Order Stacking in On-Demand Delivery Platforms

N. Bora Keskin James Scott Robert Swinney

Duke University, Fuqua School of Business, e-mail: {bora.keskin,james.scott956,robert.swinney}@duke.edu

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Abstract

On-demand delivery platforms—such as restaurant delivery services—can "stack" orders, assigning orders from multiple customers to a single driver. While this typically reduces operating costs, it also impacts customers, drivers, and service providers (e.g., restaurants) by increasing delivery times and affecting their utility from participating in the platform. To investigate the implications of order stacking, we develop a model with delay-sensitive customers that endogenizes the decisions of all platform stakeholders. We first find that for an on-demand delivery market to be economically viable, order stacking should not be excessive because stacking may lead to unacceptable delays for customers. Second, it is optimal for the platform to stack some orders as long as customer valuations for the service are not too small; we derive the platform's optimal stacking level and discuss how it depends on various market conditions. Third, customers prefer more order stacking than any other market participant, including the platform, because with order stacking the platform offers a lower price to customers, which ultimately increases both the number of customers participating and their individual utility from the service. Fourth, unless the time to drop off orders is negligible or the number of available drivers is very small, the platform can earn more profits by offering direct (non-stacked) deliveries for customers that live far from the service provider. Consequently, if implemented correctly, order stacking may benefit customers and platforms alike.

1 Introduction

Restaurant delivery platforms represent a large and increasingly important part of the food industry, their growth accelerated by the COVID-19 pandemic (McKinsey & Company 2021, Rana and Haddon 2021a). DoorDash is the largest of these platforms in the U.S., accounting for two thirds of the U.S. restaurant delivery market (Kaczmarski 2024), and delivering over 500 million orders each quarter (Business Wire 2023). The business model of such platforms is characterized by high revenue, high costs (particularly labor), and very low margins (McKinsey & Company 2021, Rana and Haddon 2021a). DoorDash and Uber, the two companies that collectively control most of the food delivery market in the U.S., have not been profitable since going public (Rana and Haddon 2021a). Because of this, operational efficiency is key to the long-term viability of restaurant delivery platforms.

Because the single largest cost facing platforms is driver compensation (Rana and Haddon 2021a), one important way they have sought to improve operational efficiency is by "stacking" two or more orders together and assigning them to a single driver (Carson 2019, McKinsey & Company 2021, Weinstein and Luo 2021, Deliveroo 2022).[1] Platforms claim that stacking orders reduces labor needs and gets orders where they need to go faster (Weinstein and Luo 2021, Deliveroo 2022, Uber 2022b), and enables drivers to earn more money more efficiently (Weinstein and Luo 2021, Deliveroo 2022, DoorDash 2022, Uber 2022a). However, some drivers complain that delivering stacked orders reduces their wages and results in lower customer ratings, because customers complain about the greater delays that result when a driver has to drop off additional orders before delivering a customer's food (see, e.g., Reddit 2018, 2020, 2021). Hence, while stacking orders can potentially reduce operating costs, it may have adverse effects on driver and customer participation, not to mention restaurant revenue. As a result, the ultimate value of stacking to platforms is unclear.

In this paper, we consider precisely this trade-off as we seek to understand the implications of order stacking for on-demand delivery platforms and customers, drivers, and service providers that use them. To study these issues, we build a game theoretic model that endogenizes the actions of a platform, customers, drivers, and a restaurant participating in the delivery service: the platform chooses the price to offer customers and the wage to offer drivers, customers choose whether to order, and drivers and the restaurant choose whether to participate. While our primary motivation is order stacking by restaurant delivery platforms such as DoorDash and UberEats,[2] our model applies broadly to any on-demand delivery platform that faces a trade-off between operating costs and customer service when stacking delivery orders (e.g., as with delivery from grocery or convenience stores via platforms like Instacart).

We consider the following key research questions. First, does order stacking impact the viability of an on-demand delivery market? If so, how does it affect the conditions for the existence of a market equilibrium with positive demand? Second, what is the optimal stacking level for the platform, and how does it depend on the underlying market parameters? Third, how does the platform's optimal stacking level compare to the preferred stacking level of the other market participants? Fourth, when is it most valuable for the platform to offer differentiated services to customers, i.e., by giving some customers stacked deliveries and other customers direct (non-stacked) deliveries?

We begin by analyzing the equilibrium with an exogenous level of order stacking. We find that a market with positive demand exists only if the order stacking level is not too large. In other words, order stacking makes it more difficult for a delivery market to be economically viable, and excessive stacking can result in the market breaking down. The equilibrium further falls into one

[1] Throughout the paper, we use the term "stacking," which has been used by some firms in the industry; other terms for this practice include "batching" and "bundling."

[2] The problem of order stacking reducing customer utility via increased waiting times—and thereby adversely impacting customers, drivers, and restaurants—is particularly salient in restaurant deliveries, as customers ordering meals are especially time-sensitive compared to, e.g., customers ordering packaged goods from a retailer.

of two qualitatively different regimes characterized by the relative number of available delivery drivers: in the first regime, sufficient drivers are available to allow the platform to set its preferred price and wage levels—we refer to this regime as the *abundant driver regime*. In the second regime, the platform is constrained by the number of available drivers and so must set a sub-optimal price and wage—this regime is called the *scarce driver regime*. While increasing the stacking level makes market existence less likely, it can make it more likely that the equilibrium falls into the abundant driver regime, which in turn increases the platform's profit.

Building on this, we then consider the optimal level of order stacking for all market participants. We find that the platform's optimal stacking level is positive if customer valuations for the service are not too small. Hence, even though order stacking can make it more difficult for a market to exist, some degree of stacking can be beneficial for the platform, as stacking decreases operating costs and makes it easier for the platform to deal with a shortage of drivers. When drivers are abundant, the preferred stacking levels of all market participants can be strictly ordered: customers prefer the *highest* level of stacking, followed by the platform, the restaurant, and drivers. Thus, even though stacking increases delivery times of individual customers, in equilibrium it benefits the customers the most because the platform can use stacking to serve more customers at a lower price than it can without stacking. On the other hand, stacking (at the platform's optimal level) negatively impacts driver surplus, due to the reduced wages and lower number of drivers recruited by the platform. In other words, the precise operational cost savings that benefit the platform come at the expense of drivers.

Finally, we consider an alternative service model for the platform—motivated by emerging practice at some delivery platforms—in which the platform stacks orders for customers located close to the restaurant (for whom delays are minimal) and offers direct deliveries for customers located far from the restaurant (for whom delays are long). We find that this alternative model can strictly increase profit for the platform unless the drop-off time of each order is negligible (which is unlikely in practice) or the driver pool is very small. Hence, unless there is a significant shortage of drivers, the platform can increase its profit by offering differentiated levels of service to customers based on their distance from the restaurant.

2 Literature

Our paper is related to the literature on order stacking—or, more generally, pooling or bundling—in ride-hailing and delivery platform settings. Both settings differ from taxi and centralized delivery services in that drivers are not employees of the companies; instead, drivers are market participants that decide when and how much to work. We begin our discussion of the literature by first considering stacking in the more established ride-hailing literature, then survey the more nascent literature on delivery platform operations.

In ride-hailing platforms, assigning two or more riders to be in the same car at the same time is called "(ride) pooling." Pooling has significant operational importance—in areas where Uber and Lyft offer pooling, 20% and 40% of rides, respectively, were pooled as of December 2017 (Shaheen and Cohen 2019). Because of this, pooling has attracted substantial attention in the research literature on ride-hailing (see, e.g., Hu et al. 2020, Jacob and Roet-Green 2021, Zhang and Nie 2021a,b, Gopalakrishnan et al. 2019, Taylor 2024). Wang and Yang (2019) present a recent survey on ride-hailing platforms, and Shaheen and Cohen (2019) provide an introduction to pooling in various contexts, including ride-hailing platforms. Of most relevance to our work, Wang and Zhang (2022) study how the surpluses of all participants in a ride-hailing platform are affected by ride pooling. They model a pooled ride as having a lower rider valuation than a solo ride, and find that the platform is better off offering pooling if the gap between the valuations of pooled and solo rides is not too large. Riders always benefit from pooling being offered, but drivers can be hurt by it. In fact, if driver reservation utilities are uniformly distributed (as they are in our model), drivers are *always* worse off when the platform allows pooling. Wang and Zhang (2022) also present a numerical study indicating that social welfare is increased by offering pooling. Taking the perspective of a platform, Bahrami et al. (2022) consider how a ride-hailing platform should set fares, target vehicle occupancy, and fleet size to maximize its profit, also characterizing when the platform should offer solo or pooled rides together or in isolation. Unlike Wang and Zhang (2022) and our paper, Bahrami et al. (2022) assume that the platform sets the fleet size as a decision variable, and thus do not model drivers as rational agents. While ride pooling is conceptually similar to order stacking in food delivery platforms, there are significant differences between the two settings. Perhaps the biggest difference is that *restaurants* participate in food delivery platforms, in addition to drivers and customers that participate in both types of platforms. The presence of restaurants makes a food delivery platform a three-sided market in which customers, drivers, and restaurants must be incentivized to participate. In addition to the economic differences between the platforms, there is also an operational difference. In ride-hailing platforms, drivers are dispatched to customers' individual locations to pick them up and transport them. By contrast, food delivery platforms dispatch drivers to one of a fixed set of restaurants (in our case, to a *single* restaurant) to pick up the orders and deliver them. Thus, the delivery problem for food delivery platforms is anchored at the restaurant(s) in a way that the routing problem for ride-hailing platforms is not.

In contrast to pooling in ride-hailing platforms, there has been little research about the effects of stacking in food delivery platforms, where prior work has focused more on the impact of platforms on restaurants (e.g., Chen et al. 2022, Feldman et al. 2023). The closest works to ours in this literature are those by Bahrami et al. (2023), Liu et al. (2023), Chen and Hu (2024), Ke et al. (2024), and Ye et al. (2024). Both Bahrami et al. (2023) and Liu et al. (2023) model restaurants, customers, and drivers as price-sensitive agents on a platform. Bahrami et al. (2023) compare the equilibria attained when the platform is respectively a profit-maximizer and a social welfare-maximizer. They find that the socially optimal equilibrium has lower restaurant commissions and higher driver wage than the profit-maximizing equilibrium. Considering the platform to be a profit-maximizer, Liu et al. (2023) compare restaurant surplus and social welfare in the presence and absence of a platform, finding that the presence of a platform increases both. Neither paper examines the effects of order stacking, which is the heart of our paper. Chen and Hu (2024), on the other hand, compare two online fleet routing policies of a platform in a disk service area centered at the restaurant. Considering uniformly distributed customer orders following a Poisson process, they examine whether it is better for the platform to assign one order or two orders to drivers under a first-come-first-served policy. However, they model drivers as employees rather than economic agents, so their paper is more appropriately considering in-house drivers rather than platform drivers. In the transportation literature, Ke et al. (2024) and Ye et al. (2024) develop spatial models to study the effect of order assignment policies on a food delivery platform's profit and social welfare. Our work is differentiated from the aforementioned three studies because we provide a theoretical analysis on how order stacking impacts the payoffs of all market participants—the platform itself as well as customers, drivers, and a restaurant.

Our research is also broadly related to the growing operations research and management science (OR/MS) literature on managing platforms (see, e.g., Cachon et al. 2017, Taylor 2018, Bai et al. 2019, Bimpikis et al. 2019, Guda and Subramanian 2019, Bernstein et al. 2021, Besbes et al. 2021, Astashkina et al. 2022, Benjaafar et al. 2022, Castillo et al. 2022, Freund et al. 2022, Hu et al. 2022, Ma et al. 2022, Siddiq and Taylor 2022, Lian and van Ryzin 2023). Chen et al. (2020) provide a detailed overview of various modeling approaches in this literature. Some of the studies in this context focus on autonomous vehicles in ride-hailing platforms (see, e.g., Freund et al. 2022, Siddiq and Taylor 2022, Lian and van Ryzin 2023), whereas several ride-hailing studies consider drivers as independent agents (see, e.g., Cachon et al. 2017, Taylor 2018, Bimpikis et al. 2019, Jacob and Roet-Green 2021). We also consider drivers as independent agents and investigate the use of pricing to match supply with demand in a platform setting. Nevertheless, as noted at the beginning of this section, our focus on food delivery platforms, especially the inclusion of deliveries from a restaurant in our model, significantly differentiates our work from the aforementioned studies.

3 Model

Consider a food delivery platform operating in a linear city (Hotelling 1929) with a restaurant located at the origin. A continuum of infinitesimal (potential) customers are distributed over the positive real line according to Lebesgue measure. In our base model, the platform offers delivery service to all customers on the positive real line—in Appendix E.1, we consider an extension of this model where the platform has a fixed service radius, where customers outside of the service radius are not permitted to participate.

We let L be the number of deliveries made by each driver on the platform. If $L > 1$, then the platform is said to be "stacking" orders; if $L = 1$, the platform offers direct deliveries. Thus, we call L the *order stacking level* of the platform. (We first treat L as an exogenous variable and then analyze the optimal value of L.)

We model deliveries as a game between four market participants—the platform, customers, drivers, and the restaurant—based on the following sequence of events:

1. The platform determines the price P to charge customers for the delivery service.[3]
2. All customers on the real line observe P and determine whether to order.
3. Based on the number of customer orders and the order stacking policy, the platform determines how many drivers to recruit, and the platform adjusts driver pay to induce participation by the appropriate number of drivers at the minimum cost to the platform.
4. The platform assigns drivers to orders. We assume that each driver is assigned to deliver a mass of L each located within a continuous interval, i.e., neighboring orders are grouped together for delivery.
5. After all the deliveries are made, collected revenue is shared between the platform and the restaurant.

Next, we describe each of the above market participants in more detail, beginning with customers.

3.1 Customers

Customers are expected utility maximizers who, after observing the price P, choose between ordering delivery from the restaurant via the platform and forgoing consumption (or, equivalently, consuming an exogenous outside option). Customers have homogeneous valuations V for their orders, and experience a disutility proportional to the time it takes their orders to be delivered, i.e., the waiting time. A customer's waiting time is the total time between the customer ordering a meal and the ordered meal arriving at the customer's location. In practice, this time is determined by (i) the time it takes a driver to reach the restaurant, (ii) the time it takes the restaurant to fulfill the order, and (iii) the time it takes the driver to bring the meal to the customer. In our base model, for simplicity and to focus on the effects of order stacking on (iii), we assume that both (i) and (ii) are equal to zero; however, we extend the base model to allow for a positive meal preparation time (or, equivalently, driver travel time to the restaurant) in Appendix E.2. Thus, in our base model,

[3] In practice, the price might be set jointly by the restaurant and the platform. For instance, the restaurant might set the the menu price of the food, while the platform sets the delivery fees. For simplicity, we assume the platform maintains all pricing power. This is meant to reflect the fact that even if the restaurant sets the menu price, the platform has the ability to raise or lower the price beyond this value, by either charging additional service fees (to raise the price) or offering discounts or promotions (to lower the price).

the waiting time for a particular customer consists entirely of driver delivery time, which in turn is comprised of two components: the travel time from the restaurant to the focal customer (which is incurred regardless of the stacking level), and the drop-off time associated with any orders the driver delivers prior to the focal customer (which depends on the stacking level).

Each customer knows their location on the real line, x. Moreover, travel is assumed to occur at unit speed; hence, the travel time from restaurant to the customer is simply x. To determine their wait time, customers must also estimate the number of drop-offs that occur prior to their delivery. We assume that customers are aware of the order stacking policy and hence the number of deliveries, L, in each order, but are unaware of their exact position in the driver's sequence of drop-offs. This is reflective of the fact that it is difficult for a customer to precisely anticipate their place in the drop-off sequence on a particular delivery run. Because customers do not know their exact position in the drop-off sequence, they assign equal probability to any position, and hence in expectation they assume they are in the middle of the drop-off sequence. Specifically, every customer assumes there is a mass of orders equal to $(L-1)/2$ that their driver needs to drop off before reaching them.[4] Each order is assumed to require d units of time to drop off with a customer (e.g., due to the driver having to park, exit their vehicle, physically deliver the food to the customer's door, and return to their vehicle), hence a total delay of $d(L-1)/2$ is incurred from stacked deliveries. When $L = 1$ (i.e., the platform offers direct deliveries), there is zero stacking delay.[5]

Combining these facts, the total expected wait time of a customer at location $x > 0$ is

$$W(x) = x + \frac{d(L-1)}{2}. \tag{1}$$

The marginal disutility of waiting is δ per unit time. Hence, the total utility a customer located at x receives is their valuation for service minus the price and the disutility from waiting, i.e.,

$$\mathcal{U}(x) = V - P - \delta W(x). \tag{2}$$

(In Appendix E.3, we study an extension where the disutility from waiting is a general convex function of wait time.) The customer orders if and only if $\mathcal{U}(x) = V - P - \delta W(x) \geq 0$, or equivalently, if and only if $W(x) \leq (V-P)/\delta$. Because $W(x)$ is increasing in x, there exists a positive real number C such that $W(x) \leq (V-P)/\delta$ if and only if $x \leq C$. Therefore, in equilibrium, only customers located in $[0, C]$ live close enough to the restaurant to receive positive

[4]When L is an integer, each of the L orders for the driver to drop off is equally likely to be the first, second, ..., Lth, implying that the number of orders that a customer would expect to be dropped off before them is $0L^{-1} + 1L^{-1} + 2L^{-1} + \cdots + (L-1)L^{-1} = (L-1)/2$. When L is not an integer, this is an approximation.

[5]We assume that customers do not include any drop-off time incurred with the delivery of their *own* order when calculating the total delay time; because this is the same regardless of the stacking policy, it is a fixed cost for all customers participating in the delivery platform, which we normalize to zero.

utility from ordering through the platform; see Figure 1. Under the wait time given in (1), the mass of participating customers is

$$C = \frac{V-P}{\delta} - \frac{d(L-1)}{2}. \tag{3}$$

Aggregate consumer surplus S_C is thus

$$S_C = \int_0^C \mathcal{U}(x)dx = \frac{\delta}{2}C^2.$$

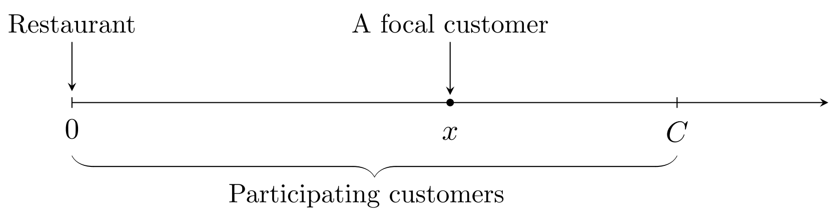


Figure 1. Linear city model and participating mass of customers.

3.2 Drivers

In food delivery platforms, driver pay is based on total trip distance, the time to complete the trip, and other proprietary factors (DoorDash 2021, Uber 2021). Consequently, we model driver pay based on trip distance $\mathcal{D}$ and time $\mathcal{T}$ as equal to $\rho_d \, g_d(\mathcal{D}) + \rho_t g_t(\mathcal{T})$, where $g_d(\cdot)$ and $g_t(\cdot)$ are positive and increasing (possibly non-linear) functions of distance and time, respectively, and $\rho_d \geq 0$ and $\rho_t \geq 0$ are non-negative coefficients. The other factors unrelated to time or distance that the platforms use do not apply to our setting. For example, the driver pay could depend on how well-known the restaurant is among drivers (DoorDash 2021). Because our city has only one restaurant, we assume that the delivery drivers are all paid at the same rate for deliveries.

We further assume that drivers are assigned randomly to orders. Hence, for each driver, the (ex-ante) expected utility from participating is

$$\mathcal{V} = \rho_d \mathbb{E}\left[g_d(\mathcal{D})\right] + \rho_t \mathbb{E}\left[g_t(\mathcal{T})\right]. \tag{4}$$

To reflect that drivers have heterogeneous alternatives for how to spend their time outside of delivering for the platform, the reservation utility v of drivers is assumed to be uniformly distributed on $[0, \overline{v}]$, where $\overline{v} > 0$. Every driver participates when their utility from doing so, $\mathcal{V}$, exceeds their reservation utility, v. Potential drivers are infinitesimal with total mass $\overline{D}$. Thus, the mass of participating drivers is

$$D = \begin{cases} \frac{\mathcal{V}}{\overline{v}}\overline{D} & \text{if } \mathcal{V} \in [0, \overline{v}], \\ \overline{D} & \text{if } \mathcal{V} \geq \overline{v}. \end{cases} \tag{5}$$

If, in equilibrium, the platform hires the whole driver pool, we say that drivers are *scarce*. If the platform hires a fraction of the driver pool in equilibrium, we say that drivers are *abundant*.

3.3 Platform and Restaurant

In practice, the platform typically keeps a fixed-percentage commission on the total sales made through the platform, returning the remaining fraction to the restaurant (Feldman et al. 2023). Let $F \in [0, 1]$ be the platform's percentage in the context of our model.

Given this contractual structure, if the restaurant participates in the platform, the restaurant's surplus is

$$S_R = (1 - F)CP. \tag{6}$$

The preceding quantity is also referred to as the restaurant's utility. In our base model, the restaurant participates whenever its utility is non-negative, i.e., when $S_R \geq 0$.

The platform generates revenue from customer orders, and its only expense is the compensation paid to drivers. Note that, given an exogenous stacking level L and a mass of participating customers C given by (3), it follows that the required number of drivers D is determined by the following equation:

$$L = \frac{C}{D} \;\Rightarrow\; D = \frac{C}{L}. \tag{7}$$

Let S_D denote the total payment made to drivers in equilibrium (note that this is also aggregate driver surplus). The platform earns a total revenue of FPC, and therefore has profit function $S_P = FPC - S_D$. From (5), the equilibrium aggregate driver pay is

$$S_D = D\mathcal{V} = \begin{cases} \frac{\overline{D}}{\overline{v}}\mathcal{V}^2 & \text{if } \mathcal{V} \in [0, \overline{v}], \\ \overline{D}\mathcal{V} & \text{if } \mathcal{V} \geq \overline{v}. \end{cases} \tag{8}$$

In (4), $\mathcal{V}$ is zero when $\rho_d = \rho_t = 0$, and continuous and strictly increasing in ρ_d and ρ_t, implying that the platform can make $\mathcal{V}$ equal to any value by appropriately setting ρ_d and ρ_t. The mass of drivers that participate is determined solely by the value of $\mathcal{V}$, and in Section 4, all parties' payoffs are specified, from which it becomes clear that the value of $\mathcal{V}$ is payoff-relevant, while the specific values of ρ_d, ρ_t, $g_d(\cdot)$, and $g_t(\cdot)$ are not. We therefore take $\mathcal{V} \geq 0$ as the relevant decision variable for the platform, instead of the coefficients ρ_d and ρ_t and functions $g_d(\cdot)$ and $g_t(\cdot)$. In fact, this further implies that any compensation function that allows the platform to make $\mathcal{V}$ equal to any positive real number by appropriately setting the contract parameters satisfies this property—even if it is not separable in distance and time, as in (4)—and can be represented by our model.

