( \varepsilon_2 / \varepsilon_1 \) is real, as can be seen from (5.29).

B. Parallel Polarization

For parallel polarization, the total transmission requires

\[
R_\parallel = -\frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1} = 0
\]

or equivalently

\[
\sin \theta_{iB} = \left[ \frac{\mu_2^* - \varepsilon_2}{\varepsilon_2 - \varepsilon_1^*} \right]^{1/2}
\]

(5.38)

For \( \theta_i \theta_r \theta_t \) to be real, \( \varepsilon_2 / \varepsilon_1 \) has to be real, and one of the two following conditions has to be satisfied:

i) \( \varepsilon_2 / \varepsilon_1 \leq \mu_2 / \mu_1 \leq \varepsilon_1 / \varepsilon_2 \),

ii) \( \varepsilon_1 / \varepsilon_2 \leq \mu_2 / \mu_1 \leq \varepsilon_2 / \varepsilon_1 \).

For the special case of non-magnetic materials (\( \mu_1 = \mu_2 \)), the Brewster angle can exist and is given by

\[
\theta_{iB} = \tan^{-1} \left( \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \right) = \cos^{-1} \left( \sqrt{\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}} \right)
\]

(5.39)

5.1.4 Total Reflection—Critical Angle

When the angle of refraction is \( \theta_i = \pi / 2 \), the corresponding angle of incidence is called the critical angle. Since

\[
\theta_i = \sin^{-1} \left( \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \sin \theta_i \right),
\]

when \( \theta_i = \pi / 2 \), we have

\[
\theta_i = \theta_c = \sin^{-1} \left( \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \right)
\]

(5.40)

For the critical angle to be real, we require that \( \varepsilon_2 / \varepsilon_1 \) is real and \( \mu_1 \varepsilon_1 \geq \mu_2 \varepsilon_2 \) (i.e., medium 1 is denser than medium 2).

For \( \theta_i \geq \theta_c \),

\[
\cos \theta_i = \sqrt{1 - \sin^2 \theta_i} = -j \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \sin^2 \theta_i - 1} = -j \sqrt{\frac{\sin^2 \theta_i}{\sin^2 \theta_c} - 1}
\]

(5.41)
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Then, the reflection coefficients

\[ |R_\perp| = \left| \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta'_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_t} \right| = 1 \]  \hspace{1cm} (5.42)

\[ |R_\parallel| = \left| \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta'_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_t} \right| = 1 \]  \hspace{1cm} (5.43)

At the critical angle, the transmission coefficient

\[ T_\perp \bigg|_{\theta_i = \theta_c} = 2 \]  \hspace{1cm} (5.44)

\[ T_\parallel \bigg|_{\theta_i = \theta_c} = 2 \]  \hspace{1cm} (5.45)

For non-magnetic materials, \( \mu_1 = \mu_2 = \mu_0 \), then

\[ \theta_c = \sin^{-1} \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}. \]  \hspace{1cm} (5.46)

- **Transmission wave for** \( \theta_i \geq \theta_c \). For post-critical angles, \( \cos \theta_i \) is (5.41) is pure imaginary if \( \varepsilon_1 \) and \( \varepsilon_2 \) are real, and \( \sin \theta_i = \sqrt{\frac{\mu_1}{\mu_2 \varepsilon_2}} \sin \theta_i \). For the perpendicular polarization, the transmitted field is

\[ \mathbf{E}_\perp = \hat{y} T_\perp E_0 \exp[-j k_2 x \sin \theta_i] \exp[-k_2 z \sqrt{\frac{\sin^2 \theta_i}{\sin^2 \theta_c} - 1}] \equiv \hat{y} T_\perp E_0 \exp(-\alpha_c z) \exp[-j \beta_c x] \]  \hspace{1cm} (5.47)

where \( \alpha_c = k_2 \sqrt{\frac{\sin^2 \theta_i}{\sin^2 \theta_c} - 1} \), \( \beta_c = \omega \sqrt{\mu_1 \varepsilon_1} \sin \theta_i \). (Note that \( \alpha_c \) and \( \beta_c \) are real only for lossless medium 2.) And

\[ \mathbf{H}_\perp = (-\hat{x} \cos \theta_i + \hat{z} \sin \theta_i) T_\perp \frac{E_0}{\eta_2} \exp(-\alpha_c z) \exp(-j \beta_c x) \]  \hspace{1cm} (5.48)

