Financial Intermediary Capital*

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Abstract

We propose a dynamic theory of financial intermediaries as collateralization specialists that are better able to collateralize claims than households. Intermediaries require capital as they can borrow against their loans only to the extent that households themselves can collateralize the assets backing the loans. We derive the collateral constraints for intermediated and direct finance from an economy with limited enforcement and limited participation. The model provides a novel notion of short-term financing as intermediated finance is extended and repaid every period and cannot be rolled over. The determinants of the capital structure of firms and intermediaries are distinct. The net worth of financial intermediaries and the corporate sector are both state variables affecting the spread between intermediated and direct finance and the dynamics of real economic activity, such as investment, and financing. The accumulation of net worth of intermediaries is slow relative to that of the corporate sector. A credit crunch has persistent real effects and can result in a delayed or stalled recovery.

Keywords: Collateral; Financial intermediation; Financial constraints; Investment

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1 Introduction

The capitalization of financial intermediaries is arguably critical for economic fluctuations and growth. We provide a dynamic model in which financial intermediaries are collateralization specialists and firms need to collateralize promises to pay with tangible assets. Financial intermediaries are modeled as lenders that are able to collateralize a larger fraction of tangible assets than households, that is, are better able to enforce their claims. Financial intermediaries require net worth as their ability to refinance their collateralized loans from households is limited, as they, too, need to collateralize their promises. Importantly, the net worth of both financial intermediaries and firms hence play a role in our model and these state variables jointly affect the dynamics of firm investment, financing, and loan spreads. Spreads on intermediated finance are high when both firms and financial intermediaries are poorly capitalized and in particular when intermediaries are moreover poorly capitalized relative to firms. One of our main results is that intermediaries accumulate net worth more slowly than the corporate sector. This has important implications for economic dynamics. For example, a credit crunch, that is, a drop in intermediary net worth, has persistent real effects and can result in a delayed or stalled recovery.

In our model, firms can raise financing either from households or from financial intermediaries. We derive collateral constraints from an environment with limited enforcement without exclusion and with limited participation.\footnote{We model limited enforcement à la Kehoe and Levine (1993) but without exclusion, as in Chien and Lustig (2010) and Rampini and Viswanathan (2010, 2013, 2015), and extend their results by introducing limited participation as well.} Financial intermediaries are better able to enforce claims than households because intermediaries participate in markets at all times, including when more of the firms’ capital is collateralizable. We show that the optimal dynamic contract with limited enforcement can be implemented with complete markets in two types of one-period ahead Arrow securities, one issued by financial intermediaries only and one issued by households or financial intermediaries. We refer to the first set of securities as intermediated finance and the second set of securities as direct finance. Limited enforcement implies separate collateral constraints for intermediated and direct finance. Since financial intermediaries are better able to collateralize promises by firms, they can extend more financing to them per unit of tangible assets collateralizing their loans. Financial intermediaries in turn are able to borrow against their corporate loans, but only to the extent that households themselves can collateralize the assets backing the loans, since the intermediaries are subject to limited enforcement, too. Intermediaries thus need to finance the additional amount that they can lend out of
their own net worth, giving a role to financial intermediary capital.

The determinants of the capital structure for firms and intermediaries differ. Firms’ capital structure is determined by the extent to which the tangible assets required for production can be collateralized. Intermediaries’ capital structure is determined by the extent to which their collateralized loans can be collateralized themselves, that is, by the difference between the intermediaries’ and households’ ability to collateralize claims. In other words, firms issue promises against tangible assets whereas intermediaries issue promises against collateralized claims, which are in turn backed by tangible assets. The substantial differences in leverage between intermediaries and firms observed in practice may be a result of these distinct determinants.

The enforcement constraints imply that intermediated finance is short term in the sense that it has to be extended and repaid every period, and cannot be rolled over. Specifically, intermediated loans need to be repaid when more of the capital is collateralizable and repayment cannot be postponed to later on, when intermediaries’ enforcement advantage has disappeared. As a consequence, intermediated loans cannot be simply rolled over. Thus, our model provides a novel notion of short-term financing.

Intermediaries are essential in our economy in the sense that allocations can be achieved with financial intermediaries, which cannot be achieved otherwise. Financial intermediaries have constant returns in our model and hence there is a representative financial intermediary. Since intermediary net worth is limited, intermediated finance commands a positive spread. We first consider the equilibrium spread on intermediated finance in a static environment with a representative firm. Importantly, the spread on intermediated finance critically depends on both firm and intermediary net worth. Given the (representative) firm’s net worth, spreads are higher when the intermediary is less well capitalized. However, spreads are particularly high when firms are poorly capitalized, and intermediaries are poorly capitalized relative to firms at the same time. Poor capitalization of the corporate sector per se does not imply high spreads, as low firm net worth reduces the demand for loans from intermediaries. Given the net worth of the intermediary sector, a reduction in the net worth of the corporate sector may reduce spreads as the intermediaries can more easily accommodate the reduced loan demand.

Our model allows the analysis of the dynamics of intermediary capital. A main result is that the accumulation of net worth of intermediaries is slow relative to that of the corporate sector. We first consider the deterministic dynamics of intermediary net worth and the spread on intermediated finance. In a deterministic steady state, intermediaries are essential, have positive net worth, and the spread on intermediated finance is positive. Dynamically, if firms and intermediaries are initially poorly capitalized, both firms and
intermediaries accumulate net worth over time. Importantly, firms in our model accumulate net worth faster than financial intermediaries, because the marginal and in particular the average return on net worth for financially constrained firms is relatively high due to the high marginal product of capital. Financial intermediaries accumulate net worth at the interest rate earned on intermediated finance, which is at most the marginal return on net worth of the corporate sector and may be below when the collateral constraint for intermediated finance binds. Thus, intermediaries, with constant returns to scale, earn at most the marginal return on all their net worth, whereas firms, with decreasing returns to scale, earn the average return on their net worth.

Suppose that firms are initially poorly capitalized also relative to financial intermediaries, say in a macroeconomic downturn. Then the dynamics of the spread on intermediated finance are as follows. Because the firms are poorly capitalized, the current demand for intermediated finance is low and the spread on intermediated finance is zero. Intermediaries save net worth by lending to households to meet higher future corporate loan demand, that is, hold “cash” at a low interest rate. As the firms accumulate more net worth, their demand for intermediated finance increases, and intermediary finance becomes scarce and the spread rises. The spread continues to rise as long as the firm’s collateral constraint for intermediated finance binds. Once the spread gets so high that the collateral constraint is slack, the spread declines again as both firms and intermediaries accumulate net worth. Since intermediary net worth accumulates more slowly, firms may temporarily accumulate more net worth and then later on re-lever as they switch to more intermediated finance when intermediaries become better capitalized. Eventually, the spread on intermediated finance declines to the steady state spread as intermediaries accumulate their steady state level of net worth.

A credit crunch, modeled as a drop in intermediary net worth, has persistent real effects in our model. While small drops to intermediary net worth can be absorbed by a cut in dividends, larger shocks reduce intermediary lending and raise the spread on intermediated finance. Real investment drops, and indeed drops even if the corporate sector is well capitalized, as the rise in the cost of intermediated finance raises firms’ cost of capital. Remarkably, the recovery of investment after a credit crunch can be delayed, or stall, as the cost of intermediated finance only starts to fall once intermediaries have again accumulated sufficient net worth.

In a stochastic economy, we provide sufficient conditions for the marginal value of intermediary and firm net worth to comove. For example, if intermediary net worth is sufficiently low, these values comove and indeed move proportionally. Thus, the marginal value of intermediary net worth may be high exactly when the marginal value of firm net
worth is high, too.

Few extant theories of financial intermediaries provide a role for intermediary capital. Notable is in particular Holmström and Tirole (1997) who model intermediaries as monitors that cannot commit to monitoring and hence need to have their own capital at stake to have incentives to monitor. In their analysis, firm and intermediary capital are exogenous and the comparative statics with respect to these are analyzed. Holmström and Tirole conclude that “[a] proper investigation ... must take into account the feedback from interest rates to capital values. This will require an explicitly dynamic model, for instance, along the lines of Kiyotaki and Moore [1997a].” We provide a dynamic model in which the joint evolution of firm and intermediary net worth and the interest rate on intermediated finance are endogenously determined. Diamond and Rajan (2001) and Diamond (2007) model intermediaries as lenders which are better able to enforce their claims due to their specific liquidation or monitoring ability in a similar spirit to our model, but do not consider equilibrium dynamics. In contrast, the capitalization of financial intermediaries plays essentially no role in liquidity provision theories of financial intermediation (Diamond and Dybvig (1983)), in theories of financial intermediaries as delegated, diversified monitors (Diamond (1984), Ramakrishnan and Thakor (1984), and Williamson (1986)) or in coalition based theories (Townsend (1978) and Boyd and Prescott (1986)).

Dynamic models in which net worth plays a role, such as Bernanke and Gertler (1989) and Kiyotaki and Moore (1997a), typically consider the role of firm net worth only, although dynamic models in which intermediary net worth matters have recently been considered (see, for example, Gertler and Kiyotaki (2010), who also summarize the recent literature, and Brunnermeier and Sannikov (2014)). However, to the best of our knowledge, we are the first to consider a dynamic contracting model in which both firm and intermediary net worth are critical and jointly affect the dynamics of financing, spreads, and economic activity.

In Section 2 we describe the model and establish the equivalence between our economy with limited enforcement and an economy with two types of collateral constraints, for intermediated and direct finance, respectively. Section 3 studies how the spread on intermediated finance varies with firm and intermediary net worth in a simplified static version of the model. The dynamics of intermediary capital are analyzed in Section 4. We first consider the deterministic steady state and dynamics of firm and intermediary capital, and the dynamic effects of a credit crunch. We then provide sufficient condi-

\[^2^\]Gromb and Vayanos (2002) and He and Krishnamurthy (2012) study the asset pricing implications of intermediary net worth in dynamic models.
tions for the comovement of the marginal value of intermediary and firm net worth in a stochastic economy. Section 5 concludes. All proofs are in Appendix A.

2 Intermediaries as Collateralization Specialists

We propose a model of financial intermediaries as collateralization specialists in an economy with collateral constraints derived from limited enforcement. We consider an economy in which firms can borrow from intermediaries and households. Limited enforcement constrains firms’ and intermediaries’ ability to make credible promises. Intermediaries, but not households, participate in markets at all times which affords intermediaries with an advantage in enforcing claims. We show that this economy with limited enforcement is equivalent to an economy with two types of collateral constraints, one for collateralized financing by intermediaries and one for collateralized financing by households.\(^3\) We derive a recursive representation of the economy with collateral constraints which is tractable. Since intermediaries are better able to enforce collateralized claims, they can lend more than households, but the additional amount that they can lend has to be financed out of their own net worth, giving a role to financial intermediary capital. Moreover, our model implies that intermediated finance is short-term in the sense that such loans are extended at the end of every period but need to be repaid at the beginning of the next period only to be taken out again at the end of that period. Thus, we provide a novel model of short-term financing.

2.1 Environment

Time is discrete and the horizon infinite. There are three types of agents: entrepreneurs, financial intermediaries, and households; we discuss these in turn. Each period has two subperiods which we refer to as morning and afternoon. All types of agents participate in markets in the afternoon. In the morning, however, only entrepreneurs and intermediaries participate in markets but not households. This is the key assumption affording intermediaries an enforcement advantage.

There is a continuum of entrepreneurs or firms with measure one which are risk neutral and subject to limited liability and discount the future at rate \(\beta \in (0, 1)\). We consider an environment with a representative firm. The representative firm (which we at times refer to simply as the firm or the corporate sector) has limited net worth \(w_0\) and has

\(^3\)In Appendix B, we establish this equivalence in a static environment. The reader may prefer to start by considering this more straightforward case first.
access to a standard neoclassical production technology; an investment of an amount \( k_t \)
of capital in the afternoon at time \( t \) yields output \( A(s^{t+1})f(k_t) \) the next morning where \( A(s^{t+1}) > 0 \) is the stochastic total factor productivity and \( f(\cdot) \) is the production function. Capital \( k_t \) depreciates at the rate \( \delta \in (0, 1) \) between the afternoon and the next morning. We assume that the production function \( f(\cdot) \) is strictly increasing and strictly concave and satisfies the usual Inada condition. Total factor productivity \( A(s^{t+1}) \) depends on the state \( s^{t+1} \) realized the next morning which follows a stochastic process to be described in more detail below. The firm can raise financing from both intermediaries and households as we discuss below.

There is a continuum of financial intermediaries with measure one which are risk neutral, subject to limited liability, and discount future payoffs at \( \beta_i \in (0, 1) \). Intermediaries participate in markets in both subperiods, the morning and afternoon. We consider the problem of a representative financial intermediary with limited net worth \( w_i \). Intermediaries can lend to and borrow from firms and households as described in more detail below.