Setting $\mathcal{V} = \overline{v}$ recruits all drivers, so to minimize its costs, the platform would never choose $\mathcal{V} > \overline{v}$. We can therefore take $\mathcal{V} \in [0, \overline{v}]$ to be a constraint in the platform's pricing problem, and under this constraint, the expression for aggregate driver surplus given in (8) simplifies to

$S_D = \overline{D}\mathcal{V}^2/\overline{v}$. Using (5) and (7), we deduce the following under the constraint $\mathcal{V} \in [0, \overline{v}]$:

$$\frac{\overline{v}}{\overline{D}L} C = \mathcal{V}. \tag{9}$$

Because the left-hand side is a function of the price P only, (9) shows that we can eliminate $\mathcal{V}$ from the model, provided that we add the constraint

$$\frac{C}{L} \leq \overline{D}, \tag{10}$$

which is equivalent to $\mathcal{V} \leq \overline{v}$ (the other constraints imply that C is non-negative). The platform maximizes its profit S_P which, given the preceding observations, yields the following optimization problem:

$$\max_{C,P\geq 0} \quad S_P = FPC - \frac{\overline{v}}{\overline{D}}\left(\frac{C}{L}\right)^2 \tag{11}$$

$$\text{subject to:} \quad C = \frac{V-P}{\delta} - \frac{d(L-1)}{2}, \tag{12}$$

$$\frac{C}{L} \leq \overline{D}. \tag{13}$$

We can then solve (12) for P, obtaining

$$P = V - \frac{d(L-1)\delta}{2} - \delta C. \tag{14}$$

We can use this expression to eliminate the decision variable P, leaving only C as a decision variable. The resulting problem can be written as

$$\max_{C} \quad S_P = F\left(V - \frac{d(L-1)\delta}{2} - \delta C\right)C - \frac{\overline{v}}{\overline{D}L^2}C^2$$

$$\text{subject to:} \;\; 0 \leq C \leq \min\left\{\frac{2V - d(L-1)\delta}{2\delta}, \overline{D}L\right\}.$$

In other words, given the market participation decisions of customers, drivers, and the restaurant, the platform's problem reduces to one of picking the participating mass of customers, C, which it accomplishes by varying the price P; given this, the platform chooses the minimum driver compensation that induces the appropriate number of drivers to participate according to the order stacking level, L.

Finally, social welfare S is defined to be the sum of the surpluses of all market participants, i.e.,

$$S = S_C + S_D + S_R + S_P = S_C + PC.$$

A summary of all notation is given in Table 1.

V	customer valuation for service
$\overline{v}$	driver maximum outside option value
d	drop-off transaction delay
$\overline{D}$	potential driver mass
ρ_d	driver per-mile pay rate
ρ_t	driver per-hour pay rate
δ	customer impatience parameter
F	revenue sharing fraction
L	order stacking level
P	equilibrium price
C	equilibrium customer mass
D	equilibrium driver mass
S_C	equilibrium aggregate customer surplus
S_D	equilibrium aggregate driver surplus
S_R	equilibrium restaurant surplus
S_P	equilibrium platform profit
S	equilibrium social welfare

Table 1. Summary of key notation.

4 Equilibrium Analysis

In this section, we analyze the equilibrium to the model with an exogenous stacking level, i.e., the solution to the platform's optimization problem derived in the preceding section. First, we note that it is clear from the constraints in the platform's pricing problem that it is only feasible if $2V - d(L-1)\delta \geq 0$. Furthermore, if this inequality holds with equality, then the only feasible choice is to set the prices high enough so that no customer participates. However, if the inequality holds strictly, then we show in Appendix A that the optimal solution always has some mass of customers (and drivers) who participate, and this further implies that both the platform and the restaurant earn strictly positive profits. These observations lead to our first main result:

Proposition 1. *Assuming all model parameters are positive, there is a market (that is, there is a positive mass of customers and drivers that participate, and the restaurant and platform find it strictly profitable to participate) if and only if*

$$2V - d(L-1)\delta > 0. \tag{15}$$

Furthermore, drivers are scarce if and only if

$$2V - d(L-1)\delta > 4\left(\frac{\overline{v}}{FL} + \overline{D}L\delta\right); \tag{16}$$

otherwise, drivers are abundant.

We refer to (15) as the *market condition* and (16) as the *driver scarcity condition*. The market condition has an intuitive explanation. Dividing by 2 and rearranging, we can express the market condition as $V > d(L-1)\delta/2$. Thus, a market exists if and only if the customer valuation is greater than the disutility induced by the mass of orders that a customer expects their driver to drop off before getting to them. As long as this is the case, there is a positive mass of customers that is close enough to the restaurant to get positive utility from ordering at some non-negative price.

When $L = 1$, (15) holds for any $V > 0$. In other words, with direct deliveries, a market always exists if customers derive positive value from the service. Because the left-hand side of (15) is decreasing in L, this immediately implies that a market exists only if L is sufficiently small. Put differently, order stacking (the case where $L > 1$) makes market existence more difficult because it increases customer waits even for customers located close to the restaurant. If the stacking level is too high (i.e., $L > 2V/d\delta + 1$) the market breaks down entirely. The parameters involving drivers (i.e., $\overline{v}$ and $\overline{D}$) and the restaurant (i.e., F) influence the equilibrium price but do not influence the existence of the market. This is because, when there is a price at which some customers and the platform are willing to participate, the restaurant makes a profit and there is always a positive mass of drivers who can make a profit.

The scarcity condition (16), on the other hand, implies that when the gap between the customer valuation and the disutility of waiting is too large, the platform is unable to serve the number of customers that it would like due to an insufficient driver pool. In turn, this implies that the available-driver constraint is binding at optimality, limiting the platform's profit. Because the left-hand side of (16) is decreasing in L and the right-hand side is convex in L, a greater stacking level can result in a transition from the scarce driver regime to the abundant driver regime. In other words, while order stacking can make market existence more difficult per (15), it can also allow the platform to more effectively operate with a limited driver pool by moving to the abundant driver regime per (16). This illustrates that order stacking is a double-edged sword that can help the platform cope with a limited driver pool, provided stacking is not so excessive that the market breaks down entirely.

4.1 Equilibrium in the Abundant Driver Regime

When drivers are abundant (i.e., (16) is violated), the platform is capable of recruiting the unconstrained optimal number of drivers to satisfy the market demand. Our first result in this case shows that, intuitively, all market participants—drivers, customers, the restaurant, and the platform—are better off when customers highly value the service (i.e., V is high) and orders are dropped off quickly (i.e., d is low):

Proposition 2. *When drivers are abundant, the surpluses of all four market participants (namely, S_D, S_C, S_R, and S_P) are strictly increasing in V and strictly decreasing in d.*

As the customer valuation V increases, the platform can both profitably reach more customers and charge them a higher price. The rise in customer participation requires more drivers and higher wages, both of which benefit the driver surplus. Consequently, all parties benefit from increasing V. On the other hand, an increase in the drop-off time d reduces the customer utility and hence the price that the platform can charge. Such an increase also reduces customer and driver participation and thus the surplus of all market participants. Because d is essentially a transaction time, it is not surprising that no party benefits from a larger d.

The next result characterizes how the platform's profit is affected by several other key market conditions.

Proposition 3. *When drivers are abundant, the platform's profit, S_P, is strictly increasing in $\overline{D}$ and F and strictly decreasing in δ and $\overline{v}$.*

Intuitively, the platform benefits when it can get more drivers for the same cost (i.e., when the driver pool size $\overline{D}$ increases, or the maximum outside option value of drivers $\overline{v}$ decreases) and when it gets a larger share of the total revenue (i.e., when the revenue sharing fraction F increases). However, the platform's profit decreases if customers are more sensitive to waiting (i.e., when the customer impatience δ increases), because the customer utility is reduced and the platform must compensate by lowering its price.

To analyze the customer surplus, we distinguish between customers' *aggregate* surplus, S_C, and *average* surplus, S_C/C. The following result examines the properties of the mass of participating customers (C), the aggregate customer surplus (S_C), the average customer surplus (S_C/C) in equilibrium.

Proposition 4. *When drivers are abundant, we have the following:*
(i) The mass of participating customers, C, and the aggregate customer surplus, S_C, are strictly increasing in F and $\overline{D}$ and strictly decreasing in $\overline{v}$.
(ii) In addition, C is strictly decreasing in δ, while S_C and S_C/C are strictly quasi-concave in δ. Furthermore, S_C is maximized at δ_{S_C} while S_C/C is maximized at $\delta_{S_C}^{avg} > \delta_{S_C}$.

Proposition 4 states that customers' participation and surplus increase in response to an increase in the driver pool size $\overline{D}$ or a decrease in the maximum outside option value of drivers $\overline{v}$. Both changes essentially make drivers cheaper for the platform. The platform responds by lowering its price and hiring more drivers, making customers better off and encouraging their participation. C and S_C also increase in the revenue sharing fraction F. As F increases, the platform earns more revenue per customer order (all else being equal), which allows the platform to lower the price and increase the number of customers served.

The impact of δ on customers' participation and surplus is more complicated. If customers become more impatient, their utility decreases and they participate less (i.e., C decreases in δ).

However, in response to this, the platform lowers the price, which increases every individual customer's utility. As a result of these two effects working in opposite directions, the aggregate customer surplus is strictly quasi-concave in δ. The value of δ that maximizes the aggregate customer surplus, δ_{S_C}, is smaller than the one that maximizes the average customer surplus, $\delta_{S_C}^{avg}$. As δ increases from δ_{S_C} to $\delta_{S_C}^{avg}$, the mass of participating customers and their aggregate surplus decrease, but their average surplus increases. This implies that there are fewer customers, each of whom is better off individually.

In our analysis, we also show that both δ_{S_C} and $\delta_{S_C}^{avg}$ are strictly decreasing in L; see Proposition C.1 in Appendix C. This observation offers a practical insight. During the COVID-19 pandemic, many people heavily relied on food deliveries (McKinsey & Company 2021, Wang et al. 2021). As the impact of the pandemic diminishes, people would likely have more dining options and their sensitivity to delay in food deliveries could increase. If food delivery platforms respond by increasing order stacking to increase efficiency, this might backfire in that there would be fewer customers, each of which may be worse off. This, in turn, may negatively affect long-term market penetration.

Next, we consider the impact of market conditions on drivers. Akin to the preceding customer analysis, we separately analyze the mass of participating drivers (D), aggregate driver surplus (S_D), and average driver surplus (S_D/D). The following proposition summarizes this analysis.

Proposition 5. *When drivers are abundant, we have the following:*
(i) The mass of participating drivers, D, and the aggregate driver surplus, S_D, are strictly increasing in F and strictly decreasing in δ.
(ii) In addition, D is strictly increasing in $\overline{D}$ and strictly decreasing in $\overline{v}$, while average driver surplus, S_D/D, has the opposite behavior. Furthermore, S_D is strictly quasi-concave in $\overline{D}$ and $\overline{v}$, with maximizers $\overline{D}_{S_D}$ and $\overline{v}_{S_D}$, respectively.

In contrast to customers, driver surplus is not necessarily increasing in the driver pool size $\overline{D}$. The aggregate driver surplus, S_D, is strictly quasi-concave in $\overline{D}$, while the average driver surplus, S_D/D, is strictly decreasing in $\overline{D}$. The quasi-concavity of S_D in $\overline{D}$ is a consequence of two effects that work in opposite directions. One is that as the driver pool gets bigger, the driver compensation rate drops, making them individually worse off. The other is that as $\overline{D}$ gets bigger, the platform can hire more drivers for the same amount of money, so the mass of participating drivers increases. As a result of these two effects, there are more drivers, each of which is individually worse off, as the number of available drivers increases. Over low values of $\overline{D}$, the aforementioned decrease in the individual driver surplus is offset in the aggregate by the surge in driver participation. The opposite happens when $\overline{D}$ is high.

Drivers' participation and surplus both increase as F grows. As noted in our customer analysis, as F increases, the platform earns more revenue per customer and hence lowers the price to expand the market. With more customers to serve, the platform also recruits more drivers, which means that the driver surplus also increases due to an increase the revenue sharing fraction F. The opposite is true of the customer impatience δ. When customers become more sensitive to waiting, their utility decreases, leading them to participate less in the platform. With less demand, the platform uses fewer drivers, and hence, the aggregate driver surplus and the mass of participating drivers are both negatively affected by an increase in δ.

Lastly, we consider how the restaurant is impacted by the market conditions in equilibrium. The analysis for the restaurant is similar to that for the platform in most respects because the revenue is split between them—any change that increases the revenue is beneficial for both the restaurant and the platform. This leads to the following result.

Proposition 6. *When drivers are abundant, we have the following:*
(i) The restaurant's surplus, S_R, is strictly increasing in $\overline{D}$ and strictly decreasing in $\overline{v}$ and δ.
(ii) In addition, S_R is strictly quasi-concave in F with a unique maximizer $F_{S_R} \in (0, 1)$. More specifically, S_R vanishes at $F = 0$, strictly increases up to F_{S_R}, and strictly decreases until it vanishes again at $F = 1$.

Intuitively, the restaurant benefits from an increase in the driver pool size $\overline{D}$, but is worse off when drivers are harder to hire (increasing $\overline{v}$) or customers are less patient (increasing δ). The impact of the revenue sharing fraction F is quite different, however. Recall that F is the fraction of the total revenue that the platform retains, with $1 - F$ of the revenue paid to the restaurant. Naïvely, one might think that the restaurant would prefer to have F as close to 0 as possible, because this maximizes the portion of the total revenue that is allocated to the restaurant. Proposition 6 shows that, in fact, the restaurant may be better off with a contract that gives them a smaller percentage of the total revenue. In decreasing its portion of the total revenue, the restaurant allocates more money to the platform, and the platform responds to this by recruiting more drivers and expanding the mass of participating customers, as discussed following Propositions 4 and 5. This means that the total revenue pie can increase, resulting in the restaurant getting more profit, even though its percentage of the total revenue is smaller. Consequently, it is not true that the restaurant always benefits from retaining a higher percentage of the delivery revenue, given the reactions of all market participants to this revenue share parameter; this suggests that platforms, restaurants, and policymakers should be careful of "commission cap" laws that place a limit on the fraction of revenue the platform can retain (see also Feldman et al. 2023), although this comes with an important caveat, as we discuss in the next section.

4.2 Equilibrium in the Scarce Driver Regime

When drivers are scarce (i.e., (16) holds), the platform is incapable of recruiting its preferred number of drivers, and hence the platform's profit is constrained at optimality. In turn, this leads to different comparative statics compared to the case of abundant drivers. These differences stem from one important feature of the scarce driver regime: in this regime, the platform recruits all drivers in equilibrium, and given that the order stacking level is L, this determines how many customers the platform serves. In other words, because the platform cannot recruit more drivers, it cannot serve as many customers as it wants, and the driver population determines how many customers participate.

Thus, the masses of participating customers and drivers, C and D, (and, hence, S_C and S_D) become insensitive to changes in many of the model parameters. For example, Proposition 2 states that each party's equilibrium surplus strictly increases in V and strictly decreases in d when drivers are abundant. In contrast, we have the following proposition in the scarce driver regime.

Proposition 7. *When drivers are scarce, the surpluses of the restaurant and the platform, S_R and S_P, are strictly increasing in V and strictly decreasing in d, but the customer surplus and the driver surplus, S_C and S_D, are independent of V and d.*

Because of the number of drivers and customers are fixed, an increase in V, which raises the customer utility (all else being equal), is fully extracted by the platform via an increase in the price. Similar, an increase in d, which decreases the customer utility, is fully compensated by the platform via a decrease in the price. Consequently, the platform (and its revenue sharing partner, the restaurant) can extract all surplus gains and absorb all surplus losses from an increase in the customer valuation or the drop-off time.

We next consider the behavior of the customer surplus, S_C, in the scarce driver regime. The following proposition is analogous to Proposition 4.

Proposition 8. *When drivers are scarce, we have the following:*
(i) The mass of participating customers, C, the aggregate customer surplus, S_C, and the average customer surplus, S_C/C, are strictly increasing in $\overline{D}$ and independent of F and $\overline{v}$.
(ii) In addition, C is independent of δ, while S_C and S_C/C are strictly increasing in δ.

In light of our discussion, it is not surprising that the only way to increase or decrease C is to change $\overline{D}$: only by increasing the potential pool of drivers can the mass of participating customers be expanded. In addition, for fixed $\overline{D}$, because the mass of customers is fixed, the more time-sensitive customers are, the less surplus the platform can extract from them, causing S_C and S_C/C to be increasing in δ.

A similar story unfolds with drivers. When drivers are scarce, $D = \overline{D}$, implying $S_D = \overline{D}\overline{v}$ and $S_D/D = \overline{v}$, which gives us the following proposition.

Proposition 9. *When drivers are scarce, the mass of participating drivers, D, and the aggregate driver surplus, S_D, are strictly increasing in $\overline{D}$, and the aggregate and average driver surpluses, S_D and S_D/D, are increasing in $\overline{v}$. They are independent of all other model parameters.*

Finally, we move to the equilibrium behavior of the platform and the restaurant in the scarce driver regime. In contrast to customers and drivers, comparative statics for the platform's profit are the same regardless of whether drivers are scarce or abundant, as the following proposition states.

Proposition 10. *When drivers are scarce, the platform's profit, S_P, is strictly increasing in $\overline{D}$ and F and strictly decreasing in δ and $\overline{v}$.*

Intuitively, even when drivers are scarce, the platform continues to benefit from an increase in the driver pool or its fraction of the revenue, and continues to be harmed by an increase in customer impatience (which reduces the price the platform can charge) and the outside option value of drivers (which increases the wage the platform must pay to recruit all drivers). Hence, the conclusions of Proposition 10 are identical to those of Proposition 3.

By contrast, the restaurant's profit does exhibit some differences in behavior in the scarce driver regime, as can be seen in the following proposition.

Proposition 11. *When drivers are scarce, the restaurant's surplus, S_R, is strictly increasing in $\overline{D}$, strictly decreasing in δ and F, and independent of $\overline{v}$.*

Akin to the case of abundant drivers, S_R increases in $\overline{D}$ and decreases in δ when drivers are scarce. However, while S_R decreases in $\overline{v}$ when drivers are abundant, it is independent of $\overline{v}$ when there is a shortage of drivers. This is because the restaurant does not bear the costs of drivers.

For the dependence of S_R on F in Proposition 11, the context of the scarce driver regime is essential. Condition (16) implies that the scarce driver regime holds for all F above a threshold $\underline{F}$. Proposition 11 states that S_R is strictly decreasing in F on the interval $[\underline{F}, 1]$. By the maximum theorem (Kreps 2013), S_R is continuous, and by Proposition 6, S_R is quasi-concave on $[0, \min\{1, \underline{F}\}]$. So, we conclude that S_R is quasi-concave on $[0, 1]$. (For a detailed discussion on the implications of S_R being quasi-concave, see the paragraph following Proposition 6.)

5 Optimal Stacking Levels

Having derived the properties of the market equilibrium under an exogenous stacking level, we now consider the *optimal* choice of the stacking level and its impact on the market equilibrium. Specifically, Section 5.1 determines and compares the optimal stacking level from the point of view of each market participant. Section 5.2 discusses how the optimal stacking levels change in response to changes in market conditions.

5.1 Comparing Optimal Stacking Levels

We begin by discussing the case where drivers are abundant, and consider how changes in L impact the market participants. To see this, consider a focal customer located at x. From (1) and (2), we see that such a customer prefers L to be smaller because this decreases their wait time. This customer may therefore want to see a decrease in L (see, e.g., Reddit 2022), but in equilibrium, the convenience of a shorter wait time comes at the cost of a higher price P, which in turn reduces the customer's surplus.

This trade-off between cost and convenience is manifested in the customer surplus being unimodal in L, and similar trade-offs affect the payoffs of the other market participants as well. Before stating our first result in this section, recall that Proposition 1 implies that a market exists if and only if $0 < L < \widehat{L} := (2V + d\delta)/(d\delta)$. The upper bound $\widehat{L}$ plays an important role in the following theorem.

Theorem 1. *When drivers are abundant, the surpluses of all four market participants, as well as social welfare, are quasi-concave in L, strictly increasing from zero at $L = 0$ to their maximum, and strictly decreasing afterwards until they equal zero at $L = \widehat{L}$.*

Hence, each market participant has an "optimal" stacking level that maximizes their surplus. These are *unconstrained* optimizers in the sense that they do not necessarily satisfy $L \geq 1$. We define the *constrained* maximizers as follows: C and S_C are maximized at L_{S_C}, D and S_D are maximized at L_{S_D}, S_R is maximized at L_{S_R}, S_P is maximized at L_{S_P}, and S is maximized at L_S. Interestingly, as the following theorem shows, these maximizers can be strictly ordered when customers highly value the service.

Theorem 2. *When drivers are abundant and the customer valuation V is sufficiently high, $(2\widehat{L})/3 \geq L_{S_C} > L_{S_P} > L_{S_R} > L_{S_D} > 1$ and $\widehat{L}/2 \geq L_{S_P}$. As the customer valuation V decreases, the strict inequalities collapse into equality from right to left.*

Figure 2 displays the platform's profit and the payoffs of customers, drivers, and the restaurant in a numerical example, in which one can see the ordering in Theorem 2.

To understand the intuition behind Theorem 2, let us first compare L_{S_C} with L_{S_D}. Theorem 2 states that customers' optimal stacking level is higher than that of drivers ($L_{S_C} > L_{S_D}$). This is because, by definition, when the platform stacks orders ($L > 1$), each driver delivers more than one order, so raising L leads to stronger growth in the equilibrium customer mass (relative to the equilibrium driver mass), which makes the marginal value of increasing L higher for S_C (relative to S_D). To see this, first recall that in equilibrium, S_C and S_D are proportional to C^2 and D^2, respectively. As $L = C/D$, we have $C^2 = D^2L^2$. Thus, at the value of L maximizing S_D, we know that S_C must be increasing. Consequently, S_C is maximized at a higher value of L than S_D.

The restaurant's equilibrium surplus depends on the total revenue, which accounts for both driver and customer incentives—the equilibrium value of S_R is essentially a weighted average of S_C and S_D, and hence, the restaurant's optimal stacking level lies between those of customers and drivers ($L_{S_C} > L_{S_R} > L_{S_D}$). The restaurant does not have to pay drivers, but the platform

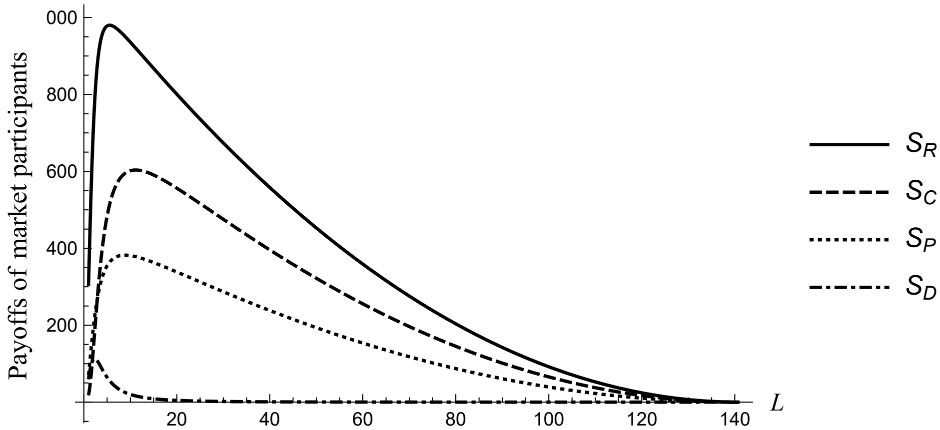


Figure 2. Optimal stacking levels in the abundant driver regime. The figure displays the payoff functions S_C, S_D, S_R, and S_P for different values of L. The plot starts at $L = 1$ and ends at $L = \widehat{L} = 141$. The problem parameters are $V = 35$, $\overline{v} = 10$, $\overline{D} = 30$, $\delta = 0.2$, $d = 2.5$, and $F = 0.3$. The constraint $D \leq \overline{D}$ does not bind for any value of L with these problem parameters.

does. Because the platform has to bear this cost, its incentives are more closely aligned (relative to the restaurant) with maximizing the customer surplus than with maximizing the driver surplus ($L_{S_C} > L_{S_P} > L_{S_R}$).