For a lossless medium 2, the phase velocity is

\[ v_{pc} = \frac{\omega}{\beta_c} \leq v_{g2} \]  \hspace{1cm} (5.49)

The complex angle of refraction is \( \theta_i = \theta_R + j \theta_X \). Now from Snell’s law

\[ \sin \theta_i = \sin(\theta_R + j \theta_X) = \sin \theta_R \cosh \theta_X + j \cos \theta_R \sinh \theta_X = \sqrt{\frac{\mu_1}{\mu_2 \varepsilon_2}} \sin \theta_i \]

Noting that for a real critical angle \( \theta_C \), \( \varepsilon_2/\varepsilon_1 \) is real. Therefore,

\[
\begin{align*}
\sin \theta_R \cosh \theta_X &= \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i \\
\sinh \theta_X \cos \theta_R &= 0
\end{align*}
\]
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Hence,

\[ \theta_R = \frac{\pi}{2}, \quad \theta_X = \cos^{-1}\left(\sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i \right) \]  

(5.50)

and

\[ \cos \theta_i = -j \sinh \theta_X. \]  

(5.51)

From the expression of \( \mathbf{E}_\perp \) in (5.47), we see that the plane wave in medium 2 is no longer uniform.

The time-averaged Poynting vector for the transmitted wave is then given by

\[
p_{av} = \frac{1}{2} \Re (\mathbf{E}_1 \times \mathbf{H}_1^*)|_{\theta_i \geq \theta_2} \\
= \frac{1}{2} |T_{\perp}|^2 \left| \frac{E_{\perp 1}}{\varepsilon_2} \right|^2 e^{-2\Re(\alpha_2) - 2\Re(\alpha_2) - 2\Re(\beta_2)} \Re \left\{ j \sqrt{\frac{\sin^2 \theta_1}{\sin^2 \theta_e}} - 1 + \frac{\sin^2 \theta_1}{\sin^2 \theta_e} \right\} e^{-j \theta_2} \\
= \frac{1}{2} |T_{\perp}|^2 \left| \frac{E_{\perp 1}}{\varepsilon_2} \right|^2 e^{-2\Re(\alpha_2) - 2\Re(\beta_2)} \left\{ \frac{\sin \theta_1}{\sin \theta_e} \sqrt{\frac{\sin^2 \theta_1}{\sin^2 \theta_e}} - 1 + \frac{\sin \theta_1}{\sin \theta_e} \right\} \]  

(5.52)

It is clear from (5.52) that in general, for post-critical incidence, there is a \( z \) component of power flow in medium 2 if it is a conductive medium. This \( z \) component is due to the conduction loss in medium 2. However, for non-conductive medium 2, \( \theta_{\eta_2} = 0 \) and thus the \( z \) component of the power flow is zero.

5.1.5 Complex Angles at an Interface Between Conductive Media

Given an incidence angle in conductive medium 1, the angle of refraction \( \theta_i \)

\[ \sin \theta_i = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin \theta_i = \frac{\beta_1 - j \alpha_1}{\beta_2 - j \alpha_2} \sin \theta_i \]  

(5.53)

is in general complex for any incidence angle. If we write \( \theta_i = \theta_R + j \theta_X \), then \( \theta_R \) and \( \theta_X \) can be solved from equation (5.53).

One particular example of interest is the transmission from a non-conductive medium to a good conductor. In that case, \( \sigma_2 > \omega \varepsilon_2 \). Hence,

\[ \sin \theta_i = \frac{k_1}{k_2} \sin \theta_i \approx \frac{\omega \sqrt{\mu_1 \varepsilon_1}}{\sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}(1 - j)}} \approx 0 \]  

(5.54)

Therefore, a direction of propagation in the good conductor is approximately along the normal direction of the interface.

5.1.6 A Note on the Local Reflection and Transmission Coefficients

In contrast to the definition of the reflection and transmission coefficients of the textbook, the definition here satisfies the duality between the perpendicular (TE\( ^\perp \)) and parallel (TM\( ^\parallel \)) polarizations.
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These coefficients defined here for a single interface between two different media are called the local reflection and transmission coefficients. It is then obvious that for both polarizations, the local reflection and transmission coefficients satisfy

\[ R_{12} = -R_{21}, \quad (5.55) \]

\[ T_{12} = 1 + R_{12}, \quad (5.56) \]

\[ T_{21} = 1 + R_{21} \quad (5.57) \]

where the subscript \( ij \) denotes that the incident wave comes from medium \( i \) to medium \( j \).