There is a continuum of households with measure one which are risk neutral and discount future payoffs at a rate \( R^{-1} \in (0, 1) \). These households participate in markets only in the afternoon, but are assumed to have a large endowment of funds and collateral in all dates and states, and hence are not subject to enforcement problems but rather are able to commit to deliver on their promises. They are willing to provide any state-contingent claim paid in the afternoon at an expected rate of return \( R \) as long as such claims satisfy the firms’ and intermediaries’ limited enforcement constraints.

We assume that \( \beta < \beta_i < R^{-1} \), that is, households are more patient than intermediaries which in turn are more patient than the firms. Since firms and intermediaries are financially constrained, they would have an incentive to accumulate net worth and save themselves out of their constraints. Assuming that firms and intermediaries are impatient relative to households is a simple way to ensure that their net worth matters even in the long run. Moreover, assuming that intermediaries are somewhat more patient that firms implies that the net worth of both the corporate sector and the intermediary sector are uniquely determined in the long run, too. We think these features are desirable properties of a dynamic model of intermediation and are empirically plausible.

The economy has limited enforcement in the spirit of Kehoe and Levine (1993) except that firms or intermediaries that default cannot be excluded from participating in financial

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4There is a representative intermediary in our model since intermediaries have constant returns to scale, making the distribution of intermediaries’ net worth irrelevant and aggregation in the intermediation sector straightforward, and thus only the aggregate capital of the intermediation sector matters.
and real asset markets going forward. Rampini and Viswanathan (2010, 2013, 2015) study this class of economies but consider only a version with one type of lender. We build on their work by considering an economy with two types of lenders, intermediaries and households. Specifically, enforcement is limited as follows: Firms can abscond both in the morning and in the afternoon. In the morning, after cash flows are realized, firms can abscond with all cash flows and a fraction $1 - \theta_i$ of depreciated capital, where $\theta_i \in (0, 1)$. In the afternoon, firms can abscond with cash flows net of payments made in the morning and a fraction $1 - \theta$ of depreciated capital, where $\theta \in (0, 1)$. Critically, we assume that $\theta_i > \theta$, which means that firms can abscond with less capital in the morning than in the afternoon. Intermediaries, too, can abscond in both subperiods, although there is no temptation for intermediaries to do so in the morning, as they will at best receive payments, and so we can ignore this constraint and focus just on the afternoon. In the afternoon, intermediaries can abscond with any payments received in the morning. To reiterate, neither firms nor intermediaries are excluded from markets after default.

The timing is summarized as follows: Each afternoon, firms and intermediaries first decide whether to make their promised payments or default. Then, firms, intermediaries, and households consume, invest, and borrow and lend. The next morning, cash flows are realized. Firms decide whether to make their promised morning payments or default. Firms carry over the cash flows net of payments made and intermediaries carry over any funds received until the afternoon. No other decisions are made until the afternoon.

We show that this economy with limited enforcement is equivalent to an economy with complete markets in two types of one-period ahead Arrow securities, claims against the next morning and against the next afternoon, each subject to state-by-state collateral constraints. Thus, in this equivalent economy, there are two types of collateral constraints: collateral constraints for loans to be repaid in the morning, which are provided by financial intermediaries, and collateral constraints for loans to be repaid in the afternoon, which can be provided by either intermediaries or households. These collateral constraints are similar to the ones in Kiyotaki and Moore (1997a), except that there are different collateral constraints for promises to pay in the morning and afternoon, and that the collateral constraints are state-by-state.

Financial intermediaries in this economy function as collateralization specialists. Intermediaries are better able to collateralize claims than households, since they participate in markets in the morning when firms can abscond with a smaller fraction of capital. Therefore, intermediaries are able to seize up to fraction $\theta_i \in (0, 1)$ of the (resale value of) collateral backing promises issued to them. It is critical however that intermediated loans backed by the additional amount of collateral that can be seized in the morning, that
is, $\theta_i - \theta$, are in fact repaid in the morning, as by the afternoon firms can abscond with that additional amount of capital and these payments are no longer enforceable. This implies that these loans are explicitly short term, that is, are extended in the afternoon and must be repaid in the morning and cannot be rolled over. It moreover implies that these loans must be extended by intermediaries, as only they participate in markets in the morning when the claims need to be enforced. And finally it means that intermediaries must finance these loans out of their own net worth, as they cannot in turn finance them by borrowing from households because they could simply default on promises to repay the households in the afternoon and abscond with the payments received in the morning.

Financial intermediaries are however able to refinance loans that they make to firms up to a fraction $\theta$ of collateral which are repaid in the afternoon by borrowing from households. In other words, intermediaries’ corporate loans up to fraction $\theta$ can be used as collateral to borrow from households, whereas loans beyond that, for fraction $\theta_i - \theta$, have to be financed by intermediaries themselves, that is, out of financial intermediary capital.\(^5\)

One interpretation of the environment is that there are three types of capital, working capital, equipment (fraction $\theta_i - \theta$), and structures (fraction $\theta$) (see Figure 1). Firms can always abscond with working capital. Firms cannot abscond with equipment in the morning, but can abscond with equipment in the afternoon.\(^6\) Firms can never abscond with structures. Structure loans can be provided by either intermediaries or households. In contrast, equipment loans have to be extended by intermediaries, have to be repaid in the morning, and have to be finance out of financial intermediary capital.

### 2.2 Economy with limited enforcement

We start by stating the representative firm’s and the representative intermediary’s problem in the economy with limited enforcement. To facilitate this we first develop some notation for states and the stochastic process for productivity, as well as state prices, stochastic discount factors, and state-dependent interest rates for both the morning and the afternoon.

Define a state at date $t$ by $s^t = \{s_0, s_1, \ldots, s_t\}$ which includes the history of realizations

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\(^5\)In contrast, an intermediary could promise corporate loans backed by $\theta_i - \theta$ to other intermediaries, as these participate in markets in the morning as well, that is, the interbank market is frictionless in our model, which is why we are able to consider a representative financial intermediary.

\(^6\)The idea is that unlike immovable assets such as structures which are always collateralizable, movable assets such as machinery and equipment are bolted down in the morning and hence collateralizable then, but are unbolted by the afternoon and hence can no longer serve as collateral.
of the stochastic process $s_\tau$ for dates $\tau = 0, 1, \ldots, t$, and the set of states at date $t$ by $S_t$. Let the transition function from state $s_t^t$ to $s_{t+1}^t$ be $\Pi(s_t^t, s_{t+1}^t)$ and let $\Pi(s_\tau^t, s_t^t)$ be the probability of state $s_t^t$ occurring at date $t$ given state $s_\tau^t$ at date $\tau < t$. Each date has two subperiods, morning and afternoon, and states are realized in the morning.

Each afternoon there are markets in Arrow securities for all subsequent mornings and afternoons. We define the (endogenous) state prices in these markets as follows: $Q_{\tau t}^m := Q_m(s_\tau^t, s_t^t)$ is the price in the afternoon in state $s_\tau^t$ of one unit of goods in the morning in state $s_t^t$, $t > \tau$, and $Q_{\tau t}^a := Q_a(s_\tau^t, s_t^t)$ the price in the afternoon in state $s_\tau^t$ of one unit of goods in the afternoon in state $s_t^t$, $t \geq \tau$. These prices are determined in equilibrium but are taken as given by firms and intermediaries. Note that $Q_{\tau \tau}^a = 1$, that is, the price of a unit of goods in the afternoon today is just one unit. We also define stochastic discount factors $q_{\tau t}^m := q_m(s_\tau^t, s_t^t)$ and $q_{\tau t}^a := q_a(s_\tau^t, s_t^t)$ as follows:

$$Q_m(s_\tau^t, s_t^t) \equiv \Pi(s_\tau^t, s_t^t) q_m(s_\tau^t, s_t^t)$$

$$Q_a(s_\tau^t, s_t^t) \equiv \Pi(s_\tau^t, s_t^t) q_a(s_\tau^t, s_t^t).$$

Moreover, we define state-dependent interest rates $R_{mt+1}^m := R_m(s_t^t, s_{t+1}^t) \equiv (q_m(s_t^t, s_{t+1}^t))^{-1}$ for the morning and $R_{at+1}^a := R_a(s_t^t, s_{t+1}^t) \equiv (q_a(s_t^t, s_{t+1}^t))^{-1}$ for the afternoon, respectively. Finally, to simplify notation, for a stochastic random variable $y_t \equiv y(s_t^t)$, we define the short hand $y_\tau \equiv \{y_t\}_{t=\tau}^\infty$. Similarly we define the short hand $Q_{\tau \tau}^a \equiv \{Q_{u \tau}^a\}_{u=\tau}^\infty$ to be all the current and future state prices for all future dates from time $\tau$ onward; $q_{\tau \tau}^a$ is defined analogously. Analogous definitions apply for the morning.

The firm’s problem ($P_{t}^{LE}$) at date $\tau$ in history $s_\tau$ in the afternoon is to choose $x_t^{LE}$ where $x_t^{LE} = (d_t, k_t, p_t, p_{mt}, p_{at})$, that is, dividends $d_t$, capital $k_t$, net payments to households $p_t$, and net payments to intermediaries in the morning $p_{mt}$ and in the afternoon $p_{at}$, given net worth $w_\tau$, to solve

$$\max_{x_t^{LE}} E_\tau \left[ \sum_{t=\tau}^\infty \beta^{t-\tau} d_t \right]$$

subject to the budget constraints for the current and all subsequent dates and states,

$$w_\tau \geq d_\tau + k_\tau + p_\tau + p_{at}, \quad \forall \tau > \tau$$

$$A_t f(k_t-1) + k_t - (1 - \delta) \geq d_t + k_t + p_t + p_{mt} + p_{at}, \quad \forall t > \tau,$$

the participation constraints for the intermediary and the household,

$$E_\tau \left[ \sum_{t=\tau+1}^\infty q_{\tau t}^m p_{mt} + \sum_{t=\tau}^\infty q_{\tau t}^a p_{at} \right] \geq 0, \quad \forall \tau$$

$$E_\tau \left[ \sum_{t=\tau}^\infty R^{-(t-\tau)} p_t \right] \geq 0, \quad \forall \tau$$
the limited enforcement constraints

\[ E_{\tilde{\tau}} \left[ \sum_{t=\tilde{\tau}}^{\infty} \beta^{t-\tau} d_t \right] \geq E_{\tilde{\tau}} \left[ \sum_{t=\tilde{\tau}}^{\infty} \beta^{t-\tau} \hat{d}_t \right], \quad \forall \tilde{\tau} > \tau, \quad (6) \]

and the non-negativity constraints

\[ d_t, p_{mt} \geq 0, \quad \forall t \geq \tau, \quad (7) \]

where \( \hat{d}_t \) together with the associated investment and financial policy \( \hat{x}^{LE}_t \) solve problem \( \text{P}^{LE}_t \) given net worth \( \hat{w}_\tau \equiv A_{\hat{\tau}} f(k_{\hat{\tau}-1}) + (1 - \theta_i)k_{\hat{\tau}-1}(1 - \delta) \) if the firm absconds in the morning and \( \hat{w}_\tau \equiv A_{\hat{\tau}} f(k_{\hat{\tau}-1}) + (1 - \theta)k_{\hat{\tau}-1}(1 - \delta) - p_{\hat{\tau}} \) if the firm absconds in the afternoon. There are therefore two sets of limited enforcement constraints, one for the morning, where the firm can abscond with cash flows and \( 1 - \theta_i \) of depreciated capital, and one for the afternoon, where the firm can abscond with cash flows minus any payments made in the morning and \( 1 - \theta \) of capital. There is however only one budget constraint for every date, rather than separate budget constraints for the two subperiods, because the firm merely carries over funds from the morning to the afternoon. Note that net payments to intermediaries in the morning are restricted to be non-negative, as intermediaries have no other funds in the morning, but there are no restrictions on \( p_t \) and \( p_{at} \).