Because of these facts, the optimal stacking levels for each market participant follow the specific ordering in Theorem 2. The quasi-concave shape of the surplus functions as well as the ordering of their maximizers immediately implies several results. Based on the recent industry trends of increasing order stacking in food delivery platforms (McKinsey & Company 2021), suppose that the platform in our model starts out with a low value of L, and seeks to increase this value by stacking more orders together. We deduce from Theorems 1 and 2 that, when drivers are abundant and the customer valuation V is not too small, increasing L from a low value (e.g., its minimum, $L = 1$) increases the surpluses of all market participants. In other words, a little bit of stacking is beneficial to everyone in this case.

However, as L increases further, drivers are the first to experience a negative impact, followed by the restaurant. This suggests that in practice, a platform's order stacking may result in drivers and restaurants becoming dissatisfied, even when the demand for the platform's service is strong. This puts the platform's long-term viability into doubt, because a food delivery platform that cannot recruit drivers and restaurants to participate cannot sustain itself.

Indeed, because the platform's optimal stacking level is higher than those of drivers and the restaurant, Theorem 2 indicates that drivers and the restaurant might feel that the platform's preferred stacking level is too high. In practice, restaurants have become increasingly wary of participating in food delivery platforms. Many restaurants have noticed that their profitability could suffer substantially when they join a platform, and have responded in several ways, most notably by ending their relationship with delivery platforms or encouraging customers to order online and pick up at the restaurant (Rana and Haddon 2021b). Theorem 2 suggests that one reason for this could be excessive stacking, which benefits the platform but, at the platform's preferred level, negatively impacts drivers and the restaurant.

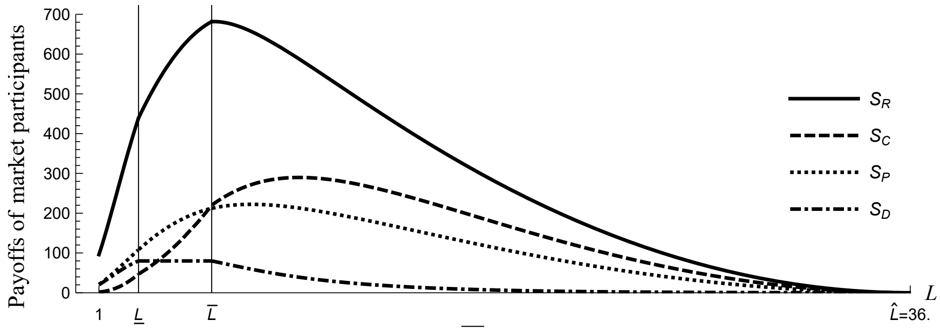
Theorem 2 also shows that customers prefer the *highest* level of stacking, even greater than the platform's optimal stacking level. This is surprising because, in a non-equilibrium setting, i.e., when P is exogenous, (3) implies that C is decreasing in L. The definition of S_C then implies that an increase in L leads to a decrease in S_C. In other words, with a fixed P, increasing L is bad for customers because of the increase in waiting time. In equilibrium, however, the platform responds to higher L by *lowering* the price, which increases the customer surplus in two ways: more customers enter the market, and customers who are already in the market pay a lower price. At the optimal stacking level for the platform, the price effect dominates the waiting time effect in the customer utility, meaning that customers prefer a higher stacking level than the platform. Hence, as long as the platform properly optimizes its price, customers can benefit from increased stacking up to (and beyond) the platform's optimal stacking level.

Lastly, we consider the scarce driver regime in the next result.

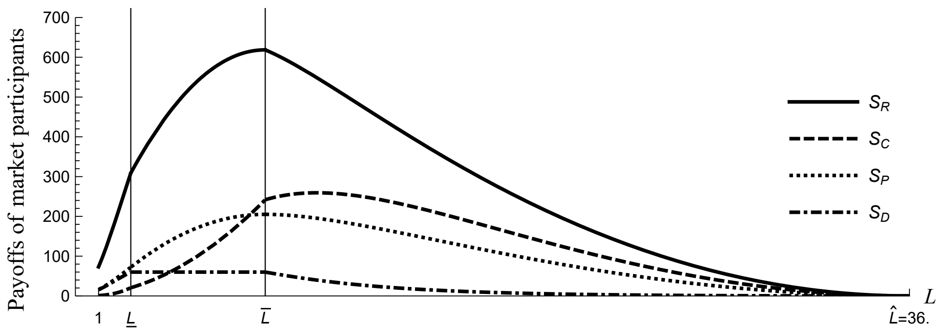
Theorem 3. *(i) There exists an interval* $[\underline{L}, \overline{L}]$ *such that if* $L \in [\underline{L}, \overline{L}]$*, then* $D = \overline{D}$ *(i.e., drivers are scarce), and therefore,* S_D *is constant.*[6]
(ii) For market participant $i \in \{C, R, P\}$*, drivers are scarce if and only if* $L_{S_i} \in [\underline{L}, \overline{L}]$*. In that case, the optimal stacking level for market participant* i *is given by a different quantity* L'_{S_i}*. These quantities satisfy* $L'_{S_R} = L'_{S_P}$ *and* $L'_{S_C} = \overline{L}$*.*

A simple verbal paraphrase of Theorem 3 is as follows. When there is a shortage of drivers, quasi-concavity of D as a function of L implies that there exists an interval $[\underline{L}, \overline{L}]$ on which D is equal to $\overline{D}$, i.e., the platform recruits all available drivers. On this interval, S_D is clearly also constant, so raising the stacking level in this interval does not affect the platform's costs. Accordingly, on this interval, both the platform and the restaurant prioritize increasing revenue as much as possible, which suppresses the misalignment in their incentives. In fact, if this interval contains L_{S_R} and L_{S_P} from Theorem 1, then both the restaurant and the platform have the *same* optimal stacking level. Another conclusion of Theorem 3 is that S_C is maximized at the right endpoint of the interval $[\underline{L}, \overline{L}]$, provided that the interval contains L_{S_C}. This is because $C = L\overline{D}$ in this interval, and so is increasing in L therein. Because S_C is proportional to C^2 in equilibrium, this implies that S_C is strictly increasing in the stacking level when there is a shortage of drivers. Figure 3 gives a graphical illustration of Theorem 3.

[6] This interval is precisely the interval for which the driver scarcity condition (16) holds.



(a) $\overline{D} = 8$



(b) $\overline{D} = 6$

Figure 3. Optimal stacking levels in the scarce driver regime. The figure displays the payoff functions S_C, S_D, S_R, and S_P for different values of L. The problem parameters are $V = 35$, $\overline{v} = 10$, $\delta = 0.2$, $d = 10$, and $F = 0.3$. In panel (a), $\overline{D} = 8$, and in panel (b), $\overline{D} = 6$.

5.2 Behavior of the Optimal Stacking Levels

We next investigate how the optimal stacking levels of different market participants depend on the underlying market conditions. Using expressions for L_{S_P}, L_{S_C}, and L_{S_D} derived in the proof of Theorem 1, we obtain the next result, which characterizes how the optimal stacking levels of the platform, customers, and drivers depend on the model parameters.[7]

Theorem 4. *(i) When drivers are abundant, the optimal stacking levels for the platform, customers, and drivers, namely, L_{S_P}, L_{S_C}, and L_{S_D}, are strictly increasing in V and $\overline{v}$ and strictly decreasing in δ, d, F, and $\overline{D}$.*
(ii) When drivers are scarce, the optimal stacking levels for the platform and the restaurant are strictly increasing in V, strictly decreasing in δ and $\overline{D}$, strictly decreasing in d if $V > \overline{D}\delta$ (strictly increasing in d if $V < \overline{D}\delta$), and independent of $\overline{v}$ and F. The optimal stacking level for customers is strictly increasing in V and F and strictly decreasing in δ, $\overline{D}$, $\overline{v}$, and d, except on a set of measure zero.

Theorem 4(i) states that, in the case of abundant drivers, *all* market participants prefer more order stacking if customers have a higher valuation V or if the maximum outside option value of drivers $\overline{v}$ is higher. All participants prefer less order stacking if customers are more impatient, the

[7] There is no closed-form expression for L_{S_R}, but we observe numerically that L_{S_R} appears to increase in V and $\overline{v}$ and decrease in δ, d, F, and $\overline{D}$, just like we show for the other optimal stacking levels.

drop-off time is longer, the platform's share of the revenue is higher, or there are more drivers. Hence, subject to an exogenous change in the market conditions (e.g., an increase or decrease in a parameter value), all market participants would prefer an adjustment in the stacking level that is directionally the same.

The story in Theorem 4(ii) is more nuanced. We start with the optimal stacking levels for the restaurant and platform, which are equal to one another. In the scarce driver regime, the mass of participating drivers does not change in response to changes in the model parameters. As mentioned in the preceding sections, this makes the platform's objective to maximize its revenue, irrespective of the cost of drivers, which is a constant in this regime. Because $\overline{v}$ and F specify the relative importance of maximizing revenue and minimizing costs in the platform's objective, they do not affect the optimal stacking level for the platform. Another difference with Theorem 4(i) is that the optimal stacking levels are increasing in the order drop-off time d when $V < \overline{D}\delta$. Note that this condition implies that customers are highly sensitive to their waiting time. An increase in the order drop-off time, therefore, reduces their utility significantly. The platform finds it optimal to respond by substantially dropping the price, thereby enticing more customers to participate.

In Theorem 4(ii), customers' optimal stacking level increases in V and decreases in δ, d, and $\overline{D}$, as in the abundant driver regime. The main difference here is that customers' optimal stacking level *increases* in F and *decreases* in $\overline{v}$ in the scarce drive regime. The reason for this difference is that in the abundant driver regime, the mass of participating drivers D is increasing in F and decreasing in $\overline{v}$ (see Proposition 5). Thus, when F is raised or $\overline{v}$ is lowered, D gets larger, expanding the set of all stacking levels for which the constraint $D \leq \overline{D}$ is binding. Because customers' payoff in the scarce driver regime is strictly increasing in the stacking level, they prefer the highest stacking level that is available to them. With the set of such stacking levels enlarged, their optimal stacking level becomes larger.

While the preferences of the market participants are directionally aligned as in Theorem 4, we observe that these preferences do not necessarily converge even under extreme market conditions. Figure 4 provides an example, showing how the maximum outside option value of drivers $\overline{v}$ affects the preferred stacking levels of each market participant. As $\overline{v} \to \infty$, L_{S_C} converges to $2(2V + d\delta)/(3d\delta)$, and L_{S_D} and L_{S_P} converge to $(2V + d\delta)/(2d\delta)$. Hence, by Theorem 2, L_{S_R} also converges to $(2V + d\delta)/(2d\delta)$ as $\overline{v} \to \infty$. In the example in Figure 4, $2(2V + d\delta)/(3d\delta) = 94$ and $(2V + d\delta)/(2d\delta) = 70.5$. Note that in Figure 4, convergence to these limits is not obvious despite the large values of $\overline{v}$ plotted; numerically observing these limits requires extending the plot to $\overline{v} \approx 100{,}000$. Hence, even for large $\overline{v}$, customers prefer more stacking than all other market participants, and the preferences of the restaurant, the platform, and drivers are not closely aligned.

Figure 5 displays how the optimal stacking levels depend on V, namely, customers' valuation for the service. As $V \to \infty$, L_{S_C} and L_{S_P} diverge to ∞, whereas L_{S_D} converges to a finite limit. Thus, Theorem 2 does not provide guidance on whether L_{S_R} tends to a finite limit or not. Nevertheless,

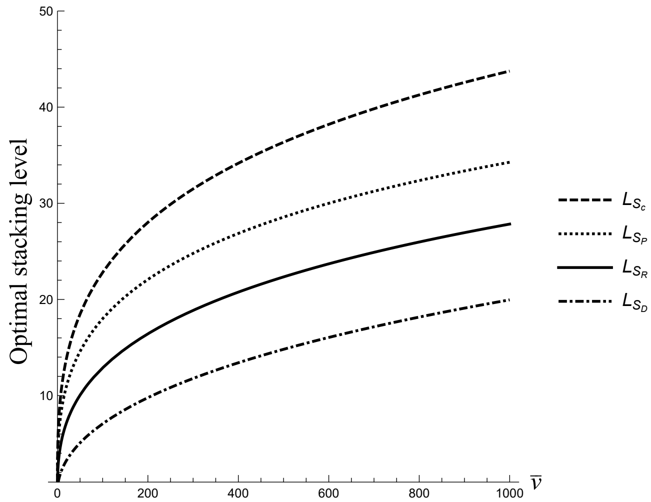


Figure 4. Impact of the maximum outside option value of drivers on optimal stacking. The figure displays L_{S_C}, L_{S_D}, L_{S_R}, L_{S_P}, and $\widehat{L}$ as functions of $\overline{v}$ for $V = 35$, $\overline{D} = 30$, $\delta = 0.2$, $d = 2.5$, and $F = 0.3$.

even if we extend the plot to $V \approx 10^{15}$, L_{S_R} does not appear to converge to a finite limit. Thus, L_{S_D} is the only optimal stacking level that provably converges to a finite limit as $V \to \infty$. This implies that as customers value the service more, the tension between preferred stacking level of drivers and of the other market participants increases: drivers always prefer the least amount of order stacking, and their preferred level becomes increasingly far from the other market participants' preferred level as V increases. This suggests that drivers may become increasingly unhappy as customers' valuation for the service increases and the platform responds by increasing the stacking level.

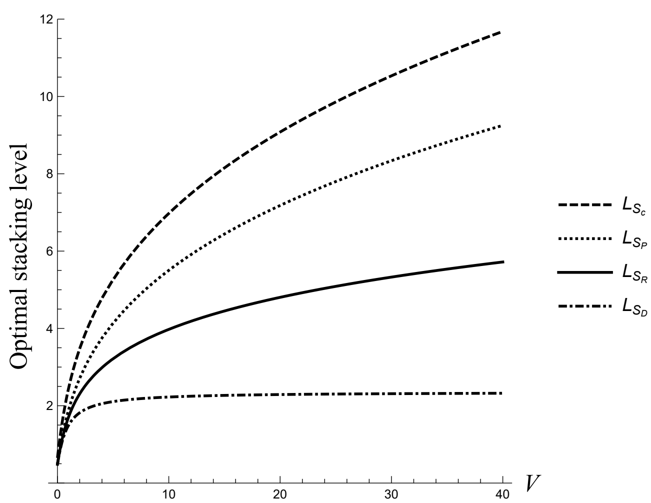


Figure 5. Impact of customers' valuation on optimal stacking. The figure displays L_{S_C}, L_{S_D}, L_{S_R}, L_{S_P}, and $\widehat{L}$ as functions of V for $\overline{D} = 30$, $\overline{v} = 10$, $\delta = .2$, $d = 2.5$, and $F = 0.3$.

6 Combining Order Stacking with Direct Deliveries

Our base model assumes that *all* customer orders are stacked by an identical amount. This provides operational efficiency for the platform, but increases customer wait times, which negatively impacts the customer utility. For the marginal customer located far from the restaurant, this increase in wait time may cause them to forgo participation (all else equal, i.e., holding the price constant). A natural question is then the following: can the platform do better by offering differentiated levels of service, i.e., making stacked deliveries to some customers and direct deliveries to others?

In this section, we analyze this practice, which is followed by several food delivery platforms, including Uber Eats (where it is called "priority delivery") and DoorDash ("express" delivery). Our main goals are to determine how differentiated service impacts the results in our base model, and when offering differentiated levels of service is most valuable to the platform.

In practice, platforms typically allow customers to choose whether they want stacked or direct deliveries, with the latter option involving an additional fee. Incorporating customer choice between these options is complicated within the framework of our model.[8] However, we can readily analyze a variation of this practice in which the platform chooses one type of delivery—either stacked or direct—to offer each customer based on their location. Clearly, the platform can do no worse following this strategy than in our base model, and further this yields an upper bound on the platform's profit if it allows all customers to choose between the two types of delivery.

With two modes of delivery, the number of orders a given driver carries depends on whether it is serving direct delivery customers or stacked delivery customers. Let C_0 be the mass of stacked delivery customers, and C_1 be the mass of direct delivery customers; see Figure 6 for an illustration. Because the stacking level of drivers assigned to direct delivery customers is always equal to 1, we take L to be the stacking level of the stacked-delivery customers. This makes the mass of participating drivers equal to $(C_0/L) + C_1$. Other modeling and analysis steps we have undertaken are similar to those for the base model in Section 3; these details are provided in Appendix D.

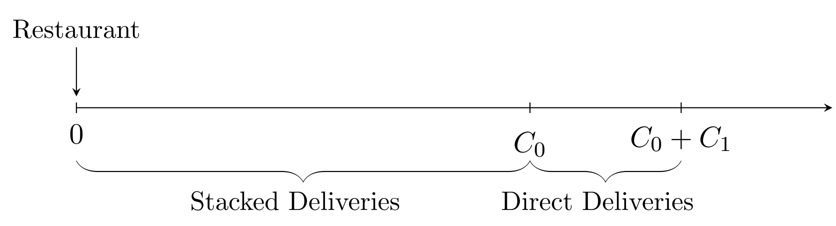


Figure 6. Linear city model and participating masses of stacked and direct delivery customers.

Figure 7 plots the equilibrium payoff functions of all parties as a function of the stacking level. The parameter values are the same as in Figure 2. In spite of the non-monotonic nature of some of these functions, the optimal stacking levels can be calculated as $L_{S_C} = 43.49$, $L_{S_P} = 8.84$,

[8] In addition, in practice, customers might choose stacked or direct deliveries due to factors outside our model, such as heterogeneous levels of impatience or heterogeneous valuations.

$L_{S_R} = 8.26$, and $L_{S_D} = 1.30$, which satisfy $L_{S_C} > L_{S_P} > L_{S_R} > L_{S_D}$.

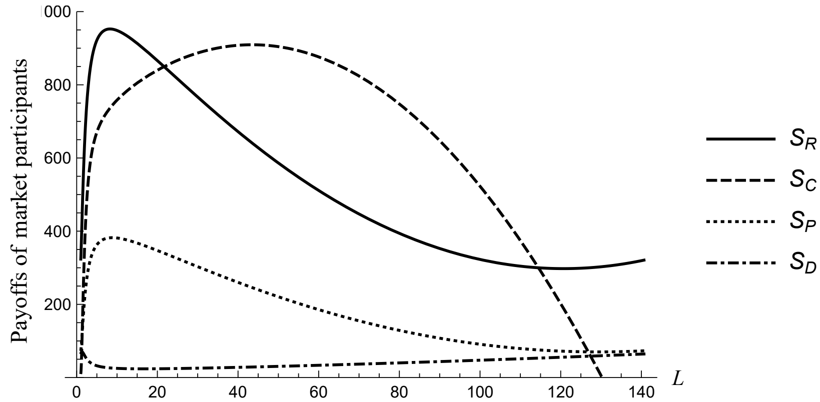


Figure 7. Optimal stacking levels with direct deliveries. The figure displays the payoff functions S_C, S_D, S_R, and S_P for different values of L when the platform can allow customers far away from the restaurant to order direct deliveries. The problem parameters are the same as in Figure 2.

We further show in Appendix D.5 that, in comparison to the base model where all customer orders are stacked, the platform is strictly better off letting at least some customers choose direct delivery except in the following two cases: the first case is when the delay imposed by stacking deliveries has a negligible effect on the customer utility—this can happen if the drop-off time d is small or the stacking level L is low. In this case, order stacking does not have a significant negative impact on the customer utility, even for customers who live far from the restaurant. The second case is characterized by multiple conditions, and one key condition is that the driver pool $\overline{D}$ is small, meaning that the platform has limited additional drivers to provide direct deliveries.

Figure 8 plots the platform's profits in the base model (only stacked deliveries) and the extended model of this section (direct deliveries may also be offered). Figure 8 shows that the value of offering direct deliveries increases in L. One can see similar behavior by fixing the stacking level and plotting the profits as functions of d, implying that offering direct deliveries is more valuable when the drop-off time is long. Intuitively, increasing d or L causes customers to experience a greater delay in receiving their food, which causes fewer customers to participate. It also leaves less customer utility on the table for the platform to extract. The platform can partially offset this by offering direct deliveries, which reduces the wait time for some customers.

7 Conclusion

Order stacking is increasingly used by on-demand delivery platforms to help reduce the cost per delivery and increase operational efficiency. However, this practice does not merely reduce operating costs of a platform. It also affects the utility of customers, drivers, and the service providers (e.g., restaurants) that use the platform. In this paper, we have analyzed these interactions and studied which market participants benefit the most (and the least) from order stacking.

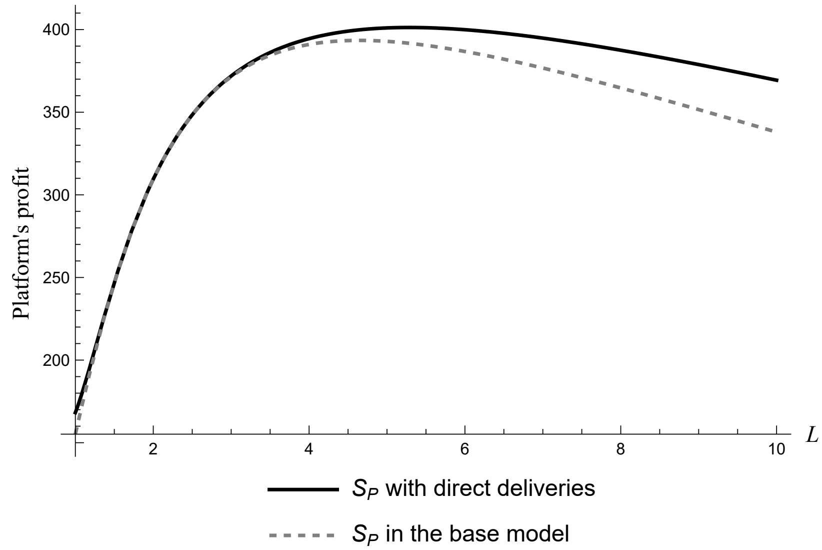


Figure 8. Value of direct deliveries. The dashed curve displays the platform's profit in the base model, and the solid curve displays the platform's profit in the extended model that allows for direct deliveries. The parameter values are $\overline{v} = 20$, $V = 100$, $\overline{D} = 20$, $\delta = 1.5$, $d = 2.5$, and $F = 0.3$.

We find that excessive order stacking can result in the market breaking down entirely, because stacking generates delays even for customers who live close to the service origin. However, stacking helps the platform in two ways: by reducing its operating costs, and by allowing it to operate more efficiently when there are insufficient drivers in the market. Thus, as long as customers' valuation for the service is not too small, the platform finds a positive stacking level optimal, and the platform's preferred stacking level increases if customer valuations or the maximum outside option value of drivers increases, or if the customer impatience, the drop-off time, the platform's revenue share, or the number of drivers decreases.

Interestingly, customers prefer even more stacking than the platform does, despite the fact that this practice increases their waiting time. This is because, if the platform properly optimizes (i.e., reduces) the price in response to stacking, more customers would participate in the market and all would pay a lower price. Hence, at the platform's optimal stacking level, customers are better off than with direct deliveries.