Now if the interface between medium 1 and medium 2 are located at \( z = z_1 \), then for the perpendicular polarization

\[ E_1 = \hat{y}E_0 e^{-jk_x z} [e^{-jk_{1z} z} + R_{12} e^{-j2k_{1z} z_i} e^{jk_{1z} z}] \equiv E_{1i} + E_{1r} \quad (5.58) \]

\[ E_2 = \hat{y}E_0 e^{-jk_x z} T_{12} e^{-j2k_{1z} (z-z_1)}. \quad (5.59) \]

\[ H_1 = \frac{1}{\eta_1} [\hat{k}_x \times E_{1i} + \hat{k}_r \times E_{1r}], \quad (5.60) \]

\[ H_2 = \frac{1}{\eta_2} \hat{k}_r \times E_2. \quad (5.61) \]

where the directions of propagation are given by equation (5.1). Similar equations apply to the parallel polarization.

5.2 Reflection and Transmission by Multiple Layers

We now consider the reflection and transmission of plane waves by multiple layers shown in Figure 5.1. The plane wave is obliquely incident from the left. The locations of the layer interfaces for the \( N \) layers are \( z = z_i \) (\( i = 1, \cdots, N-1 \)).
Figure 5.1  Oblique incidence of plane waves at multiple layers.

First, we know that if there is only one interface between layer \( i \) and layer \( i + 1 \), the local reflection and transmission coefficients \( R_{i,i+1} \), \( T_{i,i+1} \), \( R_{i+1,i} \), and \( T_{i+1,i} \) between these two layers can be found from equations (5.58)–(5.61), depending on the polarization of the incident plane wave.

Now for multiple layers, field in each layer can be written in two parts, one right- and one left-going. We can define a global reflection coefficient as the ratio of these two parts at the interface. For example, the field in layer \( i \) can be written in a way similar to (5.22) or (5.32), or (5.58)

\[
\phi_i = A_i [e^{-jk_{i+1}z} + R_{i,i+1} e^{-jk_{i,1}z} e^{jk_{i+1}z}] \tag{5.62}
\]

where \( R_{i,i+1} \) is called the global reflection coefficient between layer \( i \) and layer \( i + 1 \), and

\[
\phi_i = \begin{cases} E_{iy}, & \text{for } \text{TE}_z \\ H_{iy}, & \text{for } \text{TM}_z \end{cases} \quad A_i = \begin{cases} E_{0} e^{-jk_{i,0}z}, & \text{for } \text{TE}_z \\ H_{0} e^{-jk_{i,0}z}, & \text{for } \text{TM}_z \end{cases} \tag{5.63}
\]

The two terms in (5.62) will be denoted as the \( R_1 \) and \( L_1 \) waves, corresponding to right-going and left-going waves, respectively.

Similarly, the field in layer \( i + 1 \) can be written as in (5.62)

\[
\phi_{i+1} = A_{i+1} [e^{-jk_{i+1,1}z} + R_{i+1,i+2} e^{-jk_{i+1,1}z} e^{jk_{i+1,1}z}] \tag{5.64}
\]

The two terms in (5.64) will be denoted as the \( R_2 \) and \( L_2 \) waves, corresponding to right-going and left-going waves, respectively.
5.2. REFLECTION AND TRANSMISSION BY MULTIPLE LAYERS

Now consider the field in layer \( i + 1 \) at the interface \( z = z_i \). The right-going wave \( R_2 \) has two contributions: (i) the local transmission \( (T_{i,i+1}) \) of \( R_1 \), and (ii) the local reflection \( (R_{i+1,i}) \) of \( L_2 \). Therefore from (5.62) and (5.64) we have

\[
A_{i+1} e^{-jk_{i+1,z_i}} = T_{i,i+1} A_i e^{-jk_{i,z_i}} + R_{i+1,i} A_{i+1} R_{i+1,i+2} e^{-jk_{i+1,z_{i+1}} e^{jk_{i+1,z_i}}}
\]

(5.65)