The **intermediary’s problem** (\( \text{P}^{LE}_{it} \)) at date \( \tau \) in history \( s^\tau \) in the afternoon is to choose \( x^{LE}_{it} \) where \( x^{LE}_{it} = (d_{it}, p_{ht}, \tilde{p}_{mt}, \tilde{p}_{at}) \), that is, dividends \( d_{it} \), net payments from households \( p_{ht} \), and net payments from firms in the morning \( \tilde{p}_{mt} \) and in the afternoon \( \tilde{p}_{at} \), given net worth \( w_{it} \), to solve

\[ \max_{x^{LE}_{it}} E_{\tau} \left[ \sum_{t=\tau}^{\infty} \beta^{t-\tau} d_{it} \right] \quad (8) \]

subject to the budget constraints for the current and all subsequent dates and states,

\[ w_{it} \geq d_{it} - p_{ht} - \tilde{p}_{at} \quad (9) \]

\[ 0 \geq d_{it} - p_{ht} - \tilde{p}_{mt} - \tilde{p}_{at}, \quad \forall t > \tau, \quad (10) \]

the participation constraint for the firm and the household,

\[ -E_{\tau} \left[ \sum_{t=\tau+1}^{\infty} q_{\tau t}^m \tilde{p}_{mt} + \sum_{t=\tau}^{\infty} q_{\tau t}^a \tilde{p}_{at} \right] \geq 0, \quad (11) \]

\[ -E_{\tau} \left[ \sum_{t=\tau}^{\infty} R^{-(t-\tau)} p_{ht} \right] \geq 0, \quad (12) \]
the limited enforcement constraints

\[ E_{\hat{\tau}} \left[ \sum_{t=\hat{\tau}}^{\infty} \beta^{t-\hat{\tau}} \hat{d}_{it} \right] \geq E_{\hat{\tau}} \left[ \sum_{t=\hat{\tau}}^{\infty} \beta^{t-\hat{\tau}} \hat{d}_{it} \right], \quad \forall \hat{\tau} > \tau, \quad (13) \]

and the non-negativity constraints

\[ d_{it}, \bar{p}_{mt} \geq 0, \quad \forall t \geq \tau, \quad (14) \]

where \( \hat{d}_{it} \) together with associated lending policy \( \hat{x}_{i\hat{\tau}}^{LE} \) solve problem \( P_{i\hat{\tau}}^{LE} \) given wealth \( \hat{w}_{i\hat{\tau}} = \bar{p}_{m\hat{\tau}} \) if the intermediary absconds in the afternoon. Since the intermediary at best receives payments in the morning, the limited enforcement constraint for the morning is redundant and we hence drop it. As in the case of the firm, there is only one budget constraint for each date, rather than separate budget constraints for each subperiod. Note that we again restrict the morning payments from the firm to the intermediary to be non-negative, but there are no restrictions on \( \bar{p}_{at} \) and \( p_{ht} \), \( \forall t \geq \tau. \)

We define an equilibrium for the economy with limited enforcement as follows:

**Definition 1 (Equilibrium with limited enforcement).** An equilibrium with limited enforcement is an allocation \( x_{0}^{LE} \) for the representative firm and \( x_{i0}^{LE} \) for the representative intermediary and stochastic discount factors \( q_{00}^{m} \) and \( q_{00}^{a} \) such that: (i) \( x_{0}^{LE} \) and \( x_{i0}^{LE} \) solve the firm’s problem \( P_{0}^{LE} \) and the intermediary’s problem \( P_{i0}^{LE} \), respectively; (ii) the household participation constraints (5) and (12) hold; and (iii) markets clear, that is, the promises made by firms to intermediaries equal the promises received by intermediaries from firms, \( p_{m0} = \bar{p}_{m0} \) and \( p_{a0} = \bar{p}_{a0}. \)

### 2.3 Economy with collateral constraints

This section describes an economy with collateral constraints and one-period ahead claims (for the morning and the afternoon) which is equivalent to the economy with limited enforcement described above, as we show in the next section.

The firm’s problem \( (P_{0}^{CC}) \) at date 0 in the afternoon is to choose \( x_{0}^{CC} \) where \( x_{t}^{CC} = (d_{t}, k_{t}, b_{t}, b_{it}, b_{at}) \), that is, dividends \( d_{t} \), capital \( k_{t} \), state-contingent loans from households \( b_{t} \), and state-contingent loans from intermediaries to be repaid in the morning \( b_{it} \) and in the afternoon \( b_{at} \), given net worth \( w_{0} \) and given the stochastic interest rates \( R_{i} \), to solve

\[ \max_{x_{0}^{CC}} E_{0} \left[ \sum_{t=0}^{\infty} \beta^{t} d_{t} \right] \quad (15) \]
subject to the budget constraints for the current and all subsequent dates and states, that is, $\forall t \geq 1$,

$$w_0 \geq d_0 + k_0 - (E_0[b_1 + b_{i1} + b_{a1}]) \tag{16}$$

$$A_t f(k_{t-1}) + k_{t-1}(1 - \delta) \geq d_t + k_t + R(b_t + b_{at}) + R_{it}b_{it} - (E_t[b_{t+1} + b_{it+1} + b_{at+1}]), \tag{17}$$

the collateral constraints for loans to be repaid in the morning and afternoon, for all dates and states,

$$(\theta_i - \theta)k_t(1 - \delta) \geq R_{it+1}b_{it+1}, \tag{18}$$

$$\theta k_t(1 - \delta) \geq R(b_{t+1} + b_{at+1}), \tag{19}$$

and the non-negativity constraints for all dates and states

$$d_t, k_t, b_{it} \geq 0. \tag{20}$$

Note that there are no restrictions on $(b_t, b_{at})$. We emphasize that there are two types of collateral constraints restricting loans to be repaid in the morning and afternoon separately. Given our definition of the stochastic discount factor and the state-contingent interest rates, it is the expected value of the claims issued against the next period that enters the budget constraint in the current period. As we show below, the morning loans need to be provided by intermediaries and are provided at the equilibrium state-contingent interest rate $R_{it}$. Afternoon loans by both households and intermediaries are provided at interest rate $R$, which we show below must be the case in equilibrium.

Importantly, morning loans need to be repaid in the morning and postponing payment to the afternoon is not feasible. Morning loans can therefore not be simply rolled over but are extended every afternoon and repaid every morning. Our model thus provides a novel notion of short-term financing.

The intermediary’s problem $(P_{i0}^{CC})$ at date 0 in the afternoon is to choose $x_{i0}^{CC}$ where $x_{it}^{CC} = (d_{it}, l_t, l_{it}, l_{at})$, that is, dividends $d_{it}$, state-contingent loans to households $l_t$, and state-contingent loans to firms to be repaid in the morning $l_{it}$ and in the afternoon $l_{at}$, given net worth $w_{i0}$ and given the stochastic interest rates $R_{i}$, to solve

$$\max_{x_{i0}^{CC}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t d_{it} \right] \tag{21}$$

subject to the budget constraints for the current and all subsequent dates and states,

$$w_{i0} \geq d_{i0} + E_0[l_1 + l_{i1} + l_{a1}], \tag{22}$$

$$0 \geq d_{it} - Rl_t - R_{at}l_{it} - R_{at}l_{at} + E_t[l_{t+1} + l_{it+1} + l_{at+1}], \quad \forall t \geq 1, \tag{23}$$
the collateral constraints for all dates and states

\[ l_t + l_{at} \geq 0, \quad (24) \]

and the non-negativity constraints for all dates and states

\[ d_{it}, l_{it} \geq 0. \quad (25) \]

Note that there are no restrictions on \((l_t, l_{at})\). Critically, the collateral constraints imply that intermediaries can borrow from households only to the extent that they have corporate loans that pay off (in the afternoon) in that state. Intermediaries cannot borrow against corporate loans that pay off in the morning.

We define an equilibrium for the economy with collateral constraints as follows:

**Definition 2 (Equilibrium with collateral constraints).** An equilibrium with collateral constraint is an allocation \(x_{0,CC}\) for the representative firm and \(x_{i,CC}\) for the representative intermediary and interest rates \(R_{i,0}\) such that: (i) \(x_{0,CC}\) and \(x_{i,CC}\) solve the firm’s problem \(P_{0,CC}\) and the intermediary’s problem \(P_{i,CC}\), respectively; and (ii) markets for intermediated debt clear in each date and state, that is, \(b_{i,0} = l_{i,0}\) and \(b_{a,0} = l_{a,0}\).

The economy with collateral constraint turns out to be tractable, in part because we can restrict attention, without loss of generality, to complete markets in one-period ahead morning and afternoon Arrow securities.

### 2.4 Equivalence of limited enforcement and collateral constraints

We now show that the economy with limited enforcement and the economy with collateral constraints are equivalent. We proceed in two steps. First, limited enforcement implies that the present value of the sequence of promises issued by the firm and the intermediary can never exceed the amount of collateral that can be seized. Second, after several intermediate steps, we show that any such sequence can be implemented with the two types of one-period ahead Arrow securities subject to the collateral constraints specified above.

Theorem 1 establishes that limited enforcement implies present value collateral constraints (and vice versa).

**Theorem 1 (Enforcement constraints imply present value collateral constraints).** The firm’s limited enforcement constraints (6) for the morning and the afternoon for all dates
and states are equivalent to present value collateral constraints for all dates and states for the morning
\[ \frac{\theta_i}{\kappa_{t-1}}(1 - \delta) \geq p_t + p_{mt} + p_{at} + E_{t} \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-(t-\tau)} p_t + q_{mt}^m p_{mt} + q_{at}^a p_{at} \right) \right], \quad (26) \]
and for the afternoon
\[ \frac{\theta}{\kappa_{t-1}}(1 - \delta) \geq p_t + p_{at} + E_{t} \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-(t-\tau)} p_t + q_{mt}^m p_{mt} + q_{at}^a p_{at} \right) \right]. \quad (27) \]
Similarly, the intermediary’s limited enforcement constraints (13) for the afternoon in all dates and states are equivalent to present value collateral constraints for the afternoon for all dates and states
\[ p_{ht} + p_{at} + E_{\tau} \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-(t-\tau)} p_{ht} + q_{mt}^m p_{mt} + q_{at}^a p_{at} \right) \right] \geq 0. \quad (28) \]

Before proceeding, we define a notion of consistency of state prices. Essentially, the price of a state \( s^t \) contingent claim is the price of the state \( s^{t-1} \) contingent claim payable in the afternoon times the state \( s^{t-1} \) one-period ahead price for date \( t \) of the morning or afternoon claim, as the case may be.

**Definition 3** (Consistency of state prices). Let history \( s^{t-1} \) be such that \((s^{t-1}, s_t) \equiv s^t\), that is, history \( s^{t-1} \) occurs as part of history \( s^t \). Then, for \( t > \tau \), state prices are consistent if
\[ Q_m(s^\tau, s^t) = Q_a(s^\tau, s^{t-1})Q_m(s^{t-1}, s^t) \quad \text{and} \quad Q_a(s^\tau, s^t) = Q_a(s^\tau, s^{t-1})Q_a(s^{t-1}, s^t). \quad (29) \]

State prices in an equilibrium in the economy with limited enforcement are consistent.

**Lemma 1** (Consistency of state prices). A limited enforcement equilibrium has consistent state prices.

It turns out that the state prices of afternoon claims can be determined by no arbitrage:

**Lemma 2** (One-period ahead afternoon interest rate equals \( R \)). The interest rate on one-period ahead afternoon claims must equal the household’s interest rate, that is, \( R_a(s^\tau, s^{\tau+1}) = (q_a(s^\tau, s^{\tau+1}))^{-1} = R \) for all dates and states.

Essentially, the afternoon state-contingent interest rates must equal the riskless rate \( R \), as otherwise the intermediary could arbitrage by borrowing at \( R \) and lending at \( R_a(s^\tau, s^{\tau+1}) \) in a state-dependent way. Using Lemmas 1 and 2 recursively then yields
\[ Q_m(s^\tau, s^t) = \Pi(s^\tau, s^t) R^{-(t-\tau-1)}(R_m(s^{t-1}, s^t))^{-1} \]
\[ Q_a(s^\tau, s^t) = \Pi(s^\tau, s^t) R^{-t-\tau}. \]
It is noteworthy that multi-period ahead morning claims are priced using the morning state-contingent interest rate only from the afternoon immediately preceding the morning and are priced at the afternoon interest rate before. This is because a multi-period ahead morning claim can be replicated using an afternoon claim paying off in the preceding period and a one-period morning claim from then on. Effectively, the ability to enforce morning claims is used only once. The interest rates for one-period ahead morning claims $R_m(s^{-1}, s^*)$ are to be determined in equilibrium.

We are now ready for the second step, which establishes the equivalence of the economies with limited enforcement and with collateral constraints and one-period ahead complete markets for morning and afternoon claims.

**Theorem 2** (Equivalence of limited enforcement and collateral constraints). An equilibrium with limited enforcement $\left(x_0^{LE}, x_0^{LE}, q_{00}, q_{00}^i\right)$ is equivalent to an equilibrium with collateral constraints $\left(x_0^{CC}, x_0^{CC}, R_0\right)$. More specifically, in the economy with limited enforcement, $x_0^{LE}$ and $x_0^{LE}$ solve $P_0^{LE}$ and $P_0^{LE}$, respectively, and markets clear, $p_{m0} = \bar{p}_{m0}$ and $p_{a0} = \bar{p}_{a0}$, with stochastic discount factors $q_{00}$ and $q_{00}^i$, if and only if, in the economy with collateral constraints, $x_0^{CC}$ and $x_0^{CC}$ solve $P_0^{CC}$ and $P_0^{CC}$, respectively, and markets clear, $b_{i0} = l_{i0}$ and $b_{i0} = \bar{l}_{i0}$ where

(i) **Equivalence of state prices:** The market clearing interest rates on one-period ahead morning claims $R_{it}$ are given by $R_{it+1} = (q_{it+1}^{m})^{-1}$ for all dates and states.

(ii) **Equivalence of allocations:** Dividends and investment $(d_0, k_0, d_{i0})$ are identical in the equilibrium with limited enforcement and with collateral constraints.