On the other hand, drivers prefer the least amount of stacking of any market participant. While the operating cost savings of order stacking benefit the platform, customers (via lower prices), and the restaurant (via market expansion), these savings come directly at the expense of drivers. Hence, at the platform's preferred stacking level, there are fewer drivers, and they are paid a lower wage, than at preferred stacking level of drivers.

The key downside of stacked deliveries is that they increase customer waits, which can cause customers who are already located far from the restaurant (and thus experience a long transportation time) to exit the market. Direct deliveries reduce this effect, but are costly to implement. Hence, we find that offering differentiated services—stacked deliveries to customers close to the restaurant, and direct deliveries to customers located far from the restaurant—can achieve the best of both worlds, both reducing operating costs and expanding the customer market. This happens under some mild conditions, specifically, if the drop-off time is non-negligible and the driver pool is not too small.

Our results highlight the complex interactions that result from order stacking for on-demand delivery platforms, particularly once market participants optimally respond to the stacking level. We thus conclude that while this practice can help platforms increase their operational efficiency, care should be taken to consider its impact on all market participants.

References

Astashkina, E., Momot, R., and Salikhov, M. (2022). Impact of workforce flexibility on customer satisfaction: Empirical framework & evidence from a cleaning services platform. Working paper.

Auad, R., Erera, A., and Savelsbergh, M. (2024). Capacity requirements and demand management strategies in meal delivery. *EURO Journal on Transportation and Logistics*, 13:100135.

Bahrami, S., Nourinejad, M., Nesheli, M. M., and Yin, Y. (2022). Optimal composition of solo and pool services for on-demand ride-hailing. *Transportation Research Part E: Logistics and Transportation Review*, 161:102680.

Bahrami, S., Nourinejad, M., Yin, Y., and Wang, H. (2023). The three-sided market of on-demand delivery. *Transportation Research Part E: Logistics and Transportation Review*, 179:103313.

Bai, J., So, K. C., Tang, C. S., Chen, X., and Wang, H. (2019). Coordinating supply and demand on an on-demand service platform with impatient customers. *Manufacturing & Service Operations Management*, 21(3):556–570.

Benjaafar, S., Ding, J.-Y., Kong, G., and Taylor, T. (2022). Labor welfare in on-demand service platforms. *Manufacturing & Service Operations Management*, 24(1):110–124.

Bernstein, F., DeCroix, G. A., and Keskin, N. B. (2021). Competition between two-sided platforms under demand and supply congestion effects. *Manufacturing & Service Operations Management*, 23(5):1043–1061.

Besbes, O., Castro, F., and Lobel, I. (2021). Surge pricing and its spatial supply response. *Management Science*, 67(3):1350–1367.

Bimpikis, K., Candogan, O., and Saban, D. (2019). Spatial pricing in ride-sharing networks. *Operations Research*, 67(3):744–769.

Business Wire (2023). DoorDash releases third quarter 2023 financial results. Accessed on January 29, 2024, https://www.businesswire.com/news/home/20231101615352/en/DoorDash-Releases-Third-Quarter-2023-Financial-Results.

Cachon, G. P., Daniels, K. M., and Lobel, R. (2017). The role of surge pricing on a service platform with self-scheduling capacity. *Manufacturing & Service Operations Management*, 19(3):368–384.

Carson, B. (2019). Uber's secret gold mine: How Uber Eats is turning into a billion-dollar business to rival Grubhub. *Forbes* (February 6), https://www.forbes.com/sites/bizcarson/2019/02/06/ubers-secret-gold-mine-how-uber-eats-is-turning-into-a-billion-dollar-business-to-rival-grubhub/?sh=254469d81fa9.

Castillo, J. C., Knoepfle, D. T., and Weyl, E. G. (2022). Matching in ride hailing: Wild goose chases and how to solve them. Working paper.

Chen, M. and Hu, M. (2024). Courier dispatch in on-demand delivery. *Management Science*, 70(6):3789–3807.

Chen, M., Hu, M., and Wang, J. (2022). Food delivery service and restaurant: Friend or foe? *Management Science*, 68(9):6539–6551.

Chen, Y.-J., Dai, T., Korpeoglu, C. G., Körpeoğlu, E., Sahin, O., Tang, C. S., and Xiao, S. (2020). OM forum—Innovative online platforms: Research opportunities. *Manufacturing & Service Operations Management*, 22(3):430–445.

Deliveroo (2022). Tech round-up: Stacking orders. Accessed on August 16, 2022, https://riders.deliveroo.com.sg/en/tech-round-up-stacking-orders.

DoorDash (2020). What is a "delivery radius" or a "delivery area" on DoorDash? Accessed on January 1, 2020, https://help.doordash.com/dashers/s/article/What-is-a-Delivery-Radius-or-a-Delivery-Area-on-DoorDash?language=en_US.

DoorDash (2021). Dasher pay. Accessed on December 18, 2021, https://dasher.doordash.com/en-us/pay.

DoorDash (2022). Why did I receive multiple orders at once? Accessed on August 16, 2022, https://help.doordash.com/dashers/s/article/Why-did-I-receive-multiple-orders-at-once?language=en_US.

Feldman, P., Frazelle, A. E., and Swinney, R. (2023). Managing relationships between restaurants and food delivery platforms: Conflict, contracts, and coordination. *Management Science*, 69(2):812–823.

Freund, D., Lobel, I., and Zhao, J. K. (2022). On the supply of autonomous vehicles in open platforms. Working paper.

Gopalakrishnan, R., Tulabandhula, T., and Mukherjee, K. (2019). Sequential individual rationality in dynamic ridesharing. Working paper.

Guda, H. and Subramanian, U. (2019). Your Uber is arriving: Managing on-demand workers through surge pricing, forecast communication, and worker incentives. *Management Science*, 65(5):1995–2014.

Hotelling, H. (1929). Stability in competition. *The Economic Journal*, 39(153):41–57.

Hu, B., Hu, M., and Zhu, H. (2022). Surge pricing and two-sided temporal responses in ride hailing. *Manufacturing & Service Operations Management*, 24(1):91–109.

Hu, M., Wang, J., and Wen, H. (2020). Share or solo? Individual and social choices in ride-hailing. Working paper.

Jacob, J. and Roet-Green, R. (2021). Ride solo or pool: Designing price-service menus for a ride-sharing platform. *European Journal of Operational Research*, 295(3):1008–1024.

Kaczmarski, M. (2024). Which company is winning the restaurant food delivery war? *Bloomberg Second Measure*. Accessed on January 29, 2024, https://secondmeasure.com/datapoints/food-delivery-services-grubhub-uber-eats-doordash-postmates/.

Ke, J., Wang, C., Li, X., Tian, Q., and Huang, H.-J. (2024). Equilibrium analysis for on-demand food delivery markets. *Transportation Research Part E: Logistics and Transportation Review*, 184:103467.

Kreps, D. M. (2013). *Microeconomic Foundations I: Choice and Competitive Markets*. Princeton University Press.

Lian, Z. and van Ryzin, G. (2023). Capturing the benefits of autonomous vehicles in ride-hailing: The role of market configuration. Working paper.

Liu, Z., Shen, Y., and Sun, Y. (2023). Operating three-sided marketplace: Pricing and spatial staffing in food delivery systems. Working paper.

Ma, H., Fang, F., and Parkes, D. C. (2022). Spatio-temporal pricing for ridesharing platforms. *Operations Research*, 70(2):1025–1041.

McKinsey & Company (2021). Ordering in: The rapid evolution of food delivery. Accessed on January 29, 2024, https://www.mckinsey.com/industries/technology-media-and-telecommunications/our-insights/ordering-in-the-rapid-evolution-of-food-delivery.

Rana, P. and Haddon, H. (2021a). Doordash and Uber Eats are hot. They're still not making money. *The Wall Street Journal* (May 28), https://www.wsj.com/articles/doordash-and-uber-eats-are-hot-theyre-still-not-making-money-11622194203.

Rana, P. and Haddon, H. (2021b). Restaurants and startups try to outrun Uber Eats and DoorDash. *The Wall Street Journal* (February 21), https://www.wsj.com/articles/restaurants-and-startups-try-to-outrun-uber-eats-and-doordash-11613903401.

Reddit (2018). Stacked orders = automatic low rating. Accessed on August 16, 2022, https://www.reddit.com/r/doordash/comments/9s81mu/stacked_orders_automatic_low_rating/.

Reddit (2020). How do you explain stacked orders to customers? Accessed on August 16, 2022, https://www.reddit.com/r/doordash/comments/ez4pcy/how_do_you_explain_stacked_orders_to_customers/.

Reddit (2021). Stacked orders are going to cause people to lower their tips. Accessed on August 16, 2022, https://www.reddit.com/r/doordash/comments/mdvotv/stacked_orders_are_going_to_cause_people_to_lower/.

Reddit (2022). Stacked orders need to stop or make sense. Accessed on August 16, 2022, https://www.reddit.com/r/doordash/comments/sah0ik/stacked_orders_need_to_stop_or_make_sense/.

Shaheen, S. and Cohen, A. (2019). Shared ride services in North America: Definitions, impacts, and the future of pooling. *Transport Reviews*, 39(4):427–442.

Siddiq, A. and Taylor, T. A. (2022). Ride-hailing platforms: Competition and autonomous vehicles. *Manufacturing & Service Operations Management*, 24(3):1511–1528.

Taylor, T. A. (2018). On-demand service platforms. *Manufacturing & Service Operations Management*, 20(4):704–720.

Taylor, T. A. (2024). Shared-ride efficiency of ride-hailing platforms. *Manufacturing & Service Operations Management*. Ahead of print. https://doi.org/10.1287/msom.2021.0545.

Uber (2021). How are delivery fares calculated? Accessed on December 18, 2021, https://help.uber.com/driving-and-delivering/article/how-are-delivery-fares-calculated?nodeId=5aecf430-8e00-4608-ba0a-8bba5b104023.

Uber (2022a). Delivering multiple orders. Accessed on August 16, 2022, https://www.uber.com/us/en/deliver/basics/making-deliveries/delivering-multiple-orders/.

Uber (2022b). How do multiple (batched) orders work? Accessed on August 16, 2022, https://help.uber.com/merchants-and-restaurants/article/how-do-multiple-batched-orders-work?nodeId=ad7637cb-1fc0-4902-b097-43105c557ce3.

Wang, H. and Yang, H. (2019). Ridesourcing systems: A framework and review. *Transportation Research Part B: Methodological*, 129:122–155.

Wang, X. and Zhang, R. (2022). Carpool services for ride-sharing platforms: Price and welfare implications. *Naval Research Logistics*, 69(4):550–565.

Wang, X. C., Kim, W., Holguín-Veras, J., and Schmid, J. (2021). Adoption of delivery services in light of the covid pandemic: Who and how long? *Transportation Research Part A: Policy and Practice*, 154:270–286.

Weinstein, A. and Luo, J. (2021). Using ML and optimization to solve DoorDash's dispatch problem. Accessed on August 16, 2022, https://doordash.engineering/2021/08/17/using-ml-and-optimization-to-solve-doordashs-dispatch-problem/.

Ye, A., Zhang, K., Chen, X. M., Bell, M. G., Lee, D.-H., and Hu, S. (2024). Modeling and managing an on-demand meal delivery system with order bundling. *Transportation Research Part E: Logistics and Transportation Review*, 187:103597.

Zhang, K. and Nie, Y. M. (2021a). Inter-platform competition in a regulated ride-hail market with pooling. *Transportation Research Part E: Logistics and Transportation Review*, 151:102327.

Zhang, K. and Nie, Y. M. (2021b). To pool or not to pool: Equilibrium, pricing and regulation. *Transportation Research Part B: Methodological*, 151:59–90.

Online Supplement to
"Order Stacking in On-Demand Delivery Platforms"

Appendix A: More on the Platform's Optimization Problem

Recall that the platform's optimization problem in Section 3 is as follows:

$$\max \quad F\left(V - \frac{d(L-1)\delta}{2} - \delta C\right)C - \frac{\overline{v}}{\overline{D}L^2}C^2$$
$$\text{subject to:} \quad 0 \le C \le \min\left\{\frac{V}{\delta} - \frac{d(L-1)}{2}, \overline{D}L\right\}.$$

The unconstrained optimizer for this problem and its objective value are given by

$$C = \frac{F\overline{D}L^2(2V - d(L-1)\delta)}{4(\overline{v} + F\overline{D}L^2\delta)} \quad \text{and} \quad S_P = \frac{F^2\overline{D}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)}. \tag{A.1}$$

Note that unconstrained optimizer is non-negative and always satisfies the following:

$$C \le \frac{V}{\delta} - \frac{d(L-1)}{2}.$$

In addition, for the unconstrained optimizer, we have $C \le \overline{D}L$ if and only if

$$\overline{D} \ge \frac{V}{\delta}\left(\frac{1}{2L}\right) - d\left(\frac{L-1}{4L}\right) - \frac{\overline{v}}{F\delta}\left(\frac{1}{L^2}\right).$$

If the unconstrained optimizer does not satisfy the inequality $C \le \overline{D}L$, then the optimal solution and corresponding objective value are

$$C = \overline{D}L \quad \text{and} \quad S_P = \frac{F\overline{D}L(2V - d(L-1)\delta - 2\overline{D}L\delta)}{2} - \overline{D}\overline{v}. \tag{A.2}$$

Table 2 displays a summary of the conditions characterizing the cases of constrained and unconstrained optimizers and the corresponding objective values.

condition	optimal platform profit
$\overline{D} \ge \frac{V}{\delta}\left(\frac{1}{2L}\right) - d\left(\frac{L-1}{4L}\right) - \frac{\overline{v}}{F\delta}\left(\frac{1}{L^2}\right)$	$S_P = \frac{F^2\overline{D}L^2(2V-d(L-1)\delta)^2}{16(\overline{v}+F\overline{D}L^2\delta)}$
$\overline{D} < \frac{V}{\delta}\left(\frac{1}{2L}\right) - d\left(\frac{L-1}{4L}\right) - \frac{\overline{v}}{F\delta}\left(\frac{1}{L^2}\right)$	$S_P = \frac{\overline{D}\left(FL(2V-d(L-1)\delta-2\overline{D}L\delta)-2\overline{v}\right)}{2}$

Table 2. Optimal platform profits when orders are stacked.

Appendix B: Equilibrium Utility Expressions

B.1 Abundant Driver Regime

In the abundant driver regime, substituting the equilibrium value of C in (A.1) yields the following expressions:

$$P = \frac{\left(2\overline{v} + F\overline{D}L^2\delta\right)\left(2V - d(L-1)\delta\right)}{4(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.1}$$

$$C = \frac{F\overline{D}L^2\left(2V - d(L-1)\delta\right)}{4(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.2}$$

$$D = \frac{F\overline{D}L\left(2V - d(L-1)\delta\right)}{4(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.3}$$

$$S_R = \frac{(1-F)F\overline{D}L^2(2\overline{v} + F\overline{D}L^2\delta)(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)^2}, \tag{B.4}$$

$$S_D = \frac{F^2\overline{D}\overline{v}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)^2}, \tag{B.5}$$

$$S_C = \frac{F^2\overline{D}^2L^4\delta(2V - d(L-1)\delta)^2}{32(\overline{v} + F\overline{D}L^2\delta)^2}, \tag{B.6}$$

$$S_P = \frac{F^2\overline{D}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.7}$$

$$S = \frac{F\overline{D}L^2(4\overline{v} + 3F\overline{D}L^2\delta)(2V - d(L-1)\delta)^2}{32(\overline{v} + F\overline{D}L^2\delta)^2}, \tag{B.8}$$

$$\frac{S_D}{D} = \frac{F\overline{v}L(2V - d(L-1)\delta)}{4(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.9}$$

$$\frac{S_C}{C} = \frac{F\overline{D}L^2\delta(2V - d(L-1)\delta)}{8(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.10}$$

$$\mathcal{V} = \frac{FL\overline{v}(2V - d(L-1)\delta)}{4(\overline{v} + F\overline{D}L^2\delta)}, \tag{B.11}$$

$$\mathcal{U}(x) = \delta(C - x). \tag{B.12}$$

B.2 Scarce Driver Regime

In the scarce drive regime, substituting C in (A.2) yields the following:

$$P = \frac{(2V - d(L-1)\delta) - 2\overline{D}L\delta}{2}, \tag{B.13}$$

$$C = \overline{D}L, \tag{B.14}$$

$$D = \overline{D}, \tag{B.15}$$

$$S_R = \frac{(1-F)\overline{D}L\left((2V - d(L-1)\delta) - 2\overline{D}L\delta\right)}{2}, \tag{B.16}$$

$$S_D = \overline{D}\overline{v}, \tag{B.17}$$

$$S_C = \frac{\overline{D}^2 L^2 \delta}{2}, \tag{B.18}$$

$$S_P = \frac{F\overline{D}L\left((2V - d(L-1)\delta) - 2\overline{D}L\delta\right) - 2\overline{D}\overline{v}}{2}, \tag{B.19}$$

$$S = \frac{\overline{D}L\left((2V - d(L-1)\delta) - \overline{D}L\delta\right)}{2}, \tag{B.20}$$

$$\frac{S_D}{D} = \overline{v}, \tag{B.21}$$

$$\frac{S_C}{C} = \frac{\overline{D}L\delta}{2}, \tag{B.22}$$

$$\mathcal{V} = \overline{v}, \tag{B.23}$$

$$\mathcal{U}(x) = \delta(C - x). \tag{B.24}$$

Appendix C: Proofs and Supporting Results for Sections 4 and 5

We first present the proofs of the results in Section 4.

Proof of Proposition 1. Equations (B.1) to (B.7) show that the equilibrium price is positive and all parties participate if and only if the market condition $2V - d(L-1)\delta > 0$ holds.

Considering equation (B.3), the unconstrained optimal value of D exceeds $\overline{D}$ (i.e., there is a shortage of drivers) if and only if condition (16) holds. If all parameters are positive and condition (16) holds, then equations (B.13) to (B.19) are satisfied, implying that the equilibrium price is positive and all parties participate in this case as well. □

Proof of Proposition 2. Differentiate equations (B.4) to (B.7) with respect to V and use (15). Then, do the same with respect to d. □

Proof of Proposition 3. Differentiate equation (B.7) with respect to $\overline{D}$ and use (15). Repeat for F, δ, and $\overline{v}$. □

Proof of Proposition 4. For (i), differentiate equations (B.2) and (B.6) with respect to $\overline{D}$, F, and $\overline{v}$ and use (15).

For (ii), differentiate equation (B.2) with respect to δ, and observe that the derivatives of S_C and S_C/C with respect to δ can be written as

$$\partial_\delta S_C = \frac{F^2\overline{D}^2L^4(2V - d(L-1)\delta)}{32(\overline{v} + F\overline{D}L^2\delta)^3} g_{S_C}(\delta),$$

$$\partial_\delta \frac{S_C}{C} = \frac{F\overline{D}L^2}{8(\overline{v} + F\overline{D}L^2\delta)^2} g_{S_C/C}(\delta),$$

where

$$g_{S_C}(\delta) = 2V\overline{v} - (3d\overline{v}(L-1) + 2F\overline{D}VL^2)\delta - F\overline{D}d(L-1)L^2\delta^2,$$

$$g_{S_C/C}(\delta) = 2V\overline{v} - 2d\overline{v}(L-1)\delta - F\overline{D}d(L-1)L^2\delta^2.$$

Descartes' rule of signs implies that g_{S_C} and $g_{S_C/C}$ each have one positive real zero. Furthermore, g_{S_C} and $g_{S_C/C}$ are both positive to the left of their respective positive zeros because $g_{S_C}(0) = g_{S_C/C} = 2V\overline{v} > 0$, implying strict quasi-concavity of S_C and S_C/C. Their respective positive zeros are

$$\delta_{S_C} = \frac{\sqrt{\left(9d\overline{v}(L-1) + 2F\overline{D}VL^2\right)\left(d\overline{v}(L-1) + 2F\overline{D}VL^2\right)} - 3\left(d\overline{v}(L-1)\right) - 2\left(F\overline{D}VL^2\right)}{2dF\overline{D}(L-1)L^2}$$

$$\delta_{S_C}^{avg} = \sqrt{\frac{\overline{v}^2}{F^2\overline{D}^2L^4} + \frac{2V\overline{v}}{F\overline{D}d(L-1)L^2}} - \frac{\overline{v}}{F\overline{D}L^2},$$

by the quadratic formula. We have

$$\begin{aligned} g_{S_C}(\delta_{S_C}^{avg}) &= 2V\overline{v} - (3d\overline{v}(L-1) + 2F\overline{D}VL^2)\delta_{S_C}^{avg} - F\overline{D}d(L-1)L^2\delta_{S_C}^{avg\,2}, \\ &= g_{S_C/C}(\delta_{S_C}^{avg}) - \left(d\overline{v}(L-1) + 2F\overline{D}VL^2\right)\delta_{S_C}^{avg} \\ &= 0 - \left(d\overline{v}(L-1) + 2F\overline{D}VL^2\right)\delta_{S_C}^{avg} \\ &< 0, \end{aligned}$$

and because $g_{S_C}(0) > 0$, it must be that $\delta_{S_C}^{avg} > \delta_{S_C}$.

The quantity $\widehat{\delta} := (2V)/(d(L-1))$ is the largest δ for which (15) holds. Although not in the statement of the proposition, one can show that $\delta_{S_C}^{avg} < \widehat{\delta}$, by isolating the radical, squaring both sides of the inequality, and simplifying. □

Proof of Proposition 5. For (i), differentiate equations (B.3) and (B.5) with respect to F and δ and use (15).

For (ii), differentiate equations (B.3) and (B.9) with respect to $\overline{D}$ and $\overline{v}$ and use (15). Observe that the derivatives of S_D with respect to $\overline{v}$ and $\overline{D}$ can be written as

$$\begin{aligned} \partial_{\overline{v}} S_D &= \frac{F^2\overline{D}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)^3}(F\overline{D}L^2\delta - \overline{v}), \\ \partial_{\overline{D}} S_D &= \frac{F^2\overline{v}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)^3}(\overline{v} - F\overline{D}L^2\delta). \end{aligned}$$

Under condition (15), the preceding derivatives are positive for $\overline{v} < \overline{v}_{S_D} := F\overline{D}L^2\delta$ ($\overline{D} < \overline{D}_{S_D} := \overline{v}/(FL^2\delta)$) and negative for $\overline{v} > \overline{v}_{S_D}$ ($\overline{D} > \overline{D}_{S_D}$). □

Proof of Proposition 6. For (i), differentiate equation (B.4) with respect to $\overline{D}$, $\overline{v}$, and δ and use (15).