Similarly, consider the left-going wave in layer \( i \) at the interface \( z = z_i \). The left-going wave \( L_1 \) also has two contributions: (i) the local transmission \( (T_{i+1,i}) \) of \( L_2 \), and (ii) the local reflection \( (R_{i+1,i}) \) of \( L_1 \). Therefore,

\[
A_i R_{i,i+1} e^{-jk_{i,z_i}} = T_{i+1,i} A_{i+1} R_{i+1,i+2} e^{-jk_{i+1,z_{i+1}} e^{jk_{i+1,z_i}} + R_{i,i+1} A_i e^{-jk_{i,z_i}}}
\]

(5.66)

Solving (5.65) and (5.66) yields

\[
\bar{R}_{i,i+1} = R_{i,i+1} + R_{i+1,i+2} \frac{T_{i+1,i} P_{i+1} S_{i,i+1}}{1 - R_{i+1,i+2} P_{i+1}^2}
\]

(5.67)

\[
A_{i+1} e^{-jk_{i+1,z_i}} = A_i e^{-jk_{i,z_i}} P_i S_{i,i+1}
\]

(5.68)

where we have assumed that \( z_0 = z_1 \), and

\[
P_i = e^{-jk_{i,z} (z_i - z_{i-1})}
\]

(5.69)

\[
S_{i,i+1} = \frac{T_{i,i+1}}{1 - R_{i+1,i+2} P_{i+1}^2}
\]

(5.70)

In the above, \( P_i \) is called the propagator for layer \( i \) since it denotes the propagation from \( z_{i-1} \) to \( z_i \). Equations (5.67) and (5.68) give the recursive relations for \( \bar{R}_{i,i+1} \) and \( A_{i+1} \). Obviously, the “initial” conditions are

\[
\bar{R}_{N-1,N} = R_{N-1,N}
\]

(5.71)

\[
A_1 = \begin{cases} 
E_0 e^{-jk_{x,z}}, & \text{for TE} \\
H_0 e^{-jk_{y,z}}, & \text{for TM}
\end{cases}
\]

(5.72)

From (5.71) and (5.67), we can find \( \bar{R}_{N-2,N-1}, \ldots, \bar{R}_{1,2} \). Similarly, from (5.72) and (5.68), we can find \( A_2, \ldots, A_N \).

To find the transmission wave amplitude, we use (5.72) and (5.68) repeatedly and obtain

\[
A_N e^{-jk_{N,z_{N-1}}} = A_1 e^{-jk_{1,z_1}} \prod_{\ell=1}^{N-1} [P_\ell S_{\ell,\ell+1}] = \bar{T}_{1N} A_1 e^{-jk_{1,z_1}}
\]

(5.73)

where

\[
\bar{T}_{1N} = \prod_{\ell=1}^{N-1} [P_\ell S_{\ell,\ell+1}]
\]

(5.74)

is called the global transmission coefficient between layer 1 and layer \( N \).

Given the incident field, then the field in all layers can be obtained by (5.62). Note that for the last layer \( N, \bar{R}_{N,N+1} = 0 \). Of course, this equation gives only
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the \( y \) component of one field (\( \mathbf{E} \) or \( \mathbf{H} \) field depending on the polarization). The other field can be obtained easily by invoking the relations for plane waves. For example, for the \( \text{TE}^z \) waves, from (5.62) we obtain the electric field \( E_0^y \); Then the magnetic field in layer \( i \) is obtained by

\[
\mathbf{H}_i = \frac{1}{\eta_i k_i} E_0 e^{-jk_{ix}} \left[ (-\hat{z} k_{iz} + \hat{x} k_{ix}) e^{-jk_{ix} z_i} + R_{i,i+1}^\text{TE} (\hat{z} k_{iz} + \hat{x} k_{ix}) e^{-jk_{ix} z_i} e^{jk_{ix} z} \right]
\]

Similarly, for the \( \text{TM}^z \) waves, (5.62) gives the magnetic field \( H_{iy} \), from which the electric field can be obtained as

\[
\mathbf{E}_i = -\frac{\eta_i}{k_i} H_0 e^{-jk_{ix}} \left[ (-\hat{z} k_{iz} + \hat{x} k_{ix}) e^{-jk_{ix} z_i} + R_{i,i+1}^\text{TM} (\hat{z} k_{iz} + \hat{x} k_{ix}) e^{-jk_{ix} z_i} e^{jk_{ix} z} \right]
\]

Note that in the above, the local reflection and transmission coefficients \( R_{i,i+1} \) and \( T_{i,i+1} \) are different for different polarizations, namely \( \text{TE}^z \) (perpendicular) polarization and \( \text{TM}^z \) (parallel) polarization.