Henceforth, we work with the equivalent economy with collateral constraints which turns out to be more tractable.

### 2.5 Recursive representation with collateral constraints

For the economy with collateral constraints, we have so far stated the firm’s problem $(P_0^{CC})$ and the intermediary’s problem $(P_{i0}^{CC})$ in sequence form, albeit with one-period ahead claims. It is convenient to write these problems recursively by defining an appropriate state variable, net worth, for the firm and intermediary.\(^7\) We define the net worth of the firm (next period) as $w' = A'f(k) + k(1 - \delta) - Rb' - Rb_i'$ and the net worth of the intermediary (next period) as $w_i' = Rl' + Rl_i'$, where a prime denotes the value of

\(^7\)In our model with collateral constraints net worth, properly defined, turns out to be the most convenient state variable, whereas the state variable is typically continuation utility in dynamic contracting models in the literature.
a variable next period. We further simplify the analysis by assuming that all afternoon loans to the firm are extended by the household directly, which we refer to as the direct implementation. This is without loss of generality and we could equivalently assume that all corporate loans are extended by the intermediary who in turn borrows from the household, which we refer to as the indirect implementation.

The state of the economy \( z \equiv \{s, w, w_i\} \) includes the exogenous state \( s \) as well as two endogenous state variables, the net worth of the corporate sector \( w \) and the net worth of the intermediary sector \( w_i \). The state-contingent interest rate on intermediated finance \( R'_i \) depends on state \( s' \) and the state \( z \) of the economy, as shown below, but we suppress the argument for notational simplicity. Denote the transition probability on the induced state space for the economy by \( \Pi(z, z') \) in a slight abuse of notation.

The firm’s problem in equations (15) to (20) stated recursively is, for given net worth \( w \) and aggregate state \( z \), to maximize the discounted expected value of future dividends by choosing a dividend payout policy \( d \), capital \( k \), state-contingent promises \( b' \) and \( b'_i \) to households and intermediaries, and state-contingent net worth \( w' \) for the next period, taking the state-contingent interest rates on intermediated finance \( R'_i \) and their law of motion as given, to solve

\[
v(w, z) = \max_{\{d, k, b', b'_i, w'\}} \left\{ d + \beta E[v(w', z')|z] \right\}
\]

subject to the budget constraints for the current and next period

\[
w \geq d + k - E[b' + b'_i|z],
\]

\[
A'f(k) + k(1 - \delta) \geq w' + Rb' + R'_ib'_i,
\]

the collateral constraints for loans from intermediaries and households

\[
(\theta_i - \theta)k(1 - \delta) \geq R'_ib'_i,
\]

\[
\theta k(1 - \delta) \geq Rb',
\]

and the non-negativity constraints

\[
d, k, b'_i \geq 0.
\]

Depending on the realized state next period, the firm repays \( Rb' \) to households and \( R'_ib'_i \) to financial intermediaries as the budget constraint for the next period, equation (32), shows. The interest rate on loans from households \( R \) is constant as discussed above. The middle and right-hand side of Figure 1 illustrate the collateral constraints (33) and (34);

\[8\]

A model with two types of collateral constraints is also studied by Caballero and Krishnamurthy (2001) who consider international financing in a model in which firms can raise funds from domestic and international financiers subject to separate collateral constraints.
one interpretation of these constraints is that equation (33) is the collateral constraint for equipment loans provided by intermediaries and equation (34) is the collateral constraint for structure loans provided by households.

The first order conditions, which are necessary and sufficient, can be written as

\[ \mu = 1 + \nu_d, \quad (36) \]
\[ \mu = E [\beta (\mu' [A' f_k (k) + (1 - \delta)] + [\lambda\theta + \lambda'_i (\theta_i - \theta)] (1 - \delta)] | z], \quad (37) \]
\[ \mu = R\beta\mu' + R\beta\lambda', \quad (38) \]
\[ \mu = R_i\beta\mu' + R_i\beta\lambda' - R_i'\beta\nu'_i, \quad (39) \]
\[ \mu' = v_w(w', z'), \quad (40) \]

where the multipliers on the constraints (31) through (34) are \( \mu, \Pi(z, z')\beta\mu', \Pi(z, z')\beta\lambda', \) and \( \Pi(z, z')\beta\lambda'_i, \) and \( \nu_d \) and \( \Pi(z, z')R_i'\beta\nu'_i \) are the multipliers on the non-negativity constraints on dividends and intermediated borrowing.\(^9\) the envelope condition is \( v_w(w, z) = \mu. \)

The intermediary’s problem in equations (21) to (25) stated recursively is, for given net worth \( w_i, \) to maximize the discounted value of future dividends by choosing a dividend payout policy \( d_i, \) state-contingent loans to households \( l', \) state-contingent intermediated loans to firms \( l'_i, \) and state-contingent net worth \( w'_i \) next period to solve

\[ v_i(w_i, z) = \max_{\{d_i, l', l'_i, w'_i\}} d_i + \beta_i E [v_i(w'_i, z') | z] \quad (41) \]

subject to the budget constraints for the current and next period

\[ w_i \geq d_i + E[l' + l'_i | z], \quad (42) \]
\[ Rl' + R_l l'_i \geq w'_i, \quad (43) \]

and the non-negativity constraints

\[ d_i, l', l'_i \geq 0. \quad (44) \]

Note that we state the intermediary’s problem as if the intermediary only lends the additional amount it can collateralize to firms, that is, we focus on the direct implementation. This simplifies the notation and analysis.

\(^9\)We ignore the constraints that \( k \geq 0 \) and \( w' \geq 0 \) as they are redundant, due to the Inada condition and the fact that the firms can never credibly promise their entire net worth next period (which can be seen by combining (32) at equality with (33) and (34)).
The first order conditions, which are necessary and sufficient, can be written as

\begin{align*}
\mu_i &= 1 + \eta_d, \\
\mu_i &= R\beta_i \mu_i' + R\beta_i \eta_i', \\
\mu_i &= R_i' \beta_i \mu_i' + R_i' \beta_i \eta_i', \\
\mu_i' &= v_{i,w}(w_i', z'),
\end{align*}

where the multipliers on the constraints (42) and (43) are \( \mu_i \) and \( \Pi(z, z') \beta_i \mu_i' \), and \( \eta_d \), \( \Pi(z, z') R\beta_i \eta_i' \), and \( \Pi(z, z') R_i' \beta_i \eta_i' \) are the multipliers on the non-negativity constraints on dividends and direct and intermediated lending; the envelope condition is \( v_{i,w}(w_i, z) = \mu_i \).

We now define an equilibrium for our economy using this recursive notation. An equilibrium determines both aggregate economic activity and the cost of intermediated finance in our economy.

**Definition 4 (Equilibrium).** An equilibrium is an allocation \( x \equiv [d, k, b', b_i', w'] \) for the representative firm and \( x_i \equiv [d_i, l', l_i', w_i'] \) for the representative intermediary for all dates and states and a state-contingent interest rate process \( R_i' \) for intermediated finance such that (i) \( x \) solves the firm’s problem in (30)-(35) and \( x_i \) solves the intermediary’s problem (41)-(44) and (ii) the market for intermediated finance clears in all dates and states

\[ l_i' = b_i'. \]

Note that equilibrium promises are default free, as the promises satisfy the collateral constraints (33) and (34), which ensures that neither firms nor financial intermediaries are able to issue promises on which it is not credible to deliver. While this is of course the implementation that we study throughout, we emphasize that the promises traded in our economy are contingent claims and that these contingent claims may be implemented in practice with noncontingent claims on which issuers are expected and in equilibrium indeed do default (see Kehoe and Levine (2008) for an implementation with equilibrium default in this spirit).

### 2.6 Endogenous minimum down payment requirement

Define the **minimum down payment requirement** \( \varphi \) when the firm borrows the maximum amount it can from households only as \( \varphi = 1 - R^{-1} \theta (1 - \delta) \).\(^{10}\) Similarly, define the minimum down payment requirement when the firm borrows the maximum amount it

\(^{10}\)We use the character \( \varphi \), a script \( p \), for down payment (\( \textnormal{\LaTeX} \) available under miscellaneous symbols).
can from both households (at interest rate $R$) and intermediaries (at state-contingent interest rate $R'_i$) as $\phi_i(R'_i) = 1 - [R^{-1}\theta + E[(R'_i)^{-1}|z]\theta(\theta - \theta)](1 - \delta)$ (illustrated on the right-hand side of Figure 1). Note that the minimum down payment requirement, at times referred to as the margin requirement, is endogenous in our model. Using this definition and equations (37) through (39) the firm’s investment Euler equation can then be written concisely as

$$1 \geq E \left[ \frac{\beta \mu'}{\mu} A'(k) + (1 - \theta)(1 - \delta) \frac{\rho}{R + \rho_i} (\theta_i - \theta)(1 - \delta) \right].$$

(50)

2.7 User cost of capital with intermediated finance

We can extend Jorgenson’s (1963) definition of the user cost of capital to our model with intermediated finance. Define the premium on internal funds $\rho$ as $1/(R + \rho) \equiv E[\beta \mu'/\mu|z]$ and the premium on intermediated finance $\rho_i$ as $1/(R + \rho_i) \equiv E[(R'_i)^{-1}|z]$. Using (37) through (39) the user cost of capital $u$ is

$$u \equiv r + \delta + \frac{\rho}{R + \rho} (1 - \theta)(1 - \delta) + \frac{\rho_i}{R + \rho_i} (\theta_i - \theta)(1 - \delta),$$

(51)

where $r + \delta$ is the frictionless user cost derived by Jorgenson (1963) and $r \equiv R - 1$. The user cost of capital exceeds the user cost in the frictionless model, because part of investment needs to be financed with internal funds which are scarce and hence command a premium $\rho$ (the second term on the right hand side) and part of investment is financed with intermediated finance which commands a premium $\rho_i$, as the funds of intermediaries are scarce as well (the last term on the right hand side).\footnote{Alternatively, the user cost can be written in a weighted average cost of capital representation as $u \equiv R/(R + \rho)(r_w + \delta)$ where the weighted average cost of capital $r_w$ is defined as $r_w \equiv (r + \rho) \phi_i(R'_i) + rR^{-1}\theta(1 - \delta) + (r + \rho_i)(R + \rho_i)^{-1}(\theta_i - \theta)(1 - \delta).$ The cost of capital $r_w$ is a weighted average of the fraction of investment financed with internal funds which cost $r + \rho$ (first term on the right hand side), the fraction financed with households funds at rate $r$ (second term), and the fraction financed with intermediated funds at rate $r + \rho_i$ (third term).}

Internal funds and intermediated finance are both scarce in our model and command a premium as collateral constraints drive a wedge between the cost of different types of finance. Since the firm would never be willing to pay more for intermediated finance than the shadow cost of internal funds, the premium on internal finance is higher than the premium on intermediated finance.

**Proposition 1** (Premia on internal and intermediated finance). The premium on internal finance $\rho$ (weakly) exceeds the premium on intermediated finance $\rho_i$

$$\rho \geq \rho_i \geq 0,$$

11
and the two premia are equal, \( \rho = \rho_i \), iff the collateral constraint for intermediated finance does not bind for any state next period, that is, \( E[\lambda_i | z] = 0 \). Moreover, the premium on internal finance is strictly positive, \( \rho > 0 \), iff the collateral constraint for direct finance binds for some state next period, that is, \( E[\lambda' | z] > 0 \).

When all collateral constraints are slack, there is no premium on either type of finance, but typically the inequalities are strict and both premia are strictly positive, with the premium on internal finance strictly exceeding the premium on intermediated finance.

### 3 Effect of intermediary capital on spreads

In this section we study how the choice between intermediated and direct finance varies with firm and intermediary net worth in a static (one period) version of our model with a representative firm.\(^{12}\) We further simplify by considering the deterministic case, although the results in this section do not depend on this assumption.\(^{13}\) The equilibrium spread on intermediated finance depends on both firm and intermediary net worth. Given firm net worth, spreads are higher when the intermediary is less well capitalized. Importantly, the spread on intermediated finance depends on the relative capitalization of firms and intermediaries. Spreads are particularly high when firms are poorly capitalized and intermediaries are relatively poorly capitalized at the same time. Poor capitalization of the corporate sector does not per se imply high spreads, as firms’ limited ability to pledge may result in a reduction in firms’ loan demand which intermediaries with given net worth can more easily accommodate.\(^{14}\)

The representative firm solves

\[
\max_{\{d, k', l', w'\}} \quad d + \beta w'
\]

subject to (31) through (35). The representative intermediary solves

\[
\max_{\{d_i, l_i', w_i'\}} \quad d_i + \beta_i w_i'
\]

\(^{12}\)In Appendix C, we analyze the choice between intermediated and direct finance in the cross section of firms with different net worths in a static environment. More constrained firms borrow more from intermediaries, which is empirically plausible and similar to the results in Holmström and Tirole (1997).