For (ii), first note that equation (B.4) vanishes at $F = 0$ and $F = 1$, and is positive for $F \in (0, 1)$. Then, observe that the derivative of equation (B.4) with respect to F can be written as

$$\partial_F S_R = \frac{\overline{D}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)^3} g_{S_R}(F),$$

where

$$g_{S_R}(F) = 2\overline{v}^2 - 4F\overline{v}^2 - 3F^2\overline{D}\overline{v}L^2\delta - F^3\overline{D}^2L^4\delta^2.$$

By Descartes' rule of signs, g_{S_R} has one positive real zero. Note that $g_{S_R}(0) > 0$ and $g_{S_R}(1) < 0$, implying that this zero lies in $(0, 1)$, and that $\partial_F S_R$ is positive to the left of it and negative to the right of it. Multiplying g_{S_R} by $-\overline{D}L^2\delta$, we obtain the function

$$h(F) = (\overline{D}L^2\delta)^3F^3 + 3\overline{v}(\overline{D}L^2\delta)^2F^2 + 4\overline{v}^2(\overline{D}L^2\delta)F - 2\overline{v}^2(\overline{D}L^2\delta)$$

with the same zeros. We do the transformation $\overline{D}L^2\delta F = z - \overline{v}$ and simplify to obtain

$$i(z) = h\left(\frac{z - \overline{v}}{\overline{D}L^2\delta}\right) = z^3 + \overline{v}^2 z - 2\overline{v}^2(\overline{v} + \overline{D}L^2\delta),$$

which is a depressed cubic in z. If z_0 is one of its zeros, then $F_0 = (z_0 - \overline{v})/(\overline{D}L^2\delta)$ is a zero of $h(F)$. By Descartes' rule of signs, $i(z)$ has one positive real zero and no negative real zeros. This zero can be expressed as

$$z_{S_R} = \frac{2}{\sqrt{3}}\overline{v}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{3\sqrt{3}}{\overline{v}^3}\left(\overline{v} + \overline{D}L^2\delta\right)\right)\right].$$

Setting $F_{S_R} = (z_{S_R} - \overline{v})/(\overline{D}L^2\delta)$, we obtain the following:

$$F_{S_R} = \frac{\overline{v}}{\overline{D}L^2\delta}\left[\frac{2}{\sqrt{3}}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{3\sqrt{3}}{\overline{v}^3}\left(\overline{v} + \overline{D}L^2\delta\right)\right)\right] - 1\right].$$

This completes the proof. □

Proof of Proposition 7. The proof follows from inspecting equations (B.16) to (B.19). □

Proof of Proposition 8. The proof follows from inspecting equations (B.14), (B.18) and (B.22). □

Proof of Proposition 9. The proof follows from inspecting equations (B.15), (B.17) and (B.21). □

Proof of Proposition 10. First, differentiate equation (B.19) with respect to $\overline{D}$, and then, use (16). The rest of the proof follows from inspecting equation (B.19). □

Proof of Proposition 11. First, differentiate equation (B.16) with respect to $\overline{D}$, and then, use (16). The rest follows from inspecting equation (B.16). □

The following proposition is an auxiliary result for Section 4.

Proposition C.1. *The quantities δ_{S_C} and $\delta_{S_C}^{avg}$ in Proposition 4 are strictly decreasing in L.*

Proof. By defining $c = (2F\overline{D}V)/(d\overline{v})$, we can write

$$\frac{2F\overline{D}}{\overline{v}}\delta_{S_C} = \sqrt{\left(\frac{9}{L^2} + \frac{c}{L-1}\right)\left(\frac{1}{L^2} + \frac{c}{L-1}\right)} - \left[\frac{3}{L^2} + \frac{c}{L-1}\right],$$

$$\frac{F\overline{D}}{\overline{v}}\delta_{S_C}^{avg} = \sqrt{\frac{1}{L^4} + \frac{c}{(L-1)L^2}} - \frac{1}{L^2}.$$

By elementary analysis, we deduce that the derivative of the right-hand sides with respect to L are negative for $L > 1$. □

We now present the proofs of the results in Section 5.

Proof of Theorem 1. Differentiating equations (B.2) to (B.8) yields

$$\begin{aligned}
\partial_L C &= \frac{F\overline{D}L}{4(\overline{v} + F\overline{D}L^2\delta)^2} g_{S_C}(L), \\
\partial_L S_C &= \frac{F^2\overline{D}^2 L^3\delta(2V - d(L-1)\delta)}{16(\overline{v} + F\overline{D}L^2\delta)^3} g_{S_C}(L), \\
\partial_L D &= \frac{F\overline{D}}{4(\overline{v} + F\overline{D}L^2\delta)^2} g_{S_D}(L), \\
\partial_L S_D &= \frac{F^2\overline{D}\overline{v}L(2V - d(L-1)\delta)}{8(\overline{v} + F\overline{D}L^2\delta)^3} g_{S_D}(L), \\
\partial_L S_R &= \frac{(1-F)F\overline{D}L(2V - d(L-1)\delta)}{8(\overline{v} + F\overline{D}L^2\delta)^3} g_{S_R}(L), \\
\partial_L S_P &= \frac{F^2\overline{D}L(2V - d(L-1)\delta)}{8(\overline{v} + F\overline{D}L^2\delta)^2} g_{S_P}(L), \\
\partial_L S &= \frac{F\overline{D}L(2V - d(L-1)\delta)}{16(\overline{v} + F\overline{D}L^2\delta)^3} g_S(L),
\end{aligned}$$

where

$$\begin{aligned}
g_{S_C}(L) &= 4V\overline{v} + 2d\overline{v}\delta - 3d\overline{v}L\delta - F\overline{D}dL^3\delta^2, \\
g_{S_D}(L) &= 2V\overline{v} + d\overline{v}\delta - 2d\overline{v}L\delta - \left(2F\overline{D}V\delta + F\overline{D}d\delta^2\right) L^2, \\
g_{S_R}(L) &= 4V\overline{v}^2 + 2d\overline{v}^2\delta - 4d\overline{v}^2L\delta - 3F\overline{D}d\overline{v}L^3\delta^2 - F^2\overline{D}^2 dL^5\delta^3, \\
g_{S_P}(L) &= 2V\overline{v} + d\overline{v}\delta - 2d\overline{v}L\delta - F\overline{D}dL^3\delta^2, \\
g_S(L) &= 8V\overline{v}^2 + 4d\overline{v}^2\delta - 8d\overline{v}^2L\delta + (4F\overline{D}V\overline{v}\delta + 2F\overline{D}d\overline{v}\delta^2)L^2 - 9F\overline{D}d\overline{v}L^3\delta^2 - 3F^2\overline{D}^2 dL^5\delta^3.
\end{aligned}$$

All of the preceding g-functions are positive when $L = 0$ and negative when L is sufficiently large, implying that each function has at least one positive real zero. By Descartes' rule of signs, g_{S_C}, g_{S_D}, g_{S_R}, and g_{S_P} each have one positive real zero, and g_S has either one or three positive real zeros. However, Descartes' rule of signs implies that $g_S(-L)$ has no positive real roots (i.e., g_S has no negative real roots), and because $L = 0$ is not a zero of g_S, there are either one or three real roots, all of which must be positive. The discriminant of g_S can be written as follows:

$$\begin{aligned}
\Delta_S = 1728F^6\overline{D}^6 d^2\delta^{11} &\left(d^2\overline{v}\delta + 4F\overline{D}V^2 + 4F\overline{D}Vd\delta + F\overline{D}d^2\delta^2\right) \\
&\cdot \left(128d^2\overline{v}\delta + 12F\overline{D}V^2 + 12F\overline{D}Vd\delta + 3F\overline{D}d^2\delta^2\right) \\
&\quad \cdot \left(25d^2\overline{v}\delta + 96F\overline{D}V^2 + 96F\overline{D}Vd\delta + 24F\overline{D}d^2\delta^2\right) > 0,
\end{aligned}$$

which implies that g_S has either four or zero nonreal zeros (i.e., g_S has either one or five real zeros). However, we showed that it cannot have five real zeros, leaving us to conclude that it must have one real zero, which we noted must be positive.

In sum, each of the g-functions is positive at $L = 0$, crosses the abscissa at some $L > 0$, and is thereafter negative. We respectively label the crossing points as L_{S_C}, L_{S_D}, L_{S_R}, L_{S_P}, and L_S. Formulas are available for expressing some of these zeros for the quadratic and cubic g-functions in terms of radicals:

$$L_{S_C} = \sqrt[3]{\frac{2V\overline{v} + d\overline{v}\delta}{F\overline{D}d\delta^2} + \sqrt{\left(\frac{2V\overline{v} + d\overline{v}\delta}{F\overline{D}d\delta^2}\right)^2 + \left(\frac{\overline{v}}{F\overline{D}\delta}\right)^3}} + \sqrt[3]{\frac{2V\overline{v} + d\overline{v}\delta}{F\overline{D}d\delta^2} - \sqrt{\left(\frac{2V\overline{v} + d\overline{v}\delta}{F\overline{D}d\delta^2}\right)^2 + \left(\frac{\overline{v}}{F\overline{D}\delta}\right)^3}},$$

$$L_{S_D} = \sqrt{\left[\frac{d\overline{v}}{F\overline{D}(2V + d\delta)}\right]^2 + \frac{\overline{v}}{F\overline{D}\delta}} - \frac{d\overline{v}}{F\overline{D}(2V + d\delta)},$$

$$L_{S_P} = \sqrt[3]{\frac{2V\overline{v} + d\overline{v}\delta}{2F\overline{D}d\delta^2} + \sqrt{\left(\frac{2V\overline{v} + d\overline{v}\delta}{2F\overline{D}d\delta^2}\right)^2 + \left(\frac{2\overline{v}}{3F\overline{D}\delta}\right)^3}} + \sqrt[3]{\frac{2V\overline{v} + d\overline{v}\delta}{2F\overline{D}d\delta^2} - \sqrt{\left(\frac{2V\overline{v} + d\overline{v}\delta}{2F\overline{D}d\delta^2}\right)^2 + \left(\frac{2\overline{v}}{3F\overline{D}\delta}\right)^3}}.$$

The zeros of g_{S_C} and g_{S_P} can also be expressed using hyperbolic trigonometric functions as

$$L_{S_C} = 2\sqrt{\frac{\overline{v}}{F\overline{D}\delta}} \sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{2V + d\delta}{d\delta}\sqrt{\frac{F\overline{D}\delta}{\overline{v}}}\right)\right],$$

$$L_{S_P} = 2\sqrt{\frac{2\overline{v}}{3F\overline{D}\delta}} \sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{2V + d\delta}{d\delta}\sqrt{\frac{27F\overline{D}\delta}{32\overline{v}}}\right)\right],$$

which are sometimes easier to analyze. Similarly, it is sometimes more convenient to use the identity

$$\operatorname{arcsinh}(-x) = \ln\left(\sqrt{x^2 + 1} - x\right) \text{ for all } x \in \mathbb{R},$$

and elementary algebra to write

$$\ln L_{S_D} = \ln\left(\sqrt{\frac{\overline{v}}{F\overline{D}\delta}}\right) - \operatorname{arcsinh}\left(\frac{d\delta}{2V + d\delta}\sqrt{\frac{\overline{v}}{F\overline{D}\delta}}\right).$$

This completes the proof. □

Proof of Theorem 2. We continue to use the notation introduced in the proof of Theorem 1 as appropriate. Let $s = (V\overline{v})/(F\overline{D}d\delta^2)$ and $t = \overline{v}/(F\overline{D}\delta)$ so that

$$L_{S_C} = \sqrt[3]{(2s + t) + \sqrt{(2s + t)^2 + t^3}} + \sqrt[3]{(2s + t) - \sqrt{(2s + t)^2 + t^3}},$$

$$L_{S_P} = \sqrt[3]{\left(s + \frac{t}{2}\right) + \sqrt{\left(s + \frac{t}{2}\right)^2 + \frac{8}{27}t^3}} + \sqrt[3]{\left(s + \frac{t}{2}\right) - \sqrt{\left(s + \frac{t}{2}\right)^2 + \frac{8}{27}t^3}},$$

$$L_{S_D} = \sqrt{\left(\frac{t^2}{2s+t}\right)^2 + t} - \frac{t^2}{2s+t},$$

and $\widehat{L} = (2s+t)/t$. All roots in this proof are principal value roots. Recall from the proof of Theorem 1 that L_{S_P} is the unique real root of the cubic equation

$$g_{S_P}(L) = 2V\overline{v} + d\overline{v}\delta - 2d\overline{v}L\delta - F\overline{D}dL^3\delta^2 = 0,$$

which after dividing by the leading coefficient, can be written as follows:

$$h_{S_P}(L) \coloneqq L^3 + 2tL - (2s+t) = 0.$$

We have

$$\begin{aligned}
&h_{S_P}(L_{S_C}) \\
&= (2s+t) + \sqrt{(2s+t)^2+t^3} + 3\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{2}{3}}\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}} \\
&\qquad + 3\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}}\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{2}{3}} + (2s+t) - \sqrt{(2s+t)^2+t^3} \\
&\qquad\qquad + 2tL_{S_C} - (2s+t) \\
&= (2s+t) + 2tL_{S_C} + 3\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{2}{3}}\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}} \\
&\qquad\qquad + 3\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}}\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{2}{3}} \\
&= (2s+t) + 2tL_{S_C} + 3\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}}\left((2s+t)^2 - (2s+t)^2 - t^3\right)^{\frac{1}{3}} \\
&\qquad\qquad + 3\left((2s+t)^2 - (2s+t)^2 - t^3\right)^{\frac{1}{3}}\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}} \\
&= 2s + 2tL_{S_C} - 3t\left((2s+t) + \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}} - 3t\left((2s+t) - \sqrt{(2s+t)^2+t^3}\right)^{\frac{1}{3}} \\
&= (2s+t) + 2tL_{S_C} - 3tL_{S_C} \\
&= (2s+t) - tL_{S_C} > 0 = h_{S_P}(L_{S_P}),
\end{aligned}$$

where the inequality follows by noting that $(2s+t) - tL_{S_C}$ is equivalent to $L_{S_C} < \widehat{L}$, which is established below. To finish this argument, note that $h'_{S_P}(L) = 3L^2 + 2t > 0$, so h_{S_P} is strictly increasing on $\mathbb{R}$. So, $h_{S_P}(L_{S_C}) > h_{S_P}(L_{S_P})$ implies that $L_{S_C} > L_{S_P}$.

We now proceed indirectly to show that $L_{S_C} \leq \frac{2}{3}\widehat{L}$. Recall that L_{S_C} is the unique real root of

$$g_{S_C}(L) = 4V\overline{v} + 2d\overline{v}\delta - 3d\overline{v}L\delta - F\overline{D}dL^3\delta^2 = 0,$$

which, after dividing by the leading coefficient, can be written as the following:

$$h_{S_C}(L) \coloneqq L^3 + 3tL - (4s+2t) = 0.$$

The polynomial

$$\begin{aligned}
h_{S_C}\left(L+\frac{2}{3}\widehat{L}\right) &= \left(L+\frac{2}{3}\widehat{L}\right)^3 + 3t\left(L+\frac{2}{3}\widehat{L}\right) - (4s+2t) \\
&= L^3 + 2\widehat{L}L^2 + \frac{4}{3}\widehat{L}^2 L + \frac{8}{27}\widehat{L}^3 + 3tL + 2t\widehat{L} - (4s+2t) \\
&= L^3 + 2\widehat{L}L^2 + \left(\frac{4}{3}\widehat{L}^2 + 3t\right)L + \frac{8}{27}\widehat{L}^3 + 2t\left(\frac{2s}{t}+1\right) - (4s+2t) \\
&= L^3 + 2\widehat{L}L^2 + \left(\frac{4}{3}\widehat{L}^2 + 3t\right)L + \frac{8}{27}\widehat{L}^3
\end{aligned}$$

has no positive zeros. But, $L_{S_C} - \frac{2}{3}\widehat{L}$ is a real root of it, so we must have $L_{S_C} \leq \frac{2}{3}\widehat{L}$.

We now show that $L_{S_P} \leq \frac{1}{2}\widehat{L}$. Note that the polynomial

$$\begin{aligned}
h_{S_P}\left(L+\frac{1}{2}\widehat{L}\right) &= \left(L+\frac{1}{2}\widehat{L}\right)^3 + 2t\left(L+\frac{1}{2}\widehat{L}\right) - (2s+t) \\
&= L^3 + \frac{3}{2}\widehat{L}L^2 + \frac{3}{4}\widehat{L}^2 L + \frac{1}{8}\widehat{L}^3 + 2tL + t\widehat{L} - (2s+t) \\
&= L^3 + \frac{3}{2}\widehat{L}L^2 + \left(\frac{3}{4}\widehat{L}^2 + 2t\right)L + \frac{1}{8}\widehat{L}^3 + t\left(\frac{2s}{t}+1\right) - (2s+t) \\
&= L^3 + \frac{3}{2}\widehat{L}L^2 + \left(\frac{3}{4}\widehat{L}^2 + 2t\right)L + \frac{1}{8}\widehat{L}^3
\end{aligned}$$

has no positive roots. However, $L_{S_P} - \frac{1}{2}\widehat{L}$ is a real root of $h_{S_P}\left(L+\frac{1}{2}\widehat{L}\right)$, and therefore must be non-positive. Hence, $L_{S_P} \leq \frac{1}{2}\widehat{L}$.

To show that $L_{S_P} > L_{S_D}$, we first note that algebraically manipulating the expression for L_{S_D} allows us to write

$$(2s+t) = \frac{\sqrt{t^4 + (2s+t)^2 t} - t^2}{L_{S_D}}.$$

Therefore, we have

$$\begin{aligned}
h_{S_P}(L_{S_D}) &= L_{S_D}^3 + 2tL_{S_D} - (2s+t) \\
&= L_{S_D}^3 + 2tL_{S_D} - \frac{\sqrt{t^4 + (2s+t)^2 t} - t^2}{L_{S_D}} \\
&= L_{S_D}^3 + 2tL_{S_D} + \frac{t^2 - \sqrt{t^4 + (2s+t)^2 t}}{L_{S_D}},
\end{aligned}$$

which, because $L_{S_D} > 0$, is positive if and only if

$$L_{S_D} h_{S_P}(L_{S_D}) = L_{S_D}^4 + 2tL_{S_D}^2 + t^2 - \sqrt{t^4 + (2s+t)^2 t} > 0.$$

The equation $L_{S_D} h_{S_P}(L_{S_D}) = 0$ has a lone positive real root by Descartes' rule of signs, which can be expressed as

$$\sqrt{\sqrt[4]{t^4 + (2s+t)^2 t} - t}.$$

Because h_{S_P} is strictly increasing on $\mathbb{R}$, we have that

$$h_{S_P}(L_{S_D}) > 0 \iff L_{S_D} = \sqrt{\left(\frac{t^2}{2s+t}\right)^2 + t} - \frac{t^2}{2s+t} > \sqrt{\sqrt[4]{t^4 + (2s+t)^2 t} - t},$$

and the same with both inequalities reversed. Accordingly, we have the following

$$\begin{aligned}
&\sqrt{\left(\frac{t^2}{2s+t}\right)^2 + t} - \frac{t^2}{2s+t} > \sqrt{\sqrt[4]{t^4 + (2s+t)^2 t} - t} \\
\iff\ &\sqrt{t^4 + (2s+t)^2 t} - t^2 > (2s+t)\sqrt{\sqrt[4]{t^4 + (2s+t)^2 t} - t} \\
\iff\ &\left(\sqrt[4]{t^4 + (2s+t)^2 t} - t\right)\left(\sqrt[4]{t^4 + (2s+t)^2 t} + t\right) > (2s+t)\sqrt{\sqrt[4]{t^4 + (2s+t)^2 t} - t} \\
\iff\ &\left(\sqrt[4]{t^4 + (2s+t)^2 t} - t\right)\left(\sqrt[4]{t^4 + (2s+t)^2 t} + t\right)^2 > (2s+t)^2 \\
\iff\ &\left(\sqrt[4]{1 + \frac{(2s+t)^2}{t^3}} - 1\right)\left(\sqrt[4]{1 + \frac{(2s+t)^2}{t^3}} + 1\right)^2 > \frac{(2s+t)^2}{t^3}.
\end{aligned}$$

We show that this condition never holds and conclude that $h_{S_P}(L_{S_D})$ is always negative. To do so, we show that the function

$$x \mapsto \left(\sqrt[4]{1+x} - 1\right)\left(\sqrt[4]{1+x} + 1\right)^2 - x$$

is negative on $\mathbb{R}_{++}$. We can make the variable substitution $y = \sqrt[4]{1+x}$. With regard to this new variable, we note that

$$f(y) = (y-1)(y+1)^2 - y^4 + 1 = -y(y+1)(y-1)^2$$

is negative for $y > 1$. Hence, $h_{S_P}(L_{S_D}) < 0 = h_{S_P}(L_{S_P})$. Because h_{S_P} is strictly increasing on $\mathbb{R}$, this implies that $L_{S_P} > L_{S_D}$.

Recall from the proof of Theorem 1 that L_{S_R} is the unique positive real zero of the function

$$g_{S_R}(L) = 4V\overline{v}^2 + 2d\overline{v}^2\delta - 4d\overline{v}^2 L\delta - 3F\overline{D}d\overline{v}L^3\delta^2 - F^2\overline{D}^2 dL^5\delta^3,$$

and is therefore the unique positive real zero of the function

$$h_{S_R}(L) = L^5 + 3tL^3 + 4t^2 L - 2t(2s+t),$$

which is obtained by dividing g_{S_R} by its leading coefficient. Recall also that L_{S_P} is the unique positive real zero of $h_{S_P}(L)$, whence

$$L_{S_P}^3 + 2tL_{S_P} - (2s+t) = 0.$$

Therefore, we have

$$h_{S_R}(L_{S_P}) = L_{S_P}^5 + 3tL_{S_P}^3 + 2t\left(2tL_{S_P} - (2s+t)\right)$$

$$\begin{aligned} &= L_{S_P}^5 + 3tL_{S_P}^3 - 2tL_{S_P}^3 \\ &= L_{S_P}^5 + tL_{S_P}^3 > 0 = h_{S_R}\left(L_{S_R}\right). \end{aligned}$$

Because h_{S_R} is increasing on $\mathbb{R}$, this implies that $L_{S_P} > L_{S_R}$.

We now show that $L_{S_R} > L_{S_D}$. Recall that L_{S_D} is the unique positive real zero of the function

$$g_{S_D}(L) = 2V\overline{v} + d\overline{v}\delta - 2d\overline{v}L\delta - \left(2F\overline{D}V\delta + F\overline{D}d\delta^2\right)L^2,$$

which, after dividing by the leading coefficient, can be written

$$h_{S_D}(L) = L^2 + \frac{2t^2}{2s+t}L - t.$$

Algebraically manipulating the expression $h_{S_D}\left(L_{S_D}\right) = 0$ yields

$$-2(2s+t)L_{S_D}^2 = 4t^2L_{S_D} - 2t(2s+t).$$

Therefore, we have

$$\begin{aligned} h_{S_R}\left(L_{S_D}\right) &= L_{S_D}^5 + 3tL_{S_D}^3 + \left(4t^2L_{S_D} - 2t(2s+t)\right) \\ &= L_{S_D}^5 + 3tL_{S_D}^3 - 2(2s+t)L_{S_D}^2 \\ &= L_{S_D}^2 h_{S_C}\left(L_{S_D}\right). \end{aligned}$$

As a result, we have that $L_{S_C} > L_{S_P} > L_{S_D}$, which implies that $h_{S_C}\left(L_{S_D}\right) < 0$ (because h_{S_C} is increasing on $\mathbb{R}$). However, we also have that $h_{S_R}\left(L_{S_D}\right) < 0 = h_{S_R}\left(L_{S_R}\right)$, and because h_{S_R} is increasing on $\mathbb{R}$, we deduce that $L_{S_R} > L_{S_D}$.