\(^{13}\)With one period only, the interest rate on intermediated finance is independent of the state \( s' \), as the marginal value of net worth next period for financial intermediaries and firms equals 1 for all states, that is, \( \mu' = \mu_i' = 1 \), rendering the model effectively deterministic.

\(^{14}\)Note that in contrast to our model in Holmström and Tirole (1997) aggregate investment only depends on the sum of firm and intermediary capital.
subject to (42) through (44). An equilibrium is defined in Definition 4. In addition to the equilibrium allocation, the spread on intermediated finance, $R'_i - R$, is determined in equilibrium.

The following proposition summarizes the characterization of the equilibrium spread. Figures 2 through 4 illustrate the results. The key insight is that the spread on intermediated finance depends on both the firm and intermediary net worth. Importantly, low capitalization of the corporate sector does not necessarily result in a high spread on intermediated finance. Indeed, it may reduce spreads. Similarly, while low capitalization of the intermediation sector raises spreads, spreads are substantial only when the corporate sector is poorly capitalized and intermediaries are poorly capitalized relative to the corporate sector at the same time.

**Proposition 2** (Firm and intermediary net worth). *(i)* For $w_i \geq w_i^*$, intermediaries are well capitalized and there is a minimum spread on intermediated finance $\beta_i^{-1} - R > 0$ for all levels of firm net worth. *(ii)* Otherwise, there is a threshold of firm net worth $\bar{w}(w_i)$ (which depends on $w_i$) such that intermediaries are well capitalized and the spread on intermediated finance is $\beta_i^{-1} - R > 0$ as long as $w \leq \bar{w}(w_i)$. For $w > \bar{w}(w_i)$, intermediated finance is scarce and spreads are higher. For $w_i \in [\tilde{w}_i, w_i^*)$, spreads are increasing in $w$ until $w$ reaches $\tilde{w}(w_i)$, at which point spreads stay constant at $\hat{R}^i(w_i) - R \in (\beta_i^{-1} - R, \beta_i^{-1} - \beta_i^{-1} - R]$. For $w_i \in (0, \tilde{w}_i)$, spreads are increasing in $w$ until $w$ reaches $\check{w}(w_i)$, then decreasing in $w$ until $\check{w}(w_i)$ is reached, at which point spreads stay constant at $\beta_i^{-1} - R$. As $w_i \to 0$, $\bar{w}(w_i) \to 0$.

Figure 2 displays the cost of intermediated finance as a function of firm net worth ($w$) and intermediary net worth ($w_i$). Figure 3 displays the contours of the various areas in Figure 2. Figure 4 displays the cost of intermediated finance as a function of firm net worth for different levels of intermediary net worth, and is essentially a projection of Figure 2. When financial intermediaries are well capitalized the spread on intermediated finance is at its minimum, $\beta_i^{-1} - R > 0$. This is the case when financial intermediary net worth is high enough ($w_i \geq w_i^*$) so that they can accommodate the loan demand of even a well capitalized corporate sector or when corporate net worth is relatively low so that the financial intermediary sector is able to accommodate demand despite its low net worth ($w \leq \bar{w}(w_i)$). When intermediary capital is below $w_i^*$ and the corporate sector is not too poorly capitalized ($w > \bar{w}(w_i)$), spreads on intermediated finance are higher. Indeed, when intermediary capital is in this range, higher firm net worth initially raises spreads as loan demand increases (until firm net worth reaches $\hat{w}(w_i)$). This effect can be substantial when $w_i < \tilde{w}_i$. Indeed, interest rates in our example increase to around 200% when financial intermediary net worth is very low, albeit our example is not calibrated.
If firm net worth is still higher, spreads decline as the marginal product of capital and hence firms’ willingness to borrow at high interest rates declines. When corporate net worth exceeds $\bar{w}(w_i)$, the cost on intermediated finance is constant at $\beta^{-1}$, which equals the shadow cost of internal funds of well capitalized firms.

To sum up, spreads are determined by firm and intermediary net worth jointly. Spreads are higher when intermediary net worth is lower. But firm net worth affects both the demand for intermediated loans and, via investment, the collateral available to back such loans. When collateral constraints bind, lower firm net worth reduces spreads.

4 Dynamics of intermediary capital

Our model allows the analysis of the joint dynamics of the capitalization of the corporate and intermediary sector. We first characterize a deterministic steady state and then analyze the deterministic dynamics of firm and intermediary capitalization. Following an adverse shock to net worth, both firms and intermediaries accumulate capital over time, but the corporate sector initially accumulates net worth faster than the intermediary sector, which has important implications for the dynamics of spreads on intermediated finance. The model displays compelling dynamics. When the corporate sector is very constrained, intermediaries hold “cash” at a low interest rate by lending to households to conserve net worth to meet future corporate loan demand. Moreover, in a credit crunch, when intermediaries are very constrained, firms’ investment remains depressed even as firms are paying dividends and the economy may hence be slow to recover. Finally, we provide sufficient conditions for the marginal values of firm and intermediary net worth to comove.

4.1 Intermediaries are essential in a deterministic economy

We first show that intermediaries always have positive net worth, that is, they never choose to pay out their entire net worth as dividends if the economy is deterministic or eventually deterministic, that is, deterministic from some time $T < +\infty$ onward.

Proposition 3 (Positive intermediary net worth). Financial intermediaries always have positive net worth in an equilibrium in a deterministic or eventually deterministic economy.

Since intermediaries always have positive net worth, the interest rate on intermediated finance $R'_i$ must in equilibrium be such that the representative firm never would want to lend at that interest rate, as the following lemma shows:
Lemma 3. In any equilibrium, (i) the cost of intermediated funds (weakly) exceeds the cost of direct finance, that is, $R'_i \geq R$; (ii) the multiplier on the collateral constraint for direct finance (weakly) exceeds the multiplier on the collateral constraint for intermediated finance, that is, $\lambda' \geq \lambda'_i$; and (iii) the constraint that the representative firm cannot lend at $R'_i$ never binds, that is, $\nu'_i = 0$ w.l.o.g. Moreover, in a deterministic economy, (iv) the constraint that the representative intermediary cannot borrow at $R'_i$ never binds, that is, $\eta'_i = 0$; and (v) the collateral constraint for direct financing always binds, that is, $\lambda' > 0$.

We define the essentiality of intermediaries as follows:

Definition 5 (Essentiality of intermediation). Intermediation is essential if an allocation can be supported with a financial intermediary but not without.\textsuperscript{15}

The above results together imply that financial intermediaries must always be essential. First note that firms are always borrowing the maximal amount from households. If firms moreover always borrow a positive amount from intermediaries, then they must achieve an allocation that would not otherwise be feasible. If $R'_i = R$, then the firm must be collateral constrained in terms of intermediated finance, too, that is, borrow a positive amount. If $R'_i > R$, then intermediaries lend all their funds to the corporate sector and in equilibrium firms must be borrowing from intermediaries. We have proved the following:

Proposition 4 (Essentiality of intermediaries). In an equilibrium in a deterministic economy, financial intermediaries are always essential.

4.2 Intermediary capitalization and spreads in a steady state

We define a deterministic steady state in the economy with an infinite horizon as follows:

Definition 6 (Steady state). A deterministic steady state equilibrium is an equilibrium with constant allocations, that is, $x^* \equiv [d^*, k^*, b^*, b'^*, w^*]$ and $x^*_i \equiv [d^*_i, l^*_i, l'^*_i, w^*_i]$, and a constant interest rate on intermediated finance $R'^*_i$.

In the deterministic steady state, intermediaries are essential, have positive capital, and spreads are positive.

Proposition 5 (Steady state). In a deterministic steady state, intermediaries are essential, have positive net worth, and pay positive dividends. The spread on intermediated

\textsuperscript{15}This definition is analogous to the definition of essentiality of money in monetary theory (see, e.g., Hahn (1973)).
finance is $R_i^* - R = \beta_i^{-1} - R > 0$. Firms borrow the maximal amount from intermediaries. The relative (ex dividend) intermediary capitalization is

$$\frac{w_i^*}{w^*} = \frac{\beta_i(\theta_i - \theta)(1 - \delta)}{\theta_i(\beta_i^{-1})}.$$  

The relative (ex dividend) intermediary capitalization, that is, the ratio of the representative intermediary’s net worth (ex dividend) relative to the representative firm’s net worth (ex dividend), is the ratio of the intermediary’s financing (per unit of capital) to the firm’s down payment requirement (per unit of capital). In a steady state, the shadow cost of internal funds of the firm is $\beta - 1$ while the shadow cost of internal funds of the intermediary is $\beta_i^{-1} - 1$ and equals the interest rate on intermediated finance $R_i^* - 1$. Since $\beta_i > \beta$, intermediated finance is cheaper than internal funds for firms in the steady state, and firms borrow as much as they can. In a steady state equilibrium, financial intermediaries have positive capital and pay out the steady state interest income as dividends $d_i^* = (R_i^* - 1)l_i^*$. Both firms and intermediaries have positive net worth in the steady state despite the fact that their rates of time preference differ and both are less patient than households.

The determinants of the capital structure of firms and intermediaries are distinct. In a steady state, firm leverage, that is, the total value of debt relative to total tangible assets, is $1 - \varphi_l(R_i^*) = (R^{-1}\theta + (R_i^*)^{-1}(\theta_i - \theta))(1 - \delta)$ and is determined by the extent to which the firm can collateralize tangible assets, as emphasized in Rampini and Viswanathan (2013). In contrast, intermediary leverage can be defined in our indirect implementation as the value of total direct finance divided by the total value of debt, that is, $R^{-1}\theta(1 - \delta)$ divided by $(R^{-1}\theta + (R_i^*)^{-1}(\theta_i - \theta))(1 - \delta)$, which is approximately equal to $\theta/\theta_i$. Intermediary leverage is therefore determined by the relative enforcement ability of households and intermediaries. To illustrate this distinction, suppose that $\delta = 0.1$, $\theta = 0.4$, $\theta_i = 0.5$, $R = 1.05$ and $\beta_i = 0.94$, then corporate leverage is roughly 43% whereas intermediary leverage is roughly 80%. Thus, the substantial difference in leverage between firms and intermediaries in practice may simply be a consequence of the fact that their capital structures are determined by different factors. Further, the capital of financial institutions comprises about 23% of the global equity market so in practice the ratio of intermediary capitalization to firm capitalization $w_i^*/w$ is about 1 to 3.4. Given the parametrization above, the value in the model would be roughly 1 to 6.7, so somewhat low compared to the data, in part because the intermediaries in the model provide only corporate loans and do not lend to households unlike in practice. Needless to say, these calculations are illustrative in nature, but they should give a sense of the potential of models that provide consistent guidance on the financial structure of firms and intermediaries.
4.3 Deterministic dynamics of intermediary capital and spreads

Consider the dynamics of both firm and intermediary capitalization in an equilibrium converging to the steady state. We show that the equilibrium dynamics evolve in two main phases, an initial one in which the corporate sector pays no dividends and a second one in which the corporate sector pays dividends. Intermediaries do not pay dividends until the steady state is reached, except that they may pay an initial dividend (at time 0), if they are well capitalized relative to the corporate sector at time 0. We first state these results formally and then provide an intuitive discussion of the equilibrium dynamics.

Proposition 6 (Deterministic dynamics). Given $w$ and $w_i$, there exists a unique deterministic dynamic equilibrium which converges to the steady state characterized by a no dividend region (ND) and a dividend region (D) (which is absorbing) as follows:

Region ND $w_i \leq w_i^*$ (w.l.o.g.) and $w < \bar{w}(w_i)$, and (i) $d = 0$ ($\mu > 1$), (ii) the cost of intermediated finance is

$$R'_i = \max \left\{ R, \min \left\{ \frac{(\theta_i - \theta)(1 - \delta)}{\varphi}, \frac{A'f_k \left( \frac{w + w_i}{\varphi} \right)}{\varphi} \right\} \right\},$$

(iii) investment $k = \frac{(w + w_i)}{\varphi}$ if $R'_i > R$ and $k = \frac{w}{\varphi_i(R)}$ if $R'_i = R$, and (iv) $w'/w_i' > w/w_i$, that is, firm net worth increases faster than intermediary net worth.

Region D $w \geq \bar{w}(w_i)$ and (i) $d > 0$ ($\mu = 1$). For $w_i \in (0, \bar{w}_i)$, (ii) $R'_i = \beta^{-1}$, (iii) $k = \bar{k}$ which solves $1 = \beta[A'f_k(\bar{k}) + (1 - \theta)(1 - \delta)]/\varphi$, (iv) $w'_ex/w_i' < w'_ex/w_i$, that is, firm net worth (ex dividend) increases more slowly than intermediary net worth, and (v) $\bar{w}(w_i) = \varphi\bar{k} - w_i$. For $w_i \in [\bar{w}_i, w_i^*]$, (ii) $R'_i = (\theta_i - \theta)(1 - \delta)k/w_i$, (iii) $k$ solves $1 = \beta[A'f_k(k) + (1 - \theta)(1 - \delta)]/(\varphi - w_i/k)$, (iv) $w'_ex/w_i' < w'_ex/w_i$, that is, firm net worth (ex dividend) increases more slowly than intermediary net worth, and (v) $\bar{w}(w_i) = \varphi_i(R'_i)k$. For $w_i \geq w_i^*$, $\bar{w}(w_i) = w^*$ and the steady state of Proposition 5 is reached with $d = w - w^*$ and $d_i = w_i - w_i^*$.

Figure 5 displays the contours of the two regions in terms of firm net worth $w$ and intermediary net worth $w_i$ and Figure 6 illustrates the dynamics of firm and intermediary net worth, the interest rate on intermediated finance, and investment over time. The representative intermediary’s dividend policy is characterized as follows:

Lemma 4 (Initial intermediary dividend). The representative intermediary pays at most an initial dividend and no further dividends until the steady state is reached. If $w_i > w_i^*$, the initial dividend is strictly positive.
To understand the intuition, suppose both firms and financial intermediaries are initially poorly capitalized, and assume moreover that firms are poorly capitalized even relative to financial intermediaries, say in a macroeconomic downturn. The dynamics of financial intermediary net worth are relatively simple, since as long as no dividends are paid (which is the case until the steady state is reached, except possibly at time 0), the intermediaries’ net worth evolves according to the law of motion $w_i' = R_i'w_i$, that is, intermediary net worth next period is simply intermediary net worth this period plus interest income. When no dividends are paid, intermediaries lend out all their funds at the interest rate $R_i'$. Of course, the interest rate $R_i'$ is determined in equilibrium.

Given our assumptions, the corporate sector’s net worth, investment and loan demand evolve in several phases, which are reflected in the dynamics of the equilibrium interest rate. If firms are initially poorly capitalized even relative to financial intermediaries, as we assume, loan demand is low and intermediaries are relatively well capitalized. In this case, except for a potential initial dividend, intermediaries conserve net worth to meet future loan demand by lending some of their funds to households (see Panel B3 of Figure 6) and spreads are zero, that is, $R_i' = R$ (see Panel B1); in this sense, intermediaries are holding “cash” at an interest rate below their discount rate for some time. In fact, the intermediaries’ lending to households exceeds their lending to the corporate sector early on. Corporate investment is then $k = w/\varphi_i(R)$. Intermediaries accumulate net worth at rate $R$ in this phase while the corporate sector accumulates net worth at a faster rate, given the high marginal product; thus, the net worth of the corporate sector rises relative to the net worth of intermediaries. In Figure 6, this phase last from time $t = 0$ to $t = 3$, except that the intermediary pays an initial dividend at $t = 0$, since Figure 6 considers an initial drop in corporate net worth only.

Eventually, the increased net worth of the corporate sector raises loan demand so that intermediated finance becomes scarce. The corporate sector then borrows all the funds intermediaries are able to lend and invests $k = (w + w_i)/\varphi$. The interest rate on intermediated finance is determined by the collateral constraint, which is binding, and equals $R_i' = (\theta_i - \theta)(1 - \delta)(w/w_i + 1)/\varphi$. Note that since corporate net worth increases faster than intermediary net worth, the interest rate on intermediated finance rises in this phase. As the corporate sector accumulates net worth, it can pledge more and the equilibrium interest rate rises. In Figure 6, this occurs between $t = 3$ and $t = 4$.

As the net worth and investment of the corporate sector continues to rise faster than intermediary net worth, the increase in firms’ collateral means that firms’ ability to pledge no longer constrains their ability to raise intermediated finance. Intermediated finance is scarce in this phase because of limited intermediary net worth, however, and so spreads are
high but declining. The law of motion of investment is as in the previous phase \( k = (w + w_i)/\varphi \), while the equilibrium interest rate on intermediated finance is determined by \( R_i' = [A'f_k(k) + (1 - \theta)(1 - \delta)]/\varphi \). Both firm and intermediary net worth continue to increase, and hence investment increases and the equilibrium interest rate on intermediated finance decreases. In Figure 6, this occurs between \( t = 4 \) and \( t = 5 \).

Eventually, the interest rate on intermediated finance reaches \( \beta^{-1} \), the shadow cost of internal funds of the corporate sector. At that point, corporate investment stays constant, that is, the recovery stalls, while at the same time firms start to pay dividends. That is, in this phase, firms seem to be well capitalized because they are paying dividends, yet the economy has not fully recovered. However, intermediaries continue to accumulate net worth and as they do so, the corporate sector reduces its net worth by paying dividends. Essentially, the corporate sector relevers as the supply of intermediated finance increases when financial intermediary net worth increases. This is the case at \( t = 5 \) and \( t = 6 \) in Figure 6.

Once intermediary capital is sufficiently high to accommodate the entire loan demand of the corporate sector at an interest rate \( \beta^{-1} \), the cost of intermediated funds decreases further. As the interest rate on intermediated finance is now below the shadow cost of internal funds of the corporate sector, the collateral constraint binds again. Investment increases due to the reduced cost of intermediated financing and the recovery resumes. This phase lasts from \( t = 7 \) to \( t = 9 \) in Figure 6. Eventually, intermediaries accumulate their steady state level of net worth and the cost of intermediated finance reaches \( \beta_i^{-1} \), the intermediaries’ shadow cost of internal funds. The steady state is reached at \( t = 9 \) in Figure 6.

We emphasize two key aspects of the dynamics of intermediary capital, beyond the fact that intermediary and firm net worth affect the dynamics jointly. First, intermediary capital accumulates more slowly than corporate net worth in our model. Second, when the corporate sector is temporarily relatively poorly capitalized, the interest rate on intermediated finance is low and intermediaries conserve net worth by lending to households at a low interest rate to meet the higher subsequent corporate loan demand. And vice versa, the corporate sector accumulates additional net worth and spreads remain higher (and investment lower than in the steady state) as the corporate sector “waits” for intermediary net worth to rise and eventually reduce spreads, at which point firms relever. Of course, the second two observations are a reflection of the relatively slow pace of intermediary capital accumulation.
4.4 Dynamics of a credit crunch

Suppose the economy experiences a credit crunch, which we model here as an unanticipated one-time drop in intermediary net worth $w_i$. We assume that the economy is otherwise deterministic and is in steady state when the credit crunch hits. Figure 7 illustrates the effects of such a credit crunch on interest rates, net worth, intermediary lending, and investment. The effect of a credit crunch depends on its size. Intermediaries can absorb a small enough credit crunch simply by cutting dividends. But a larger drop in intermediary net worth results in a reduction in lending and an increase in the spread on intermediated finance. Moreover, the higher cost of intermediated finance increases the user cost of capital (51) (as the premium on internal finance is either unchanged or increases) and so investment drops. Thus, a credit crunch has real effects in our model. Remarkably, investment drops even if the corporate sector is still well capitalized (that is, even if $w^* > \bar{w}$). The reason is that the cost of capital increases even if the corporate sector is well capitalized, as intermediaries’ capacity to extend relatively cheap financing is reduced. In that case, the credit crunch results in a jump in the interest rate on intermediated finance to $R_i = \beta^{-1} > R_i^* = \beta^{-1}$ and an immediate drop in investment (and capital, which drops to $\bar{k} < k^*$). The real effects in our model are moreover persistent, even if the corporate sector remains well capitalized. Indeed, the recovery of the real economy can be delayed. After a sufficiently large credit crunch, investment and capital remain constant at the lower level, and spreads remain constant at the elevated level, until the intermediary sector accumulates sufficient capital to meet the loan demand. At that point, intermediary interest rates start to fall and investment begins to recover, until the economy eventually recovers fully.

If the corporate sector is no longer well capitalized after the credit crunch, the spread on intermediated finance rises further and investment drops even more. This is the case in Figure 7 at time 0 (see Panel B1 and B4). Moreover, after an initial partial recovery, the recovery stalls, potentially for a long time (from time 1 to time 23 in Figure 7), in the sense that the interest rate on intermediated finance remains at $R_i = \beta^{-1}$ and investment remains constant below its steady state level (in fact, capital remains constant at $\bar{k}$), until the intermediaries accumulate sufficient capital. Then the recovery resumes.

The firms’ response to a credit crunch is also striking. In response to a credit crunch, firms cut dividends to substitute retained earnings for intermediated loans. That is, firms delever and temporarily accumulate more net worth than they retain in the steady state. Only as the intermediaries recover, do firms gradually relever.

If net worth of both the intermediaries and the corporate sector drop at the same time, for example, because of a one-time depreciation shock to capital, then investment and
output fall more substantially. The dynamics of the recovery from such a downturn are as described in Section 4.3. It is noteworthy, though, that the spreads on intermediated finance may or may not go up in such a general downturn, and in fact may well go down despite the scarcity of intermediary capital. The point is that the lower net worth of the corporate sector reduces loan demand, possibly by more than the drop in intermediary net worth reduces loanable funds. If corporate loan demand drops sufficiently, intermediaries may pay a one time dividend when the downturn hits, and then cut dividends to zero until the economy recovers.

Our model can be used as a laboratory to evaluate a policy that provides some net worth to either the firms or intermediaries. If the policy maker’s objective is to speed up the recovery, that is, reach the steady state as soon as possible, this is best accomplished by giving the net worth to the intermediaries, as long as they do not currently pay dividends. If the policy maker were to give the additional net worth to dividend-paying intermediaries, they would simply pay it out as additional dividends. If the intermediaries are holding cash but not paying dividends, they respond by lending the additional net worth to households, that is, simply increase their cash holdings, as long as the additional amount is not too large; if the additional amount of net worth provided is large, the intermediaries would increase their cash holdings and pay out a one time dividend. Note that in this case, firm investment is unaffected as long as intermediaries would have held cash anyway, that is, there is no effect on the real economy for some time. Thus, our model shows that there are subtle issues in evaluating the policy choice of whether to provide net worth to the corporate sector or the financial intermediary sector.

### 4.5 Comovement of firm and intermediary capital

Do the marginal value of firm and intermediary net worth comove? We consider this question in a stochastic economy which is deterministic from time 1 onward. Importantly, this allows both firms and intermediaries to engage in risk management at time 0 and hedge the net worth available to them in different states $s' \in S$ at time 1. We first show that the representative firm optimally engages in incomplete risk management, that is, the collateral constraint for direct finance against at least one state $s' \in S$ must bind. We then provide sufficient conditions for the marginal value of net worth of the representative firm and the representative intermediary to comove.

**Proposition 7** (Comovement of the value of firm and intermediary capital). In an economy that is deterministic from time 1 onward and has constant expected productivity, (i) the representative firm must be collateral constrained for direct finance against at least
one state at time 1; (ii) the marginal value of firm and intermediary net worth comove, in fact $\mu(s')/\mu(s'_+)$ = $\mu_i(s')/\mu_i(s'_+)$, $\forall s', s'_+ \in S$, if $\lambda_i(s') = 0$, $\forall s' \in S$. (iii) Suppose moreover that there are just two states, that is, $S = \{\hat{s}', \check{s}'\}$. If only one of the collateral constraints for direct finance binds, $\lambda(\check{s}') > 0 = \lambda(\hat{s}')$, then the marginal values must comove, $\mu(\hat{s}') > \mu(\check{s}')$ and $\mu_i(\hat{s}') \geq \mu_i(\check{s}')$.

Proposition 7 implies that the marginal values of firm and intermediary net worth comove, for example, when the intermediary has very limited net worth and hence the collateral constraints for intermediated finance are slack for all states. They also comove if the firm hedges one of two possible states, as then the intermediary effectively must be hedging that state, too. Thus, the marginal value of intermediary net worth may be high exactly when the marginal value of firm net worth is high, too. The marginal values may however move in opposite directions, for example, if a high realization of productivity raises firm net worth substantially, which lowers the marginal product of capital and hence the marginal value of firm net worth, while it may raise loan demand substantially and hence raise the marginal value of intermediary net worth.

5 Conclusion

We develop a dynamic theory of financial intermediation and show that the capital of both the financial intermediary and corporate sector affect real economic activity, such as firm investment, financing, and the spread between intermediated and direct finance. We derive collateral constraints from an explicit model of limited enforcement in which financial intermediaries play the role of collateralization specialists. Financial intermediaries participate in markets at all times allowing them to enforce claims when more of the capital is collateralizable. This advantage in enforcement enables financial intermediaries to lend more than households, but they cannot in turn finance such loans by borrowing from households and hence have to finance these loans out of their own net worth; we argue that this is why financial intermediaries need capital. In our view, the enforcement of payment is a key rationale for the existence of financial intermediaries, in addition to the monitoring and liquidity provision motives emphasized in the previous literature.

Our model implies that intermediated finance is explicitly short term in nature and must be repaid every period rather than being rolled over; thus, intermediated loans are short term in a novel sense here. The determinants of capital structure of firms and intermediaries in the model are distinct. Firm leverage is determined by the extent to which firms can pledge their assets as collateral to intermediaries and households. In contrast, the leverage of intermediaries is determined by the relative ability of intermediaries and
households to enforce collateralized claims. When firms have limited tangible assets and the difference between the intermediaries’ and households’ ability to enforce claims is not too large, firm leverage is substantially lower than intermediary leverage, as is the case in practice.