For the second sentence of the theorem, note that if L_{S_C}, L_{S_D}, L_{S_R}, and L_{S_P} are at least 1, then they are the maximizers of their respective payoffs; otherwise, $L = 1$ is the maximizer. The condition that $L_{S.} \geq 1$ is equivalent to $g_{S.}$ being non-negative at $L_{S.}$ (i.e., the maximum is to the right of 1 if the derivative is increasing at 1). We have

$$\begin{aligned} \frac{g_{S_C}(1)}{d\overline{v}\delta} &= \frac{4V}{d\delta} - 1 - \frac{F\overline{D}\delta}{\overline{v}}, \\ \frac{g_{S_D}(1)}{\overline{v}} &= 2V - d\delta - \frac{F\overline{D}\delta\left(2V + d\delta\right)}{\overline{v}}, \\ \frac{g_{S_R}(1)}{d\overline{v}^2\delta} &= 2\left(\frac{2V}{d\delta} - 1\right) - \frac{F\overline{D}\delta}{\overline{v}}\left(3 + \left(\frac{F\overline{D}\delta}{\overline{v}}\right)\right), \\ \frac{g_{S_P}(1)}{d\overline{v}\delta} &= \frac{2V}{d\delta} - 1 - \frac{F\overline{D}\delta}{\overline{v}}, \end{aligned}$$

whence

$$\begin{aligned} L_{S_C} \geq 1 &\iff \frac{2V}{d\delta} \geq \frac{1}{2} + \frac{F\overline{D}\delta}{2\overline{v}}, \\ L_{S_D} \geq 1 &\iff \frac{2V - d\delta}{2V + d\delta} \geq \frac{F\overline{D}\delta}{\overline{v}}, \end{aligned}$$

$$
\begin{aligned}
L_{S_R} \geq 1 &\iff \frac{2V}{d\delta} \geq 1 + \frac{F\overline{D}\delta}{2\overline{v}}\left(3 + \left(\frac{F\overline{D}\delta}{\overline{v}}\right)\right), \\
L_{S_P} \geq 1 &\iff \frac{2V}{d\delta} \geq 1 + \frac{F\overline{D}\delta}{\overline{v}}.
\end{aligned}
$$

The left-hand sides are strictly increasing in V, implying that the inequalities become violated as V decreases. The chain of implications

$$
L_{S_C} \leq 1 \implies L_{S_P} \leq 1 \implies L_{S_R} \leq 1 \implies L_{S_D} \leq 1,
$$

which implies the second sentence of the theorem, is trivial to establish, with the exception of the final implication. To prove that one, note that $L_{S_D} \leq 1$ is equivalent to

$$
\frac{2V}{d\delta} \leq 1 + \frac{F\overline{D}\delta}{\overline{v}}\left(1 + \frac{2V}{d\delta}\right),
$$

which is implied by $L_{S_R} \leq 1$ if

$$
\frac{3}{2} + \frac{F\overline{D}\delta}{2\overline{v}} \leq 1 + \frac{2V}{d\delta}.
$$

But this last inequality is equivalent to $L_{S_C} \geq 1$, so the final implication holds if $L_{S_C} \geq 1$. This is all we need since $L_{S_C} \leq 1$ implies $L_{S_D} < 1$ by the first part of the theorem. □

Proof of Theorem 3. For part (i), note that the driver scarcity condition (16) can be rewritten as $L \in [\underline{L}, \overline{L}]$ for some $\underline{L}$ and $\overline{L}$. By elementary algebra, we can express $\overline{L}$ as

$$
\overline{L} = \frac{2V + d\delta + \sqrt{(2V + d\delta)^2 - 16\overline{v}(d\delta + 4\overline{D}\delta)/F}}{2(d\delta + 4\overline{D}\delta)}, \tag{C.1}
$$

observing that the interval $[\underline{L}, \overline{L}]$ exists only if the expression in the radical in equation (C.1) is non-negative. (If the expression in the radical is negative, then for all values of L, the unconstrained optimal solution satisfies the constraint $D \leq \overline{D}$, meaning that there is no scarce driver regime.)

Differentiating equations (B.13) to (B.20) yields

$$
\begin{aligned}
\partial_L P &= -\frac{d\delta + 2\overline{D}\delta}{2}, \\
\partial_L C &= \overline{D}, \\
\partial_L S_C &= \overline{D}^2 L\delta, \\
\partial_L D &= 0, \\
\partial_L S_D &= 0, \\
\partial_L S_R &= \frac{(1 - F)\overline{D}}{2} g_{S_R}(L), \\
\partial_L S_P &= \frac{F\overline{D}}{2} g_{S_P}(L), \\
\partial_L S &= \frac{\overline{D}}{2} g_S(L),
\end{aligned}
$$

where

$$g_{S_R}(L) = 2V - d(L-1)\delta - dL\delta - 4\overline{D}L\delta,$$
$$g_{S_P}(L) = g_{S_R}(L),$$
$$g_S(L) = 2V - d(L-1)\delta - dL\delta - 2\overline{D}L\delta.$$

The respective zeros of these linear functions are

$$L'_{S_R} = \frac{2V + d\delta}{2d\delta + 4\overline{D}\delta},$$
$$L'_{S_P} = \frac{2V + d\delta}{2d\delta + 4\overline{D}\delta},$$
$$L'_{S} = \frac{2V + d\delta}{2d\delta + 2\overline{D}\delta}.$$

All the g-functions above are positive at $L = 0$, so $\partial_L S_R$, $\partial_L S_P$, and $\partial_L S$ are strictly positive before L'_{S_R}, L'_{S_P}, and L'_S, respectively, and strictly negative thereafter. Furthermore, because S_C is increasing on the interval as a function of L, it is maximized over $[\underline{L}, \overline{L}]$ at its right endpoint, implying that $L'_{S_C} = \overline{L}$ is its maximizer. Then L'_{S_i} is the constrained optimal solution if L_{S_i} satisfies the driver scarcity condition (16); otherwise, the unconstrained optimizer L_{S_i} is feasible. □

Proof of Theorem 4. We first prove part (i), beginning with the analysis of L_{S_P} and L_{S_C}. Recalling the expressions of L_{S_P} and L_{S_C} in the proof of Theorem 2, and noting that sinh and arcsinh are strictly increasing on $\mathbb{R}_{++}$, we observe that L_{S_P} and L_{S_C} are strictly increasing in V and strictly decreasing in d. To study the dependence on other model parameters, we define

$$f(x) = \frac{\sinh\left(\frac{1}{3}\operatorname{arcsinh} x\right)}{x},$$

and note that

$$f'(x) = \frac{\cosh\left(\frac{1}{3}\operatorname{arcsinh} x\right)}{3x\sqrt{1+x^2}} - \frac{\sinh\left(\frac{1}{3}\operatorname{arcsinh} x\right)}{x^2} < 0 \iff \frac{x}{3\sqrt{1+x^2}} < \tanh\left(\frac{1}{3}\operatorname{arcsinh} x\right)$$
$$\iff \operatorname{arctanh}\left(\frac{x}{3\sqrt{1+x^2}}\right) < \frac{1}{3}\operatorname{arcsinh} x.$$

However, $\operatorname{arctanh} y = \frac{1}{2}\ln\left((1+y)/(1-y)\right)$ for $y \in [-1, 1]$; thus, by elementary algebra, we have

$$\operatorname{arctanh}\left(\frac{x}{3\sqrt{1+x^2}}\right) = \frac{1}{2}\ln\left(\frac{3\sqrt{1+x^2}+x}{3\sqrt{1+x^2}-x}\right).$$

In addition, $\operatorname{arcsinh} x = \ln\left(\sqrt{x^2+1}+x\right)$; hence,

$$f'(x) < 0 \iff \frac{1}{2}\ln\left(\frac{3\sqrt{1+x^2}+x}{3\sqrt{1+x^2}-x}\right) < \frac{1}{3}\ln\left(\sqrt{x^2+1}+x\right)$$
$$\iff \left(3\sqrt{1+x^2}+x\right)^3 - \left(3\sqrt{1+x^2}-x\right)^3\left(\sqrt{x^2+1}+x\right)^2 < 0.$$

By elementary algebra, we deduce that for $x > 0$,

$$\left(3\sqrt{1+x^2}+x\right)^3 - \left(3\sqrt{1+x^2}-x\right)^3\left(\sqrt{x^2+1}+x\right)^2 = -16x^3\left(1+x^2+x\sqrt{1+x^2}\right) < 0.$$

Hence, f is decreasing on $\mathbb{R}_{++}$. This implies that $a \sinh\left(\frac{1}{3}\operatorname{arcsinh}\left(\frac{b}{a}\right)\right)$ is increasing in a for any $b > 0$. Based on the expressions of L_{S_P} and L_{S_C} in the proof of Theorem 2, we have that L_{S_P} and L_{S_C} are strictly increasing in $\overline{v}$ and strictly decreasing in F and $\overline{D}$.

With regard to L_{S_D}, we deduce by inspection that L_{S_D} strictly increases in $\overline{v}$ and strictly decreases in F and $\overline{D}$. From the expression for $\ln(L_{S_D})$ in the proof of Theorem 1, it is clear that L_{S_D} is strictly increasing in V and strictly decreasing in d.

To analyze the dependence of L_{S_C}, L_{S_P}, and L_{S_D} on δ, we note that

$$\partial_\delta\left[\frac{2V+d\delta}{d\delta}\sqrt{\frac{\overline{v}}{F\overline{D}\delta}}\right] = -\frac{6V+d\delta}{2d\delta^2}\sqrt{\frac{\overline{v}}{F\overline{D}\delta}} < 0,$$

so the bracketed quantity is strictly decreasing in δ. From the hyperbolic trigonometric expressions for L_{S_C}, L_{S_P}, and $\ln L_{S_D}$ in the proof of Theorem 1, along with the fact that arcsinh is strictly positive and increasing on $\mathbb{R}_+$, this implies that L_{S_C}, L_{S_P}, and L_{S_D} are strictly decreasing in δ.

To prove part (ii), we differentiate the expression for L'_{S_P} (which is equal to L'_{S_R}) in the proof of Theorem 3 to find that it is strictly increasing in V, strictly decreasing in δ and $\overline{D}$, and strictly decreasing in d if and only if $V > \overline{D}\delta$. The expression is independent of $\overline{v}$ and F. Recall from the same proof that L'_{S_C} is given by equation (C.1), which is clearly increasing in V and F and decreasing in $\overline{v}$ and $\overline{D}$.

For the dependence of L'_{S_C} on δ, we use equation (C.1) to get that L'_{S_C} is (strictly) increasing (decreasing) in δ if and only if

$$(d+4\overline{D})L'_{S_C} - \frac{d}{2} = \frac{V}{\delta} + \sqrt{\frac{V^2}{\delta^2} + \frac{Vd - 4\overline{v}(d+4\overline{D})/F}{\delta} + \frac{d^2}{4}}$$

is (strictly) increasing (decreasing) in δ. For exposition, define

$$g(\delta) = \frac{a}{\delta} + \sqrt{\frac{a^2}{\delta^2} + \frac{b}{\delta} + c},$$

where $a = V$, $b = Vd - 4\overline{v}(d+4\overline{D})/F$, and $c = d^2/4$. From the discussion around equation (C.1), we deduce that L'_{S_C} exists only if the expression in the radical is non-negative. Using algebra, we express the condition under which L'_{S_C} does not exist as follows:

$$b^2 > 4a^2c \text{ and } \delta^-_{L'_{S_C}} < \delta < \delta^+_{L'_{S_C}},$$

where

$$\delta^-_{L'_{S_C}} := \frac{-b-\sqrt{b^2-4a^2c}}{2c} \text{ and } \delta^+_{L'_{S_C}} := \frac{-b+\sqrt{b^2-4a^2c}}{2c}.$$

Differentiating and using elementary algebra, we find that

$$\begin{aligned} g'(\delta) < 0 &\iff b \geq 0 \text{ or } b^2 < 4a^2c \text{ or } \delta < \delta^-_{L'_{S_C}}, \\ g'(\delta) = 0 &\iff b < 0 \text{ and } b^2 = 4a^2c \text{ and } \delta > \delta^+_{L'_{S_C}} = \delta^-_{L'_{S_C}}, \\ g'(\delta) > 0 &\iff b < 0 \text{ and } b^2 > 4a^2c \text{ and } \delta > \delta^+_{L'_{S_C}}. \end{aligned}$$

By substituting the original variables, we note that the constraint $L'_{S_C} \geq 1$ rules out the latter two cases; hence, L'_{S_C} is strictly decreasing in δ where it is defined.

For the dependence of L'_{S_C} on d, we use equation (C.1) to deduce that L'_{S_C} is (strictly) increasing (decreasing) in d if and only if

$$2L'_{S_C} = \tilde{g}(d) := \frac{\tilde{a}+d}{\tilde{b}+d} + \sqrt{\left(\frac{\tilde{a}+d}{\tilde{b}+d}\right)^2 - \frac{\tilde{c}}{\tilde{b}+d}}$$

is (strictly) increasing (decreasing) in d, where $\tilde{a} = 2V/\delta$, $\tilde{b} = 4\overline{D}$, and $\tilde{c} = 16\overline{v}/(F\delta)$. As before, L'_{S_C} exists only if the radical in the preceding equation is non-negative, and we express the condition under which L'_{S_C} does not exist as follows:

$$\tilde{a} - \tilde{b} < \frac{\tilde{c}}{4} \text{ and } d^-_{L'_{S_C}} < d < d^+_{L'_{S_C}},$$

where

$$d^-_{L'_{S_C}} := \frac{\tilde{c}}{2} - \tilde{a} - \sqrt{\tilde{c}\left(\tilde{b} - \tilde{a} + \frac{\tilde{c}}{4}\right)} \text{ and } d^+_{L'_{S_C}} := \frac{\tilde{c}}{2} - \tilde{a} + \sqrt{\tilde{c}\left(\tilde{b} - \tilde{a} + \frac{\tilde{c}}{4}\right)}.$$

We differentiate $\tilde{g}(d)$ and use elementary algebra to find that $\tilde{g}'(d) < 0$ if and only if any of the following conditions hold:

$$\begin{aligned} &\text{(case 1)} \quad \tilde{a} - \tilde{b} > \frac{\tilde{c}}{4}, \\ &\text{(case 2)} \quad \tilde{a} - \tilde{b} > 0 \text{ and } d < d^-_{L'_{S_C}}, \\ &\text{(case 3)} \quad \tilde{a} - \tilde{b} < 0 \text{ and } d < d^-_{L'_{S_C}}. \end{aligned}$$

In addition, $\tilde{g}'(d) > 0$ if and only if

$$\text{(case 4)} \quad \tilde{a} - \tilde{b} < \frac{\tilde{c}}{4} \text{ and } d > d^+_{L'_{S_C}}.$$

For completeness, we note that $\tilde{g}'(d) = 0$ if and only if

$$\text{(case 5)} \quad \tilde{a} - \tilde{b} = \frac{\tilde{c}}{4} \text{ and } d > d^-_{L'_{S_C}} = d^+_{L'_{S_C}}.$$

(If $\tilde{a}-\tilde{b} = \tilde{c}/4$ and $d < d^-_{L'_{S_C}}$, then case 2 holds, implying that $\tilde{g}(d)$ is strictly decreasing on $(0, d^-_{L'_{S_C}})$ and constant on $(d^-_{L'_{S_C}}, \infty)$.) Moreover, the constraint $L'_{S_C} \geq 1$ (i.e., $\tilde{g}(d) \geq 2$) holds if and only if either

$$\tilde{a} - \tilde{b} \geq \frac{\tilde{c}}{4}, \text{ or}$$
$$\tilde{a} - \tilde{b} > 0 \text{ and } d \leq d^-_{L'_{S_C}},$$

which eliminates cases 3 and 4. Hence, as a function of d, L'_{S_C} is decreasing where it is defined. □

Appendix D: Analysis of the Extension to Direct Deliveries

D.1 Details of the Extended Model

This subsection provides further details of the extended model in Section 6. In this extended model, if a customer at location $x > 0$ experiences a stacked delivery, then their wait time is the same as in the base model, which is

$$W_0(x) = x + \frac{d(L-1)}{2}. \tag{D.1}$$

If, instead, this customer experiences a direct delivery, their wait time is simply

$$W_1(x) = x. \tag{D.2}$$

The platform can set different prices for customers receiving stacked and direct deliveries. Because all deliveries are stacked in the base model, we also refer to stacked deliveries as *regular* deliveries. Let P_0 and P_1 be the prices charged to a customer for regular and direct delivery, respectively. Customers continue to experience a disutility of $\delta > 0$ per unit of waiting time. Thus, if regular delivery is offered at location x, then a customer at location x receives the following utility:

$$\mathcal{U}_0(x) = V - P_0 - \delta W_0(x). \tag{D.3}$$

Similarly, if direct delivery is offered at location x, then a customer at location x receives the following utility:

$$\mathcal{U}_1(x) = V - P_1 - \delta W_1(x). \tag{D.4}$$

As in the base model, a customer at location x orders if and only if $\mathcal{U}_i(x) = V - P_i - \delta W_i(x) \geq 0$, or equivalently, if and only if $W_i(x) \leq (V - P_i)/\delta$, where i is type of delivery offered at location x (i.e., i equals 1 for regular delivery and 0 for direct delivery). Because $W_0(x)$ is increasing in x, there exists a positive real number C_0 such that $W_0(x) \leq (V - P_0)/\delta$ if and only if $x \leq C_0$. Therefore, in equilibrium, only the customers located in $[0, C_0]$ live close enough to the restaurant to benefit from ordering with regular delivery through the platform.

Therefore, it follows that the platform only has an incentive to offer direct deliveries to customers who do *not* find ordering regular deliveries optimal, i.e., to customers with $x > C_0$. In other words, the platform should offer stacked deliveries to customers who live close to the restaurant (and who receive positive utility from stacked orders), and direct deliveries to customers who live far from the restaurant (who would receive negative utility from stacked orders). Thus, letting C_1 be the number for which $W_1(x) \leq (V - P_1)/\delta$ if and only if $x \leq C_0 + C_1$, we note that only the customers on $(C_0, C_0 + C_1]$ benefit from ordering direct delivery (this interval is empty if $C_1 = 0$); see Figure 6. Under the wait times given in (D.1) and (D.2), C_0 and C_1 are given by the following system of equations:

$$C_0 = \frac{V - P_0}{\delta} - \frac{d(L^C - 1)}{2}, \tag{D.5}$$

$$C_1 + C_0 = \frac{V - P_1}{\delta}. \tag{D.6}$$

Customers who receive regular delivery live on $[0, C_0]$, and those who receive direct delivery live on $(C_0, C_0 + C_1]$, either of which may be empty. We denote the mass of drivers that serve the regular- and direct-delivery customers by D_0 and D_1, respectively. Direct-delivery customers each have a dedicated driver; thus, $D_1 = C_1$. Because drivers are randomly assigned to orders, a driver's expected utility from participating is as follows:

$$\mathcal{V} = \frac{D_0}{D_0 + D_1}\left[\rho_d\left(\frac{C_0 + L}{2}\right) + \rho_t\left(\frac{C_0 + L}{2} + dL\right)\right] + \frac{D_1}{D_0 + D_1}(\rho_d + \rho_t)\left(C_0 + \frac{C_1}{2}\right). \tag{D.7}$$

Similar to the base model, we take $\mathcal{V} \geq 0$ as a decision variable for the platform.

In what follows, when discussing the stacking level L, we mean the stacking level among the customers who have stacked orders—this is not the *average* stacking level because the direct deliveries are not taken into account. As in the base model, we have the following in equilibrium:

$$L = \frac{C_0}{D_0}. \tag{D.8}$$

The restaurant's profit with direct deliveries is $S_R = (1 - F)(C_0 P_0 + C_1 P_1)$, and the platform's profit is $S_P = F(C_0 P_0 + C_1 P_1) - S_D$. From our previous analysis, the equilibrium aggregate driver pay is $S_D = (\overline{D}\mathcal{V}^2)/\overline{v}$ under the constraint that $\mathcal{V} \in [0, \overline{v}]$. The aggregate consumer surplus S_C is

$$S_C = \int_0^{C_0} \mathcal{U}_0(x)dx + \int_{C_0}^{C_0+C_1} \mathcal{U}_1(x)dx = \frac{\delta}{2}C_0^2 + \frac{\delta}{2}C_1^2 + \frac{d(L-1)\delta}{2}C_1.$$

Using (5) and (7), we deduce that when $\mathcal{V} \in [0, \overline{v}]$,

$$\frac{\overline{v}}{\overline{D}L}C_0 + \frac{\overline{v}}{\overline{D}}C_1 = \mathcal{V}. \tag{D.9}$$

Because the left-hand side is a function of the prices P_0 and P_1 only, (D.9) implies that we can eliminate $\mathcal{V}$ from the model, provided that we add the following constraint:

$$\frac{C_0}{L} + C_1 \le \overline{D}, \tag{D.10}$$

which is equivalent to $\mathcal{V} \le \overline{v}$ (the other constraints imply that C_0 and C_1 are non-negative).

The platform's optimization problem in this case is

$$\max_{C_0,C_1,P_1,P_0\ge 0} \quad F\left(P_0 C_0 + P_1 C_1\right) - \frac{\overline{v}}{\overline{D}}\left(\frac{C_0}{L} + C_1\right)^2 \tag{D.11}$$

$$\text{subject to:} \quad C_0 = \frac{V - P_0}{\delta} - \frac{d(L-1)}{2}, \tag{D.12}$$

$$C_1 + C_0 = \frac{V - P_1}{\delta}, \tag{D.13}$$

$$\frac{C_0}{L} + C_1 \le \overline{D}. \tag{D.14}$$

Similar to the base model, we can solve (D.12) and (D.13) for P_0 and P_1, obtaining the following:

$$P_0 = V - \frac{d(L-1)\delta}{2} - \delta C_0, \tag{D.15}$$

$$P_1 = V - \delta\left(C_0 + C_1\right). \tag{D.16}$$

We use these expressions to eliminate the decision variables P_0 and P_1, leaving only C_0 and C_1 as decision variables. The resulting problem can be written as follows:

$$\max_{C_1,C_0\ge 0} \quad F\left(V - \frac{d(L-1)\delta}{2} - \delta C_0\right) C_0 + F\left(V - \delta\left(C_0 + C_1\right)\right) C_1 - \frac{\overline{v}}{\overline{D}}\left(\frac{C_0}{L} + C_1\right)^2 \tag{D.17}$$

$$\text{subject to:} \quad C_0 + C_1 \le \frac{V}{\delta}, \tag{D.18}$$

$$\frac{C_0}{L} + C_1 \le \overline{D}, \tag{D.19}$$

$$C_0 \le \frac{2V - d(L-1)\delta}{2\delta}. \tag{D.20}$$

Akin to the base model, the platform sets the prices so as to choose the optimal masses of customers that order regular and direct delivery, subject to the preceding three constraints. Constraints (D.18) and (D.20) ensure that the prices that induce these customer masses are non-negative, and constraint (D.19) ensures that the customer masses do not require more drivers than are available. Once these masses are determined, the prices that induce them are given by (D.15) and (D.16).

D.2 The Unconstrained Optimal Solution

In this subsection, we characterize the optimal solutions of the platform's decision problem in (D.17)-(D.20). Clearly, this problem is feasible if and only if $2V - d(L-1)\delta \ge 0$, and is strictly feasible if and only if this inequality holds strictly. Furthermore, by examining the Hessian matrix, we see that the objective function in (D.17) is strictly concave on $\mathbb{R}_+^2$; hence, its optimizer is unique. The unique unconstrained optimizers and objective values are the following:

$$C_0 = \frac{L\left(F\overline{D}L\delta(V - d(L-1)\delta) + \overline{v}(L-1)(2V - dL\delta)\right)}{\delta\left(4(L^2 - L + 1)\overline{v} + 3F\overline{D}L^2\delta\right)},$$
$$C_1 = \frac{F\overline{D}L^2\delta(2V + d(L-1)\delta) - 2\overline{v}(L-1)(2V - dL\delta)}{2\delta\left(4(L^2 - L + 1)\overline{v} + 3F\overline{D}L^2\delta\right)},$$
$$S_P = \frac{F\left(F\overline{D}L^2\delta(4V^2 - 2Vd(L-1)\delta + d^2(L-1)^2\delta^2) + \overline{v}(L-1)^2(2V - dL\delta)^2\right)}{4\delta\left(4(L^2 - L + 1)\overline{v} + 3F\overline{D}L^2\delta\right)}.$$

This solution satisfies

(D.18) always,

(D.19) if and only if $\overline{D} \geq \dfrac{2V(L+1)}{\delta L} + \dfrac{d(L^2 - 3L + 2)}{L} - \dfrac{8\overline{v}(L^2 - L + 1)}{F\delta L^2}$,

(D.20) always,

$C_0 \geq 0$ if and only if $\dfrac{d}{V} \leq \dfrac{2\overline{v}(L-1) + F\overline{D}L\delta}{(L-1)L\delta(\overline{v} + F\overline{D}\delta)}$,

$C_1 \geq 0$ if and only if $\dfrac{d}{V} \geq \dfrac{2(2\overline{v}(L-1) - F\overline{D}L^2\delta)}{(L-1)L\delta(2\overline{v} + F\overline{D}L\delta)}$,

and therefore is the constrained optimal solution if and only if

$$\overline{D} \geq \frac{2V(L+1)}{\delta L} + \frac{d(L^2 - 3L + 2)}{L} - \frac{8\overline{v}(L^2 - L + 1)}{F\delta L^2},$$
$$\frac{2\overline{v}(L-1) + F\overline{D}L\delta}{(L-1)L\delta(\overline{v} + F\overline{D}\delta)} \geq \frac{d}{V} \geq \frac{2(2\overline{v}(L-1) - F\overline{D}L^2\delta)}{(L-1)L\delta(2\overline{v} + F\overline{D}L\delta)}.$$

In the subsequent subsections, we study the optimal solution in different cases.

D.3 The Case Where the Unconstrained Optimizer Violates (D.19)

Suppose that the unconstrained optimizer violates (D.19); i.e.,

$$\overline{D} < \frac{2V(L+1)}{\delta L} + \frac{d(L^2 - 3L + 2)}{L} - \frac{8\overline{v}(L^2 - L + 1)}{F\delta L^2}.$$

Because this solution always satisfies (D.18) and (D.20), constraint (D.19) must bind at optimality. Consequently, noting that $C_1 = \overline{D} - (C_0/L)$, we eliminate the variable C_1 and reduce the optimization problem to the following:

$$\max_{C_0} \quad F\left(V - \frac{d(L-1)\delta}{2} - \delta C_0\right)C_0 + F\left(V - \delta\left(C_0 + \overline{D} - \frac{C_0}{L}\right)\right)\left(\overline{D} - \frac{C_0}{L}\right) - \overline{D}\overline{v}$$
$$\text{subject to:} \qquad 0 \leq C_0 \leq a,$$

where

$$a = \begin{cases} \frac{L}{L-1}\left(\frac{V}{\delta} - \overline{D}\right) & \text{if } \overline{D} \geq \frac{V}{\delta L} + \frac{d(L-1)^2}{2L}, \\ \frac{V}{\delta} - \frac{d(L-1)}{2} & \text{if } \frac{V}{\delta L} + \frac{d(L-1)}{2L} \leq \overline{D} < \frac{V}{\delta L} + \frac{d(L-1)^2}{2L}, \\ \overline{D}L & \text{if } \overline{D} < \frac{V}{\delta L} + \frac{d(L-1)}{2L}. \end{cases}$$

For the preceding problem, the unconstrained optimal solution and the corresponding values of C_1 and S_P are as follows:

$$C_0 = \frac{L\big((L-1)(2V - dL\delta) - 2\overline{D}(L-2)\delta\big)}{4(L^2 - L + 1)\delta},$$

$$C_1 = \frac{2\overline{D}(2L-1)L\delta - (L-1)(2V - dL\delta)}{4(L^2 - L + 1)\delta},$$

$$S_P = \frac{F\big(4V^2(L-1)^2 - 4VL(d(L-1)^2 - 2\overline{D}(L+1))\delta + L(4\overline{D}d(L^2 - 3L + 2) + d^2(L-1)^2 - 12\overline{D}^2 L)\delta^2\big)}{16(L^2 - L + 1)\delta}.$$

This solution satisfies

$$C_0 \leq \frac{L}{L-1}\left(\frac{V}{\delta} - \overline{D}\right) \iff \overline{D} \leq \frac{V(L^2+1)}{\delta(L+1)L} + \frac{d(L-1)}{2}, \tag{D.21}$$

$$C_0 \leq \frac{V}{\delta} - \frac{d(L-1)}{2} \iff \overline{D}(L-2)L \geq \frac{d(L^3 - 3L^2 + 4L - 2)}{2} - \frac{V(L^2 - L + 2)}{\delta}, \tag{D.22}$$

$$C_0 \leq \overline{D}L \iff \overline{D} \geq \frac{4V(L-1)}{\delta(2L-1)L} - \frac{d(L-1)}{2(2L-1)}, \tag{D.23}$$

$$C_0 \geq 0 \iff \overline{D}(L-2) \leq \frac{V(L-1)}{\delta} - \frac{d(L-1)L}{2}. \tag{D.24}$$

This is the constrained optimal solution if and only if the relevant inequalities hold to make it feasible. If the unconstrained solution is negative, then the optimal solution and objective value are given by

$$C_0 = 0, \quad C_1 = \overline{D}, \quad S_P = \overline{D}(FV - (\overline{v} + F\overline{D}\delta)).$$

If instead $C_0 \geq 0$, then we consider the preceding conditions on C_0 and the corresponding inequalities. If the inequality that corresponds to a given condition is not satisfied, then the upper bound on C_0 is the optimal solution. The solutions and objective values can be calculated for each respective case as follows: in the case of the first condition, if (D.21) is not satisfied, then

$$C_0 = \frac{L}{L-1}\left(\frac{V}{\delta} - \overline{D}\right), \quad C_1 = \frac{\overline{D}L\delta - V}{(L-1)\delta}, \quad S_P = \frac{FL(\overline{D}\delta - V)(2V + (d(L-1)^2 - 2\overline{D}L)\delta)}{2(L-1)^2\delta} - \overline{D}\overline{v}.$$

Regarding the case of the second condition, if (D.22) is not satisfied, then

$$C_0 = \frac{V}{\delta} - \frac{d(L-1)}{2}, \quad C_1 = \overline{D} - \frac{1}{L}\left(\frac{V}{\delta} - \frac{d(L-1)}{2}\right),$$

$$S_P = \frac{F\big(-4V^2 + \big(d^2(L-1)^3 - 4\overline{D}^2L^2 + 2\overline{D}d(L^2-3L+2)L\big)\delta^2 + V\left(8\overline{D}L\delta - 2d(L^2-3L+2)\delta\right)\big)}{4L^2\delta} - \overline{D}\overline{v}.$$

Finally, in the case of the third condition, if (D.23) is not satisfied, then

$$C_0 = \overline{D}L, \quad C_1 = 0, \quad S_P = \frac{F\overline{D}L(2V - d(L-1)\delta - 2\overline{D}L\delta)}{2} - \overline{D}\overline{v}.$$

D.4 The Case Where the Unconstrained Optimizer is Negative

Suppose that the unconstrained optimal solution is such that $C_0 < 0$ or $C_1 < 0$.[9] Then, the optimal solution to the constrained problem lies on one or both of the lines defined by $C_0 = 0$ and $C_1 = 0$.

D.4.1 The Case Where $C_0 < 0$

If the unconstrained optimal solution is such that $C_0 < 0$, then the constraint $C_0 \geq 0$ binds at optimality. This happens when

$$\overline{D} \geq \frac{2V(L+1)}{\delta L} + \frac{d(L^2-3L+2)}{L} - \frac{8\overline{v}(L^2-L+1)}{F\delta L^2} \quad \text{and} \quad \frac{d}{V} > \frac{2\overline{v}(L-1) + F\overline{D}L\delta}{(L-1)L\delta(\overline{v} + F\overline{D}\delta)}.$$

Note that, in this case, the optimization problem reduces to the following:

$$\begin{aligned} \max_{C_1} \quad & F\,(V - \delta C_1)C_1 - \frac{\overline{v}}{\overline{D}}C_1^2 \\ \text{subject to:} \quad & 0 \leq C_1 \leq a, \end{aligned}$$

where

$$a = \begin{cases} \frac{V}{\delta} & \text{if } \overline{D} \geq \frac{V}{\delta}, \\ \overline{D} & \text{if } \overline{D} < \frac{V}{\delta}. \end{cases}$$

The unconstrained optimizer and its objective value are given by

$$C_0 = 0, \quad C_1 = \frac{F\overline{D}V}{2(\overline{v} + F\overline{D}\delta)}, \quad S_P = \frac{F^2\overline{D}V^2}{4(\overline{v} + F\overline{D}\delta)}.$$

This solution is non-negative and satisfies

$$\begin{aligned} & C_1 \leq \frac{V}{\delta} \text{ always,} \\ & C_1 \leq \overline{D} \text{ if and only if } \overline{D} \geq \frac{V}{2\delta} - \frac{\overline{v}}{F\delta}. \end{aligned}$$

[9] Based on Appendix D.2, the unconstrained optimizer cannot be such that $C_0 < 0$ and $C_1 < 0$ simultaneously.

If the unconstrained optimizer does not satisfy $C_1 \leq \overline{D}$, then the optimal solution and objective value are

$$C_0 = 0, \quad C_1 = \overline{D}, \quad S_P = \overline{D}(FV - (\overline{v} + F\overline{D}\delta)).$$

D.4.2 The Case Where $C_1 < 0$

If the unconstrained optimizer is such that $C_1 < 0$, then the constraint $C_1 \geq 0$ binds at optimality, which happens when

$$\overline{D} \geq \frac{2V(L+1)}{\delta L} + \frac{d(L^2 - 3L + 2)}{L} - \frac{8\overline{v}(L^2 - L + 1)}{F\delta L^2},$$
$$\frac{2(2\overline{v}(L-1) - F\overline{D}L^2\delta)}{(L-1)L\delta(2\overline{v} + F\overline{D}L\delta)} > \frac{d}{V}.$$

In this case, the optimization problem reduces to the following:

$$\max_{C_0} \quad F\left(V - \frac{d(L-1)\delta}{2} - \delta C_0\right) C_0 - \frac{\overline{v}}{\overline{D}L^2} C_0^2$$
$$\text{subject to: } 0 \leq \; C_0 \leq a,$$

where

$$a = \begin{cases} \frac{V}{\delta} - \frac{d(L-1)}{2} & \text{if } \overline{D} \geq \frac{V}{\delta L} - \frac{d(L-1)}{2L}, \\ \overline{D}L & \text{if } \overline{D} < \frac{V}{\delta L} - \frac{d(L-1)}{2L}. \end{cases}$$

Accordingly, the unconstrained optimizer and its objective value are given by

$$C_0 = \frac{F\overline{D}L^2(2V - d(L-1)\delta)}{4(\overline{v} + F\overline{D}L^2\delta)}, \quad C_1 = 0, \quad S_P = \frac{F^2\overline{D}L^2(2V - d(L-1)\delta)^2}{16(\overline{v} + F\overline{D}L^2\delta)}.$$

This solution is always non-negative. It satisfies

$$C_0 \leq \frac{V}{\delta} - \frac{d(L-1)}{2} \text{ always,}$$
$$C_0 \leq \overline{D}L \text{ if and only if } \overline{D} \geq \frac{V}{2\delta L} - \frac{d(L-1)}{4L} - \frac{\overline{v}}{F\delta L^2}.$$

If this unconstrained optimizer does not satisfy $C_0 \leq \overline{D}L$, then the optimal solution and objective value are

$$C_0 = \overline{D}L, \quad C_1 = 0, \quad S_P = \frac{F\overline{D}L(2V - d(L-1)\delta - 2\overline{D}L\delta)}{2} - \overline{D}\overline{v}.$$

D.5 Comparison of Platform Profits Between the Two Models

In the extended model, setting $C_1 = 0$ is always feasible. Thus, the objectives of the extended and base models are equal only when $C_1 = 0$ in the extended model; otherwise, the optimal objective value of the extended model is strictly greater than that of the base model. Based on the analysis in the preceding subsections, we note that their optimal objective values match only when the problem parameters satisfy one of the following two systems of inequalities:

$$\text{(I)} \begin{cases} \overline{D} \geq \frac{2V(L+1)}{\delta L} + \frac{d(L^2-3L+2)}{L} - \frac{8\overline{v}(L^2-L+1)}{F\delta L^2}, \\ \frac{2V}{\delta}\left(\frac{2\overline{v}(L-1)}{F\delta} - \overline{D}L^2\right) \geq d\left(\frac{2\overline{v}(L-1)L}{F\delta} + \overline{D}(L-1)L^2\right), \end{cases}$$

$$\text{(II)} \begin{cases} \overline{D} \leq \frac{2V(L+1)}{\delta L} + \frac{d(L^2-3L+2)}{L} - \frac{8\overline{v}(L^2-L+1)}{F\delta L^2}, \\ \overline{D}(L-2) \leq \frac{V(L-1)}{\delta} - \frac{d(L-1)L}{2}, \\ \overline{D} \leq \frac{V}{\delta L} + \frac{d(L-1)}{2L}, \\ \overline{D} \leq \frac{4V(L-1)}{\delta(2L-1)L} - \frac{d(L-1)}{2(2L-1)}. \end{cases}$$

To obtain the preceding conclusion, we use the fact that

$$\frac{2(2\overline{v}(L-1) - F\overline{D}L^2\delta)}{(L-1)L\delta(2\overline{v} + F\overline{D}L\delta)} \geq \frac{d}{V}$$

if and only if

$$\frac{2V}{\delta}\left(\frac{2\overline{v}(L-1)}{F\delta} - \overline{D}L^2\right) \geq d\left(\frac{2\overline{v}(L-1)L}{F\delta} + \overline{D}(L-1)L^2\right).$$

Using the analysis in the preceding subsections, we observe that the first system corresponds to when the drop-off time d and stacking level L are so small that the extra time imposed by stacking orders is negligible. For the second system, recall that the scarce driver regime holds when

$$\overline{D} \leq \frac{V}{2\delta L} - \frac{d(L-1)}{4L} - \frac{\overline{v}}{F\delta L^2},$$

and this inequality implies the third inequality of the system.

Appendix E: Further Extensions of the Base Model

E.1 A Service Radius

The contract between the restaurant and the platform typically specifies a *service radius* (DoorDash 2020), which is the maximum distance from the restaurant that the platform delivers to. In our model, a service radius of $\overline{C}$ limits the extent of the interval of participating customers to $[0, \overline{C}]$. We analyze the base model with the addition of a service radius constraint in Appendix F. From that analysis, we obtain the following proposition.

Proposition E.1. *As in the base model, there is a market if and only if* (15) *holds. Drivers are scarce if and only if* $\overline{C} \geq \overline{D}L$ *and* (16) *holds, while the service radius binds if and only if* $\overline{C} \leq \overline{D}L$ *and*

$$2V - d(L-1)\delta \geq 4\left(\frac{\overline{v}}{FL^2} + \delta\overline{C}\right). \tag{E.1}$$

Recall that C is strictly increasing in L for $L < L_{S_C}$. Thus, if the service-radius constraint ($C \leq \overline{C}$) binds, then there exists $\overline{L}$ such that, as L starts to exceed $\overline{L}$, the constraint starts to bind. Because C is continuous and strictly increases from 0 to its maximum, we deduce that $\overline{L}$ can lie anywhere in $(0, L_{S_C})$. Where it falls dictates the effect that adding the constraint has on each platform participant's payoff, as explained in the following result.

Proposition E.2. *The constraint $C \leq \overline{C}$ binds for L in a subinterval of $(0, \widehat{L})$. Define $\widehat{L}_{S_P} = \sqrt[3]{(4\overline{v}\overline{C})/(dF\overline{D}\delta)}$. Then, as functions of L on $(\overline{L}, L_{S_C})$, C and S_C are constant; D, S_D, S_D/D, and PC are strictly decreasing; S_P is quasiconcave with a maximizer $\widehat{L}_{S_P}$. Furthermore, there exists a constant $\overline{C}^\star \in (0, \max_L\{C\}]$ such that*

$$\text{if } \overline{C} > \overline{C}^\star, \text{ then } \widehat{L}_{S_P} > L_{S_P};$$
$$\text{if } \overline{C} < \overline{C}^\star, \text{ then } \widehat{L}_{S_P} < L_{S_P}.$$

Figure 9 illustrates the effects characterized in Proposition E.2. Intuitively, if the driver or customer surplus peaks *after* $\overline{L}$, then adding the constraint $C \leq \overline{C}$ causes it to peak *at* $\overline{L}$. The effect on the platform is more complicated, because, as Proposition E.2 states, the platform's profit is still quasiconcave after the constraint is added, and the imposition of the constraint can cause its peak to occur before or after its original peak of L_{S_P}. In Figure 9, the peaks lower as the radius tightens. Hence, a service radius appears to hurt all market participants *if* the radius is smaller than the equilibrium radius without the constraint.

Suppose that we temporarily fix all equilibrium quantities and impose a binding service radius of $\overline{C}$. Then, the marginal customer (on the boundary at $\overline{C}$) has excess utility. The platform can take two actions to claw away some of this excess utility from the customer. It can raise the price P to increase its revenue, or it can increase the stacking level L, which is equivalent to hiring fewer drivers, to reduce its expenses. In equilibrium, *both* of these happen. As Proposition E.2 indicates, if the service radius is sufficiently large (i.e., not very restrictive), then the platform focuses on cutting costs by hiring fewer drivers (increasing the stacking level). When the service radius is smaller (i.e., more restrictive), the platform focuses on increasing revenue by hiring more drivers (which increases the available customer utility) and raising the price to extract more of their surplus.

Observe that in Figure 9, the ordering of L_{S_C} and L_{S_P} switch for sufficiently small $\overline{C}$. It is interesting to note that in this case, the platform wants to stack orders at a higher level than any other market participant.

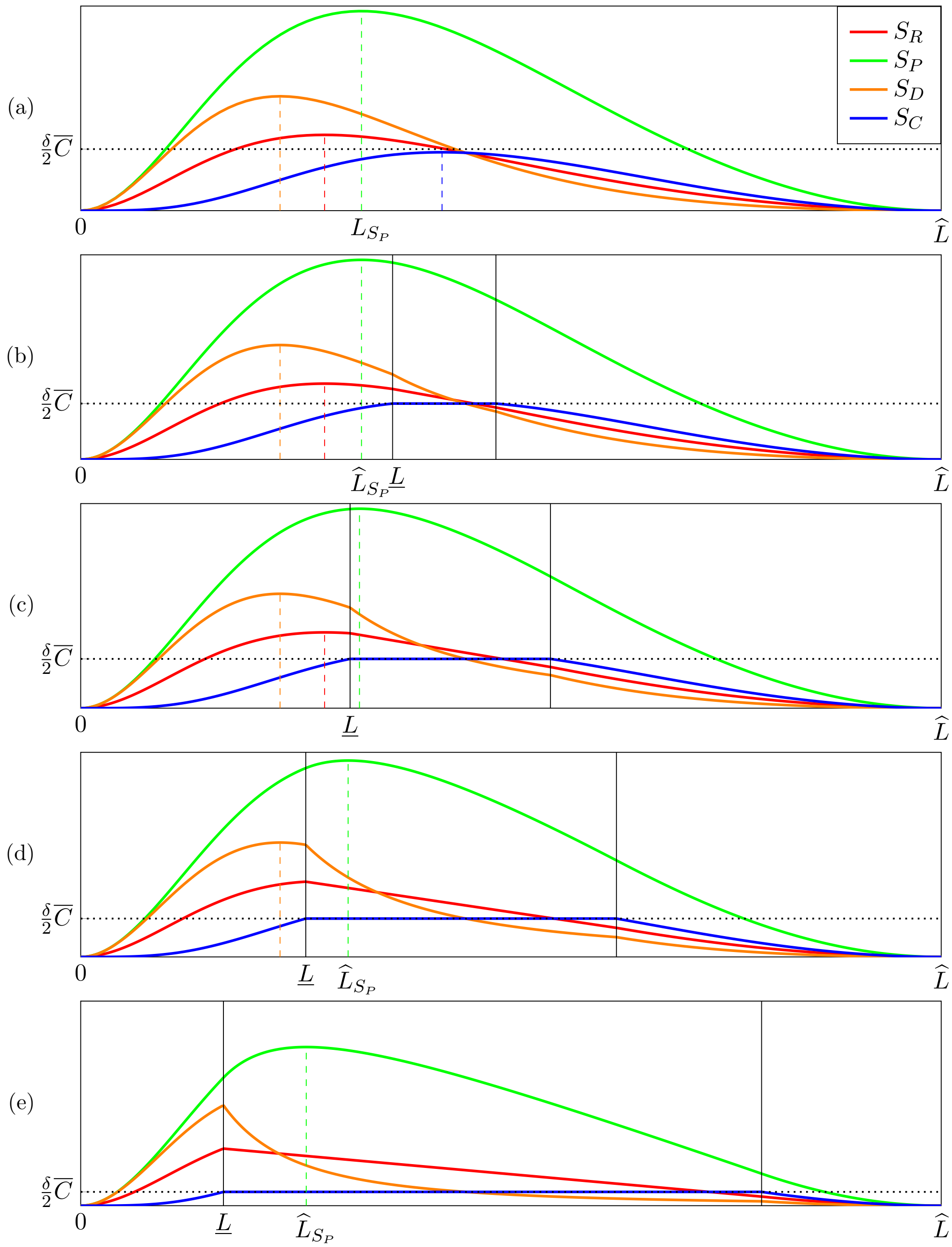


Figure 9. Impact of a service radius. The figure displays the payoff functions S_C, S_D, S_R, and S_P for different values of L in the case of a service radius $\overline{C}$. From (a) to (e), $\overline{C} = 0.19$, 0.181, 0.17, 0.15, and 0.09, respectively. The problem parameters are $V = \overline{v} = \overline{D} = \delta = 10$, $d = 2$. The maximizer of each colored curve is shown as a vertical dashed line of the same color. If no matching dashed line is present for a colored curve, the maximizer of that curve is $\underline{L}$.

There are benefits to using service radii for restaurants that are not captured in our model. For example, Auad et al. (2024) study how a dynamic service radius could be used by the platform to control demand. Furthermore, service radii localize certain aspects of the driver pricing problem, apparently rendering the pricing problem more tractable. For these and other reasons, we hesitate to conclude that service radii hurt all or almost all market participants. In any case, most of the original results hold, with slight modification. For example, the ordering of the optimal stacking levels for the market participants is preserved with the exception of L_{S_P}, which switches places with L_{S_C} as the service radius becomes more restrictive. However, for moderate values of $\overline{C}$, the original result appears to hold with weak inequalities.

E.2 Positive Meal Preparation Time

The analysis in Sections 3 and 5 assumes that orders become ready immediately. We now relax this assumption and suppose that each order takes time $t > 0$ to prepare. Once all the orders are placed at the beginning, the only effect this has is to add t time units on to each customer's wait time $W(x)$. Let $V' = V - \delta t$, and note that it is a function of δ and t. Then, following our original arguments, we obtain

$$C = \frac{V - P}{\delta} - \frac{d(L-1)}{2} - t = \frac{V' - P}{\delta} - \frac{d(L-1)}{2}.$$

Propagating the change through, we can obtain versions of all propositions and theorems of Sections 4 and 5.[10]

Because $V' = V - \delta t$, it is clear that any quantity that is decreasing (increasing) in V in the original model is increasing (decreasing) in t in the extended model. Combining this with the results in Section 5, we obtain the following proposition.