Our model features compelling dynamics. When the corporate sector is severely constrained, say after a macroeconomic downturn, intermediaries hold cash to conserve net worth to meet future corporate loan demand and spreads are low. In a credit crunch, when intermediary capital is low, investment and real activity are depressed even if the corporate sector is well capitalized and paying dividends. A key factor driving these results is that intermediary capital accumulates more slowly than the capital of firms in the model. Our model may provide a useful framework for the analysis of the dynamic interaction between financial structure and economic activity.
Appendix

Appendix A: Proofs

Proof of Theorem 1. We first consider the firm and show that the limited enforcement implies present value collateral constraints. The proof is by contraposition, that is, we show that if the present value collateral constraint is violated, then so is the limited enforcement constraint.

Define, for $\tau' \geq \tau$,

$$PV_{\tau}(p_{\tau'}) = E_{\tau} \left[ \sum_{t=\tau'}^{\infty} R^{-t-\tau} p_t \right]$$

and similarly

$$PV_{m\tau}(p_{m\tau'}) = E_{\tau} \left[ \sum_{t=\tau'}^{\infty} q_{m\tau}^t p_{mt} \right] ;$$

$$PV_{a\tau}(p_{a\tau'}) = E_{\tau} \left[ \sum_{t=\tau'}^{\infty} q_{a\tau}^t p_{at} \right].$$

Suppose that the present value collateral constraint (27) does not hold in the afternoon at time $\tau$. The firm can default and issue new promises at date $\tau$ and keep the promises, investments, and dividends in all future dates and states the same. The firm can make the same investment $k_{\tau}$ and must be able to pay a higher dividend, since, if (27) is violated,

$$\theta k_{\tau-1}(1 - \delta) - [p_{\tau} + p_{a\tau}] < PV_{\tau}(p_{\tau+1}) + PV_{m\tau}(p_{m\tau+1}) + PV_{a\tau}(p_{a\tau+1}) = -\hat{p}_{\tau} - \hat{p}_{a\tau},$$

and therefore

$$\hat{d}_\tau = A_{\tau} f(k_{\tau-1}) + (1 - \theta) k_{\tau-1}(1 - \delta) - k_{\tau} - \hat{p}_{\tau} - p_{m\tau} - \hat{p}_{a\tau}$$

$$> A_{\tau} f(k_{\tau-1}) + k_{\tau-1}(1 - \delta) - k_{\tau} - p_{\tau} - p_{m\tau} - p_{a\tau} = d_\tau,$$

which completes the proof in one direction.

Suppose now that the present value collateral constraint (27) holds in the afternoon at time $\tau$. If the firm were to default in the afternoon, the firm’s net worth $\hat{w}_{\tau}$ would be $\hat{w}_{\tau} = A_{\tau} f(k_{\tau-1}) + (1 - \theta) k_{\tau-1}(1 - \delta) - p_{m\tau}$. Consider an optimal plan given $\hat{w}_{\tau}$ say $\hat{x}_{\tau}$. One can instead implement a plan $\check{x}_{\tau}$ without defaulting that has $\check{x}_{\tau+1} = \hat{x}_{\tau+1}$, $\check{k}_{\tau} = \hat{k}_{\tau}$, and choose promises today given by

$$\check{p}_{\tau} = \hat{p}_{\tau} + p_{\tau} + PV_{\tau}(p_{\tau+1})$$
\[ \hat{p}_{ar} = \hat{p}_{ar} + p_{ar} + \text{PV}_{mr}(p_{m\tau+1}) + \text{PV}_{ar}(p_{a\tau+1}). \]

Hence, the firm could pay the present value of its current promises and make the same promises as under \( \hat{x}_\tau \). Using (4) and (5) at equality for \( \hat{x}_\tau \) and (27) for \( x_\tau \) we have

\[ \theta k_{\tau-1}(1 - \delta) \geq \hat{p}_r + \hat{p}_{ar} + \text{PV}_r(\hat{p}_r) + \text{PV}_{mr}(\hat{p}_{m\tau+1}) + \text{PV}_{ar}(\hat{p}_{a\tau+1}), \]

so (27) is satisfied for \( \hat{p}_r, \hat{p}_{m\tau+1}, \) and \( \hat{p}_{ar} \). Moreover, using (27) for \( x_\tau \), the dividend \( \hat{d}_\tau \) must satisfy

\[ \hat{d}_\tau = A_\tau f(k_{\tau-1}) + (1 - \theta)k_{\tau-1}(1 - \delta) - k_\tau - \hat{p}_r - p_{mr} - \hat{p}_{ar} \]
\[ \leq A_\tau f(k_{\tau-1}) + k_{\tau-1}(1 - \delta) - k_\tau - \hat{p}_r - p_{mr} - \hat{p}_{ar} - p_r - p_{ar} - \text{PV}_r(p_{r\tau+1}) - \text{PV}_{mr}(p_{m\tau+1}) - \text{PV}_{ar}(p_{a\tau+1}) \]
\[ = A_\tau f(k_{\tau-1}) + k_{\tau-1}(1 - \delta) - k_\tau - \hat{p}_r - \hat{p}_{ar} = \hat{d}_\tau. \]

Thus, a feasible strategy without default yields a weakly higher payoff, implying that default cannot be optimal.

Next we show that if the present value collateral constraint (26) does not hold in the morning at time \( \tau \), then the limited enforcement constraint is violated, too. If (26) is violated, the firm could default in the morning, and in the afternoon set \( \hat{x}_{\tau+1} = x_{\tau+1}, \)
\[ \hat{k}_\tau = k_\tau, \]
and issue new promises today such that (4) and (5) hold with equality and

\[ 0 = \hat{p}_r + \hat{p}_{ar} + \text{PV}_r(p_{r\tau+1}) + \text{PV}_{mr}(p_{m\tau+1}) + \text{PV}_{ar}(p_{a\tau+1}). \]

Since the present value collateral constraint is violated, we have

\[ \theta_i k_{\tau-1}(1 - \delta) - [p_r + p_{mr} + p_{ar}] < \text{PV}_r(p_{r\tau+1}) + \text{PV}_{mr}(p_{m\tau+1}) + \text{PV}_{ar}(p_{a\tau+1}) = -\hat{p}_r - \hat{p}_{ar}, \]

and the firm can set \( \hat{d}_\tau \) to

\[ \hat{d}_\tau = A_\tau f(k_{\tau-1}) + (1 - \theta_i)k_{\tau-1}(1 - \delta) - k_\tau - \hat{p}_r - \hat{p}_{ar} \]
\[ > A_\tau f(k_{\tau-1}) + k_{\tau-1}(1 - \delta) - k_\tau - p_r - p_{mr} - p_{ar} = d_\tau, \]

an improvement. Therefore, the limited enforcement constraint in the morning implies the corresponding present value collateral constraint.

The proof that the present value collateral constraint in the morning implies the limited enforcement constraint in the morning follows the argument for the afternoon closely and is hence omitted.

Next we turn to the intermediary’s problem. Since the intermediary receives a non-negative payment in the morning, intermediary default is a concern only in the afternoon. Suppose that the present value collateral constraint (28) at date \( \tau \) is violated, that is,

\[ p_{h\tau} + \hat{p}_{ar} + \text{PV}_r(p_{h\tau+1}) + \text{PV}_{mr}(\hat{p}_{m\tau+1}) + \text{PV}_{ar}(\hat{p}_{a\tau+1}) < 0. \]
The intermediary could default with \( \hat{w}_i\tau = \hat{p}_m\tau \) and issue new promises at date \( \tau \)

\[
\hat{p}_h\tau = -\text{PV}_\tau(p_{h\tau+1}) \quad \text{and} \quad \hat{p}_a\tau = -\text{PV}_{m\tau}(\hat{p}_{m\tau+1}) - \text{PV}_{a\tau}(\hat{p}_{a\tau+1})
\]

while keeping the promises and dividends in all future dates and states the same. Hence, \( \hat{p}_h\tau + \hat{p}_a\tau > p_h\tau + p_a\tau \) and the dividend at date \( \tau \) is higher since

\[
\hat{d}_i\tau = \hat{w}_i\tau + \hat{p}_h\tau + \hat{p}_a\tau > w_i\tau + p_h\tau + p_a\tau = d_i\tau,
\]

implying that the limited enforcement constraint would be violated, too.

To see that (28) implies (13) proceed analogously to the proof for the firm by assuming that the intermediary defaults and showing that the intermediary could do at least as well without defaulting. □

**Proof of Lemma 1.** Consider first consistency of state prices for afternoon payments. Suppose consistency is violated at time \( \tau \) for state \( s^{\tau+2} \) and assume w.l.o.g. that \( Q_a(s^\tau, s^{\tau+2}) > Q_a(s^\tau, s^{\tau+1})Q_a(s^{\tau+1}, s^{\tau+2}) \). The intermediary could issue a claim against state \( s^{\tau+2} \) and receive \( Q_a(s^\tau, s^{\tau+2}) \) at time \( \tau \) and at the same time purchase \( Q_a(s^{\tau+1}, s^{\tau+2}) \) units of state \( s^{\tau+1} \) claims at a per unit cost of \( Q_a(s^\tau, s^{\tau+1}) \), yielding a positive payout today. In state \( s^{\tau+1} \), the promise the intermediary issued will be worth \( Q_a(s^{\tau+1}, s^{\tau+2}) \) which equals the payoff of the one-period claim. Thus the intermediary can repurchase the claim at that time, yielding a zero payout then. Note that the payoff to the intermediary is positive at time \( \tau \) and that the present value of the future promises is zero, implying that the present value collateral constraint is satisfied throughout. This is an arbitrage and hence afternoon state prices have to be consistent for time \( \tau + 2 \), and, proceeding recursively, for all \( t \geq \tau + 2 \).

To prove the consistency claim for morning state prices, we have to maintain the constraints that \( p_{mt} \) and \( \hat{p}_{mt} \) have to be non-negative. Suppose first that \( Q_m(s^\tau, s^{\tau+2}) > Q_a(s^\tau, s^{\tau+1})Q_m(s^{\tau+1}, s^{\tau+2}) \). The firm could issue a claim against state \( s^{\tau+2} \) in the morning and receive \( Q_m(s^\tau, s^{\tau+2}) \) at time \( \tau \) and at the same time purchase \( Q_m(s^{\tau+1}, s^{\tau+2}) \) units of state \( s^{\tau+1} \) claims at a per unit cost of \( Q_a(s^\tau, s^{\tau+1}) \), yielding a positive payout today. In state \( s^{\tau+1} \), the promise the firm issued will be worth \( Q_m(s^{\tau+1}, s^{\tau+2}) \) which equals the payoff of the one-period claims bought. Thus the firm can repurchase the claim at that time, yielding a zero payout then. Note that the payoff to the firm is positive at time \( \tau \), that is, this is an arbitrage, and that the present value of the future promises is zero, implying that the present value collateral constraint is satisfied throughout. Suppose instead that \( Q_m(s^\tau, s^{\tau+2}) < Q_a(s^\tau, s^{\tau+1})Q_m(s^{\tau+1}, s^{\tau+2}) \), then the intermediary could purchase a claim against state \( s^{\tau+2} \) in the morning and proceed analogously to the firm above, reversing
all the signs of the transactions. In either case, there is thus an arbitrage and hence morning state prices have to be consistent for time $\tau + 2$, and, proceeding recursively, for all $t \geq \tau + 2$. □

**Proof of Lemma 2.** Suppose $R_a(s^\tau, s^{\tau+1}) > R$ for some $(s^\tau, s^{\tau+1})$. The intermediary could lend t the firm at $R_a(s^\tau, s^{\tau+1})$ and borrow from the household at $R$ against state $s^{\tau+1}$. This transaction satisfies the present value collateral constraints at both $s^\tau$ and $s^{\tau+1}$ and yields a zero payoff at $s^\tau$ and a strictly positive payoff at $s^{\tau+1}$, an arbitrage. Therefore, $R_a(s^\tau, s^{\tau+1}) \leq R$ for all $(s^\tau, s^{\tau+1})$. Moreover, if $R_a(s^\tau, s^{\tau+1}) < R$ for some $(s^\tau, s^{\tau+1})$ then the reverse transaction presents an arbitrage for the intermediary. Thus, the interest rate on one-period ahead afternoon claims equals $R$ in all dates and states. □

**Proof of Theorem 2.** Using Lemmas 1 and 2, we can write the present value collateral constraints (26) and (27) for the morning and afternoon, respectively, as

$$\theta_i k_{\tau-1} (1 - \delta) \geq p_t + p_{mr} + p_{ar} + E_t \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-\tau}(t) p_t + R^{-(t-\tau-1)}(R_{mt})^{-1}p_{mt} + R^{-(t-\tau)}p_{at} \right) \right]$$

and

$$\theta k_{\tau-1} (1 - \delta) \geq p_t + p_{ar} + E_t \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-\tau}(t) p_t + R^{-(t-\tau-1)}(R_{mt})^{-1}p_{mt} + R^{-(t-\tau)}p_{at} \right) \right].$$