Proposition E.3. *P, C, D, S_C, S_D, S_R, S_P, and S are all strictly decreasing in t.*

Proposition E.3 embodies the intuitive notion that the longer it takes the restaurant to fill orders, the worse off are all other market participants. An increase in the meal preparation time makes customers receive their meals later, which hurts their utility. In response, the platform must lower the price, but the combination of lower prices and fewer customers leaves the platform with less money to pay itself and drivers. It also leaves less money for the restaurant.

The upshot is that all market participants prefer a faster restaurant. This preference is manifested in practice in the advent of *virtual restaurants* and *ghost kitchens*—both of which prepare meals for delivery only. A ghost kitchen is a restaurant that has no storefront or seating area. A virtual restaurant can be either a ghost kitchen or a delivery-only restaurant operated out of an existing restaurant's kitchen. Virtual restaurants may prepare orders for several different restaurants

[10] The constants such as δ_{S_C} and $\widehat{\delta}$ that appear in the proofs of some of these results are modified.

listed on the platform. Such restaurants exist to decrease meal preparation times for online-order customers without degrading dine-in service times.

E.3 General Convex Cost of Waiting

Suppose that the customers incur a disutility of $g(W(x))$ from experiencing wait time $W(x)$ (the base model has $g(y) = \delta y$). Suppose g is convex and strictly increasing with $g(0) = 0$. Then, a customer at location x receives the following utility from ordering:

$$\mathcal{U}(x) = V - P - g(W(x)), \tag{E.2}$$

and this customer places an order if and only if $g(W(x)) \leq V - P$. Similar to the base model, $g(W(x))$ is increasing in x, so there exists a positive real number C such that $g(W(x)) \leq V - P$ if and only if $x \leq C$. Thus, in equilibrium only the customers living in $[0, C]$ are close enough to the restaurant to participate. In this case, C is defined as the solution to the indifference condition

$$V - P = g\left(C + \frac{d(L-1)}{2}\right). \tag{E.3}$$

Based on the arguments used in the analysis of the base model, we write the platform's decision problem as the following:

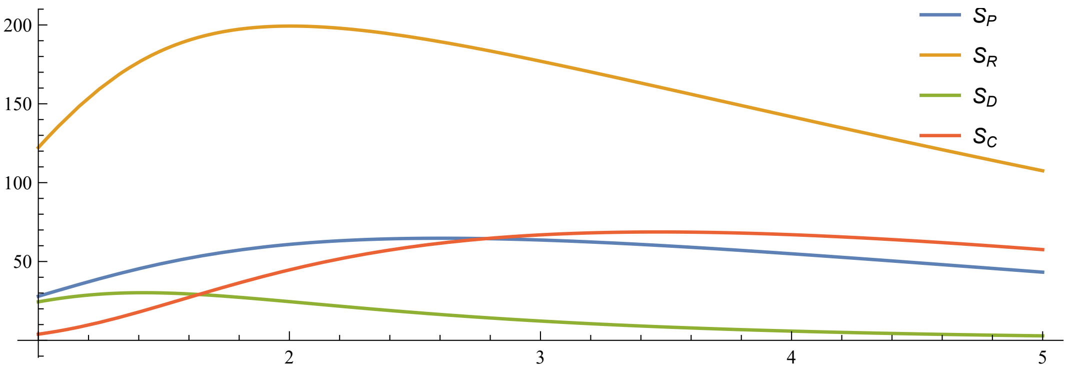
$$\begin{aligned} \max \quad & F\left(V - g\left(C + \frac{d(L-1)}{2}\right)\right)C - \frac{\overline{v}}{\overline{D}L^2}C^2 \\ \text{subject to:} \quad & 0 \leq C \leq \overline{D}L, \\ & V - g\left(C + \frac{d(L-1)}{2}\right) \geq 0. \end{aligned}$$

The corresponding market condition is

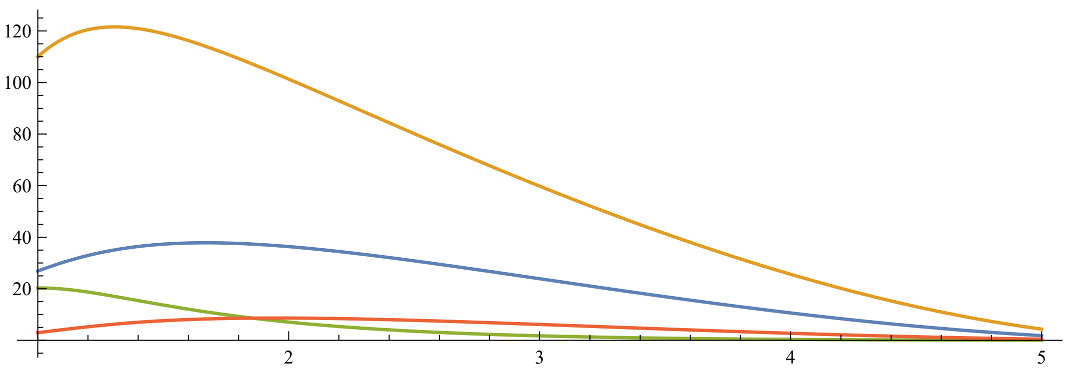
$$g\left(\frac{d(L-1)}{2}\right) < V,$$

and the problem is strictly feasible when this holds.

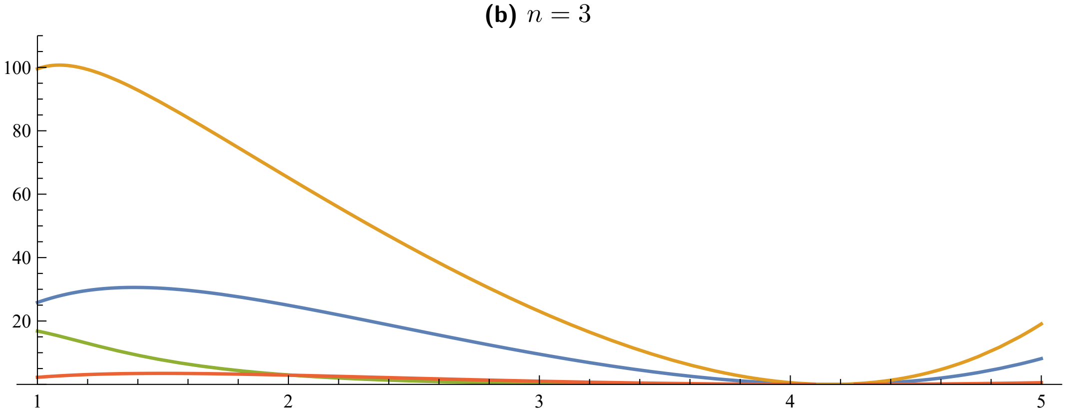
We show in Appendix G that in the case that there are abundant drivers for all values of L and g has a non-negative third derivative, S_C and S_D are strictly quasiconcave as functions of L. Suppose that S_R and S_P are as well. Because $S_D + S_P = (FS_R)/(1 - F)$, taking derivatives and evaluating at $L = L_{S_R}$ reveals that $S_D(L)$ and $S_R(L)$ have slopes of opposite sign at $L = L_{S_R}$. By the strict quasiconcavity of these functions, this implies that either $L_{S_D} > L_{S_R} > L_{S_P}$ or else $L_{S_P} > L_{S_R} > L_{S_D}$. We also show in Appendix G that L_{S_C} is greater than both L_{S_D} and L_{S_P}, so in either case, L_{S_C} is greater than all other optimal stacking levels. Figure 10 demonstrates that the ordering of Theorem 2 prevails in a selection of g that satisfies these assumptions.



(a) $n = 2$



(b) $n = 3$



(c) $n = 4$

Figure 10. Impact of convex waiting costs. The figure displays the (unconstrained) payoff functions S_C, S_D, S_R, and S_P in the case of convex waiting costs, with $g(y) = y^n$ for $n \in \{2, 3, 4\}$. The problem parameters are $V = 100$, $\overline{v} = 30$, $d = 2$, $F = 0.3$, and $\overline{D} = 4$. Note that $\widehat{L} \approx 4.16$ for panel (c).

Appendix F: Analysis for the Case of a Service Radius

When there is a service radius of $\overline{C}$, the platform's optimization problem can be written as follows:

$$\begin{aligned} \max \quad & F\left(V - \frac{d(L-1)\delta}{2} - \delta C\right) C - \frac{\overline{v}}{\overline{D}L^2} C^2 \\ \text{subject to:} \quad & 0 \le C \le \min\left\{\frac{V}{\delta} - \frac{d(L-1)}{2}, \overline{D}L, \overline{C}\right\}. \end{aligned}$$

In Appendix A, we show that the unconstrained optimizer (A.1) is non-negative and satisfies the following constraint:

$$C \le \frac{V}{\delta} - \frac{d(L-1)}{2}.$$

Proof of Proposition E.1. Because the unconstrained optimal solution is the same as that of the base model, the market condition is the same, namely (15). Note that the optimal solution is constrained to simultaneously lie in both $[0, \overline{D}L]$ and $[0, \overline{C}]$. The interval that is smaller dictates the possible type(s) of equilibria; drivers can only be scarce if $\overline{D}L \le \overline{C}$ and the service radius can only bind if $\overline{C} \le \overline{D}L$.

From the expression for the unconstrained optimizer given in (A.1), we deduce that it violates the constraint $C \le \overline{D}L$ if and only if

$$\overline{D} < \frac{V}{2\delta L} - \frac{d(L-1)}{4L} - \frac{\overline{v}}{F\delta L^2}. \tag{F.1}$$

Moreover, the same unconstrained optimizer violates the constraint $C \le \overline{C}$ if and only if

$$\overline{D} < \frac{V}{2\delta L^*} - \frac{d(L-1)}{4L^*} - \frac{\overline{v}}{F\delta L^2}. \tag{F.2}$$

Note that the driver scarcity condition given in (F.1) is the same as that of the base model, (16). We now see that drivers are scarce if and only if $\overline{C} \ge \overline{D}L$ and (F.1) holds, and that the service radius binds if and only if $\overline{C} \le \overline{D}L$ and (F.2) holds. □

Proof of Proposition E.2. If the service radius binds, then $S_P = [FL^2(2V - d(L-1)\delta - 2\overline{C}\delta)\overline{C} - 2\overline{C}\overline{v}L^*]/(2L^2)$, which is strictly concave in L. Its unconstrained maximizer is therefore the unique solution $\widehat{L}_{S_P}$ to the following first-order condition:

$$0 = -\frac{F\overline{C}d\delta}{2} + \frac{2\overline{C}^2\overline{v}}{\overline{D}} \frac{1}{L^3},$$

which can be written as

$$\widehat{L}_{S_P} = \sqrt[3]{\frac{4\overline{C}\overline{v}}{F\overline{D}d\delta}}.$$

Furthermore, we have the following:

$$S_C = \frac{\delta}{2}\overline{C}^2,$$

$$S_D = \frac{\overline{v}}{\overline{D}L^2}\overline{C}^2,$$

$$S_P = VF\overline{C} - F\delta\overline{C}^2 - \frac{dF\delta\overline{C}}{2}L - \frac{\overline{v}\overline{C}^2}{\overline{D}}\frac{1}{L^2},$$

$$S_R = (1-F)PC = (1-F)\overline{C}\left(V - \delta\left(\overline{C} + \frac{d}{2}L\right)\right) = (1-F)\left(\overline{C}V - \delta\overline{C}^2 - \frac{d\delta\overline{C}}{2}L\right),$$

$$C = \overline{C},$$

$$D = \frac{\overline{C}}{L},$$

$$\frac{S_D}{D} = \frac{\overline{v}}{\overline{D}L}\overline{C}.$$

The remainder of the proof, with the exception of showing the existence of $\overline{C}^\star$ satisfying the properties in the proposition, follows from examining the preceding expressions.

We now prove the existence of $\overline{C}^\star$ with the desired properties. As in the proof of Theorem 2, define $s = (V\overline{v})/(F\overline{D}d\delta^2)$ and $t = \overline{v}/(F\overline{D}\delta)$. Then, we can write

$$\widehat{L}_{S_P} = \sqrt[3]{\frac{4\overline{C}}{d}t},$$

$$L_{S_P} = \sqrt[3]{\left(s+\frac{t}{2}\right) + \sqrt{\left(s+\frac{t}{2}\right)^2 + \frac{8}{27}t^3}} + \sqrt[3]{\left(s+\frac{t}{2}\right) - \sqrt{\left(s+\frac{t}{2}\right)^2 + \frac{8}{27}t^3}}.$$

By elementary algebra, we deduce that

$$\widehat{L}_{S_P} < L_{S_P} \iff \overline{C} < \overline{C}^\star := \frac{d\widehat{L}}{4} - \frac{dL_{S_P}}{2}.$$

Note that the right-hand side of the preceding inequality is independent of $\overline{C}$.

Finally, we prove that $\overline{C}^\star \le \max_L\{C\}$ by contradiction. Recall that L_{S_C} is the maximizer of C as a function of L. By equation (B.2), we have that

$$\begin{aligned}
\overline{C}^\star > \max_L\{C\} &\iff \frac{d\widehat{L}}{4} - \frac{dL_{S_P}}{2} > \frac{F\overline{D}L_{S_C}^2\big(2V - d(L_{S_C}-1)\delta\big)}{4\big(\overline{v} + F\overline{D}L_{S_C}^2\delta\big)} \\
&\iff t\big(\widehat{L} - 2L_{S_P}\big) > \frac{F\overline{D}L_{S_C}^2\big(2V - d(L_{S_C}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_C}^2\delta\big)} \\
&\iff L_{S_P}^3 > \frac{F\overline{D}L_{S_C}^2\big(2V - d(L_{S_C}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_C}^2\delta\big)},
\end{aligned}$$

where the last line follows because, as shown in the proof of Theorem 2, L_{S_P} is a root of

$$L^3 + 2tL - (2s+t) = L^3 + t\big(2L - \widehat{L}\big).$$

Because C is strictly maximized at L_{S_C} (Theorem 1) and $L_{S_P} \neq L_{S_C}$ (Theorem 2), we deduce that

$$\frac{F\overline{D}L_{S_C}^2\big(2V - d(L_{S_C}-1)\delta\big)}{4\big(\overline{v} + F\overline{D}L_{S_C}^2\delta\big)} > \frac{F\overline{D}L_{S_P}^2\big(2V - d(L_{S_P}-1)\delta\big)}{4\big(\overline{v} + F\overline{D}L_{S_P}^2\delta\big)},$$

which after multiplying by $(4t)/d$, yields the following:

$$\frac{F\overline{D}L_{S_C}^2\big(2V - d(L_{S_C}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_C}^2\delta\big)} > \frac{F\overline{D}L_{S_P}^2\big(2V - d(L_{S_P}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_P}^2\delta\big)}.$$

Therefore, a necessary condition for $\overline{C}^\star > \max_L\{C\}$ is that

$$L_{S_P}^3 > \frac{F\overline{D}L_{S_P}^2\big(2V - d(L_{S_P}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_P}^2\delta\big)}.$$

This holds if and only if

$$\begin{aligned} L_{S_P} > \frac{F\overline{D}\big(2V - d(L_{S_P}-1)\delta\big)t}{d\big(\overline{v} + F\overline{D}L_{S_P}^2\delta\big)} &\iff L_{S_P} > \frac{\overline{v}\big(2V - d(L_{S_P}-1)\delta\big)}{d\delta\big(\overline{v} + F\overline{D}L_{S_P}^2\delta\big)} \\ &\iff 0 > 2V\overline{v} + d\overline{v}\delta - 2d\overline{v}L_{S_P}\delta - F\overline{D}L_{S_P}^3 d\delta^2 = g_{S_P}\left(L_{S_P}\right), \end{aligned}$$

where g_{S_P} is as in the proof of Theorem 1. However, $g_{S_P}\left(L_{S_P}\right) = 0$; hence, the aforementioned necessary condition for $\overline{C}^\star > \max_L\{C\}$ cannot hold, and it must be that $\overline{C}^\star \leq \max_L\{C\}$. □

Appendix G: Analysis for General Convex Cost of Waiting

We note that the derivative of the objective function with respect to C is positive at $C = 0$ by the market condition, and because g is convex, that this derivative is strictly decreasing on $\mathbb{R}_+$. The objective function is therefore quasiconcave in C on $\mathbb{R}_+$, so the unconstrained maximizer is given by the first-order condition, which can be written as follows:

$$V - g\left(C + \frac{d(L-1)}{2}\right) = g'\left(C + \frac{d(L-1)}{2}\right)C + \frac{2\overline{v}}{F\overline{D}L^2}C.$$

The right-hand side of this equation is non-negative for $C \geq 0$; thus, the second constraint is always satisfied and can therefore be dropped. Furthermore, evaluating the first-order condition at $C = 0$ and using the market condition, we see that $C \neq 0$ at optimality. Consequently, the only constraint that the unconstrained optimal solution can violate when the market condition holds is $C \leq \overline{D}L$. The constrained optimal solution is therefore

$$C = \min\{C^*, \overline{D}L\},$$

where C^* is the unconstrained optimal solution given by the first-order condition.

We now argue that C^* is strictly quasiconcave as a function of L. To this end, we state the following lemma.

Lemma G.1. *Suppose that for every $t \in T$, $f(x,t)$ is quasiconcave in $x \in X$. Moreover, suppose that for each $x \in X$, $\partial_x f(x,t)$ is non-negative on a connected strictly convex set T_x (which may be the empty set). Then, the unique optimal selection $x^*(t)$ is strictly quasiconcave.*

Proof. Let $\alpha \in X$. Then, $\partial_x f(\alpha,t) \geq 0$ for $t \in T_\alpha$, and the quasiconcavity of $f(x,t)$ in x implies that $x^*(t) \geq \alpha$ for all $t \in T_\alpha$. Suppose that there exists $t^* \notin T_\alpha$ such that $x^*(t^*) \geq \alpha$. Then, by the quasiconcavity of $f(x,t)$ in x, $\partial_x f(\alpha, t^*) \geq 0$, which is a contradiction. Therefore, $T_\alpha = \{t \in T : x^*(t) \geq \alpha\}$, and this is strictly convex by assumption, implying that $x^*(t)$ is strictly quasiconcave. □

Note that S_P is quasiconcave in C on $\mathbb{R}_+$, and suppose that $g''' \geq 0$. Then,

$$\phi(L) \coloneqq \partial_{CL} S_P = \frac{4\overline{v}}{\overline{D}L^3} C - \frac{Fd}{2}\left[g''\left(C + \frac{d(L-1)}{2}\right)C + g'\left(C + \frac{d(L-1)}{2}\right)\right]$$

is a strictly decreasing function, so $\partial_C S_P$ is strictly concave as a function of L; consequently, $\{L : \partial_C S_P(L) \geq 0\}$ is strictly convex, and Lemma G.1 implies that $C^*(L)$ is strictly quasiconcave. Using $C = DL$ to reexpress the platform's decision problem with decision variable D and following the same argument shows that $D^*(L)$ is also strictly quasiconcave.

Since $C^*(L) = D^*(L)L$, we have that $C^{*\prime}(L) = D^{*\prime}(L)L + D^*(L)$. Let L_C and L_D be maximizers of $C^*(L)$ and $D^*(L)$, respectively. Then, $C^{*\prime}(L_D) = D^{*\prime}(L_D)L_D + D^*(L_D) = D^*(L_D) > 0$, implying that $L_C > L_D$. Note that $S_D = (\overline{v}/\overline{D})D^2$ is a strictly monotone function of D, so $L_{S_D} = L_D$. Furthermore,

$$\begin{aligned} S_C(C) = \int_0^C \mathcal{U}(x)dx &= \int_0^C V - P - g\left(x + \frac{d(L-1)}{2}\right)dx \\ &= \int_0^C g\left(C + \frac{d(L-1)}{2}\right) - g\left(x + \frac{d(L-1)}{2}\right)dx, \end{aligned}$$

which is strictly monotone in C. Because g is convex, as L gets larger, so does the integrand (for each fixed x). Consequently, S_C is increasing as a function of L whenever C^* is, implying that $L_{S_C} \geq L_C$. Hence, $L_{S_C} > L_{S_D}$.

Suppose that S_C, S_D, S_R, S_P, and C are strictly quasiconcave in L and are maximized at an interior point of $[1, \widehat{L}]$. Then,

$$\begin{aligned} S_C(L) &= \int_0^{C(L)} g\left(C(L) + \frac{d(L-1)}{2}\right) - g\left(x + \frac{d(L-1)}{2}\right)dx \\ &= g\left(C(L) + \frac{d(L-1)}{2}\right)C(L) - \int_0^{C(L)} g\left(x + \frac{d(L-1)}{2}\right)dx \\ &= VC(L) - \frac{1}{F}S_P(L) - \frac{1}{F}S_D(L) - \int_0^{C(L)} g\left(x + \frac{d(L-1)}{2}\right)dx. \end{aligned}$$

Therefore, using Leibniz' integral rule, we deduce that

$$\begin{aligned} S_C'(L) &= VC'(L) - \frac{1}{F}S_P'(L) - \frac{1}{F}S_D'(L) - g\left(C(L) + \frac{d(L-1)}{2}\right)C'(L) \\ &\qquad\qquad\qquad\qquad - \frac{d}{2}\int_0^{C(L)} g'\left(x + \frac{d(L-1)}{2}\right)dx \\ &= \left[V - g\left(C(L) + \frac{d(L-1)}{2}\right)\right]C'(L) - \frac{1}{F}S_P'(L) - \frac{1}{F}S_D'(L) \\ &\qquad\qquad\qquad\qquad - \frac{d}{2}\left[g\left(C(L) + \frac{d(L-1)}{2}\right) - g\left(\frac{d(L-1)}{2}\right)\right]. \end{aligned}$$

Evaluating both sides at $L = L_{S_C}$, noting that $S_C'(L_{S_C}) = 0$, and rearranging yields the following:

$$\begin{aligned} \frac{1}{F}S_P'(L_{S_C}) &= \left[V - g\left(C(L_{S_C}) + \frac{d(L_{S_C}-1)}{2}\right)\right]C'(L_{S_C}) - \frac{1}{F}S_D'(L_{S_C}) \\ &\qquad\qquad - \frac{d}{2}\left[g\left(C(L_{S_C}) + \frac{d(L_{S_C}-1)}{2}\right) - g\left(\frac{d(L_{S_C}-1)}{2}\right)\right]. \end{aligned}$$

On the right-hand side, the first and third terms are non-positive and the second term is negative. To see this, we note the following: because $L_{S_C} \in (1, \widehat{L})$ by assumption, the market condition implies that the bracketed part of the first term is positive. Since $C(L)$ is strictly quasiconcave and $L_C \leq L_{S_C}$, we have that $C'(L_{S_C}) \leq 0$, so the first term is non-positive. Similarly, because $S_D(L)$ is strictly quasiconcave and $L_{S_D} < L_{S_C}$, $S_D'(L_{S_C}) > 0$, implying that the second term is negative. Since g is strictly increasing and $C(L_{S_C}) \geq 0$, the third term is non-positive. Hence, the right-hand side of the equation is negative, implying that $S_P'(L_{S_C}) < 0$. Because $S_P(L)$ is strictly quasiconcave with an interior extremum, this implies that $L_{S_C} > L_{S_P}$.