Now define $p^*_t \equiv p_t + E_t[(R_{mt+1})^{-1}p_{mt+1}]$ and rewrite the collateral constraints as

$$\theta_i k_{\tau-1} (1 - \delta) \geq p_t + p_{mr} + p_{ar}^+ + PV_{\tau}(p_{\tau+1} + p_{ar+1}^+) \quad (54)$$

and

$$\theta k_{\tau-1} (1 - \delta) \geq p_t + p_{ar}^+ + PV_{\tau}(p_{\tau+1} + p_{ar+1}^+). \quad (55)$$

Note that determining $(p_0, p_{m0}, p_{a0})$ and $(p_0, p_{m0}, p_{a0}^+)$ are equivalent. Next we show that collateral constraints (54) and (55) are equivalent to (55) and the following collateral constraint

$$(\theta_i - \theta) k_{\tau-1} (1 - \delta) \geq p_{mt} \quad (56)$$

First, observe that adding (55) and (56) yields (54) which establishes the first direction. Second, to establish the other direction, suppose (56) is violated, that is, $(\theta_i - \theta) k_{\tau-1} (1 - \delta) < p_{mt}$. Then it must be that (55) is slack, that is, the inequality must be strict, as otherwise adding (56) and (55) would imply that (54) is violated. For such a state $s^\tau$
the firm could raise the payment \( p_\tau \) to households in the afternoon by \( R \varepsilon \) and reduce the payment \( p_m \tau \) to intermediaries in the morning by \( R_m \varepsilon \) (and correspondingly reduce the payment \( p_{\tau-1} \) to households at time \( \tau - 1 \) by \( \Pi(s^\tau,s^{\tau+1})\varepsilon \) while raising the payment \( p_a^+ \tau-1 \) to intermediaries by the same amount). This would yield an additional payoff \( (R_m \tau - R)\varepsilon \) in state \( s^\tau \) which is non-negative and strictly positive if \( R_m \tau > R \); but in the latter case we obtain a strict improvement, which is not possible, and in the former case we can shift the payment to the afternoon and (73) holds without loss of generality. This establishes the equivalence of the economy with limited enforcement and an economy with collateral constraints as in (56) and (55).

We now show how to recover the collateral constraints in the one period debt form (18), (19), and (24). First, let \( R_i \tau \equiv R_m \tau \), for all \( (\tau,s^\tau) \). Second, for \( \tau \geq 1 \) define

\[
Rb_\tau \equiv PV_\tau(p_\tau), \quad b_i \tau = (R_i \tau)^{-1}p_m \tau, \quad Rb_a \tau \equiv PV_\tau(p_a^+),
\]

and rewrite (56) and (55) as

\[
(\theta_i - \theta)k_{\tau-1}(1 - \delta) \geq R_i b_i \tau \\
\theta k_{\tau-1}(1 - \delta) \geq R(b_i + b_a),
\]

that is, as in (18) and (19). Similarly, given one period borrowing \( (b_0,b_i0,b_a0) \) we can recover payments \( (p_0,p_m0,p_a^+0) \) by constructing, for all \( \tau \geq 1 \),

\[
p_\tau = Rb_\tau - E_\tau[b_{\tau+1}], \quad p_m \tau = R_i b_i \tau, \quad p_a^+ \tau = Rb_a \tau - E_\tau[b_{a\tau+1}],
\]

which can be seen for \( p_\tau \), for example, as follows:

\[
Rb_\tau \equiv PV_\tau(p_\tau) = p_\tau + R^{-1}E_\tau[PV_{\tau+1}(p_{\tau+1})] = p_\tau + R^{-1}E_\tau[Rb_{\tau+1}] = p_\tau + E_\tau[b_{\tau+1}],
\]

and analogously for \( p_a^+ \). For date 0, using the household’s participation constraint (5) at equality we have

\[
0 = PV_0(p_0) = p_0 + R^{-1}E_0[PV_1(p_1)] = p_0 + R^{-1}E_0[Rb_1] = p_0 + E_0[b_1],
\]

that is, \( p_0 = -E_0[b_1] \), and similarly using the intermediary’s participation constraint at equality we have

\[
0 = PV_0(p_a^+0) = p_a^+0 + R^{-1}E_0[PV_1(p_a^+1)] \\
\quad = p_a^0 + E_0[(R_{m1})^{-1}p_{m1}] + R^{-1}E_0[Rb_{a1}] = p_a^0 + E_0[b_{i1}] + E_0[b_{a1}],
\]

so \( p_a^0 = -E_0[b_{i1}] - E_0[b_{a1}] \), and thus (2) and (16) are equivalent.
Hence, given \( w_0 \), we have shown how to translate an allocation \( x_0^{LE} \) (with \( p_{m0} = 0 \)) into an allocation \( x_0^{CC} \) for the firm and vice versa.

Consider next the intermediary’s present value collateral constraint (28) which using Lemmas (1) and (2) can be written as

\[
p_{h\tau} + \bar{p}_{a\tau} + E_{\tau} \left[ \sum_{t=\tau+1}^{\infty} \left( R^{-\tau}(t-\tau)p_{ht} + R^{-\tau}(t-\tau) \bar{p}_{mt} + R^{-\tau}(t-\tau) \bar{p}_{at} \right) \right] \geq 0,
\]
and further simplified by proceeding as before and defining \( \bar{p}_{at}^+ \equiv \bar{p}_{at} + E_{\tau}[(R_{mt+1})^{-1} \bar{p}_{mt+1}] \) and for \( \tau \geq 1 \),

\[
Rl_{\tau} \equiv PV_{\tau}(p_{h\tau}), \quad l_{\tau} = (R_{i\tau})^{-1} \bar{p}_{m\tau}, \quad RL_{\tau} \equiv PV_{\tau}(\bar{p}_{at}^+),
\]
reducing the collateral constraint for the intermediary to

\[
l_t + l_{at} \geq 0,
\]
which is (24), the collateral constraint with one period loans. Moreover, by mimicking the proof for the firm above, we can show that for \( \tau \geq 1 \)

\[
p_{h\tau} = RL_{\tau} - E_{\tau}[l_{\tau+1}], \quad \bar{p}_{m\tau} = R_{i\tau} l_{\tau}, \quad \bar{p}_{at}^+ = RL_{\tau} - E_{\tau}[l_{a\tau+1}],
\]
and, using the participation constraints for the firm and household at equality, that \( p_{h0} = -E_0[l_1] \) and \( \bar{p}_{a0} = -E_0[l_{i1}] - E_0[l_{a1}] \).

Hence, given \( w_{i0} \), we have shown how to translate an allocation \( x_0^{LE} \) (with \( \bar{p}_{m0} = 0 \)) into an allocation \( x_0^{CC} \) for the intermediary and vice versa.

Consider now a given equilibrium with limited commitment. Using the interest rates \( R_{m0} \) implied by the state prices for morning payments, in the equivalent collateral constraint problem with \( R_{i0} = R_{m0} \) our construction ensures that the same dividends and investment and the one period borrowing defined above are optimal for the firm. Similarly, the same dividends and the one period loans defined above are optimal for the intermediary. Hence, given one period interest rates \( R_{i0} \) for intermediary loans repaid in the morning and the interest rate \( \bar{R} \) for loans repaid in the afternoon, the market clears and we have an equilibrium with collateral constraints. The converse argument obviously obtains as well. Therefore, we have shown the equivalence of the economy with limited enforcement and the economy with collateral constraints. \( \square \)

**Proof of Proposition 1.** Using (39) and the fact that \( \nu_i^i = 0 \) (proved below in Lemma 3, part (iii)), we have \((R_i')^{-1} = \frac{\beta \mu'}{\mu} + \frac{\beta \lambda_i}{\mu}\) and, taking conditional expectations,

\[
\frac{1}{R + \rho_i} \equiv E \left[ (R_i')^{-1} \left| z \right| \right] = \frac{1}{R + \rho} + E \left[ \frac{\lambda_i}{\mu} \left| z \right| \right]
\]
and hence \( \rho \geq \rho_i \) with equality iff \( E[\lambda_i'|z] = 0 \). Moreover, since \( R'_i \geq R \) (proved below in Lemma 3, part (i)), \( \rho_i \geq 0 \). Finally, using (38), we have \( 1/(R + \rho) = E[\beta \mu'/\mu|z] = 1/R - E[\beta \lambda'/\mu|z] \), implying that \( \rho > 0 \) iff \( E[\lambda'|z] > 0 \). \( \square \)

**Proof of Proposition 2.** First, consider the intermediary’s problem. The first order conditions are (45)-(47) and \( \mu'_i = 1 + \eta'_i \), where \( \beta \eta'_i \) is the multiplier on the constraint \( w'_i \geq 0 \). Since (43) holds with equality, the non-negativity constraints on \( l' \) and \( l'_i \) render the non-negativity constraint on \( w'_i \) redundant and hence \( \mu'_i = 1 \). Using (46) we have \( \eta' = (R\beta_i)^{-1}\mu_i - 1 \geq (R\beta_i)^{-1} - 1 > 0 \) (and \( l' = 0 \)) and similarly using (47) \( \eta'_i > 0 \) as long as \( R'_i < \beta_i^{-1} \). Therefore, for \( l'_i > 0 \) it is necessary that \( R'_i \geq \beta_i^{-1} \). If \( R'_i > \beta_i^{-1} \), then \( \mu'_i > 1 \) (and \( l'_i = w_i \)) while if \( R'_i = \beta_i^{-1}, 0 \leq l'_i \leq w_i \).

Now consider the representative firm’s problem. The first order conditions are (45)-(47) and \( \mu' = 1 + \nu'_d \), where \( \beta \nu'_d \) is the multiplier on the constraint \( w' \geq 0 \). Proceeding as in the proof of Proposition 8 one can show that \( \mu' = 1 \). Suppose \( \nu'_d > 0 \) (and hence \( b'_i = 0 \)). Since \( k > 0 \), (33) is slack and \( \lambda'_i = 0 \). Using (36) and (39) we have \( 1 \leq \mu < R'_i \beta \) which implies that \( R'_i > \beta^{-1} \). But at such an interest rate on intermediated finance \( l'_i = w_i > 0 \), which is not an equilibrium as \( b'_i = 0 \). Therefore, \( \nu'_d = 0 \) and \( R'_i \leq \beta^{-1} \). Moreover, if \( R'_i < \beta^{-1} \), then \( \lambda'_i = (R'_i \beta^{-1} - 1) > 0 \) and hence \( \nu'_d = (R'_i)^{-1}(\theta_i - \theta)k(1 - \delta) > 0 \). Since \( l'_i = 0 \) if \( R'_i < \beta_i^{-1} \), we have \( R'_i \in [\beta_i^{-1}, \beta^{-1}] \) in equilibrium. The firm’s investment Euler equation (50) simplifies to \( 1 = \beta(1/\mu)[A'f_k(k) + (1 - \theta_i)(1 - \delta)]/\varphi_i(R'_i) \). Given the interest rate on intermediated finance, the firm’s problem induces a concave value function and thus \( \mu \) (weakly) decreases in \( w \), implying that \( k \) (weakly) increases.

We first show that intermediaries are well capitalized and there is a minimum spread on intermediated finance \( \beta_i^{-1} - R > 0 \) for all levels of firm net worth when \( w_i \geq w^*_i \) and for levels of firm net worth \( w \leq w(w_i) \) when \( w_i < w^*_i \). If \( R'_i = \beta_i^{-1} \), a well capitalized firm invests \( k^* \) which solves (50) specialized to \( 1 = \beta[A'f_k(k^*) + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta_i^{-1}) \), while less well capitalized firms invests \( k \leq k^* \). The intermediary can meet the required demand for intermediated finance for any level of firm net worth \( w \) if \( w_i \geq w^*_i \equiv \beta_i(\theta_i - \theta)k^*(1 - \delta) \). Suppose instead that \( w_i < w^*_i \). In this case the intermediary is able to meet the firm’s loan demand at \( R'_i = \beta_i^{-1} \) only if the firm is sufficiently constrained; the constrained firm invests \( k = w/\varphi_i(\beta_i^{-1}) \) using (31), (34), and (33) at equality, and thus \( b'_i = \beta_i(\theta_i - \theta)k(1 - \delta) \); the intermediary can meet this demand as long as \( w \leq w(w_i) \equiv \varphi_i(\beta_i^{-1})/[\beta_i(\theta_i - \theta)(1 - \delta)]w_i \).

Suppose now that \( w_i < w^*_i \) and \( w > w(w_i) \) as defined above. First, consider \( w_i \in [\bar{w}_i, w^*_i] \) where \( \bar{w}_i \equiv \beta(\theta_i - \theta)k(1 - \delta) \) and \( 1 = \beta[A'f_k(\bar{k}) + (1 - \theta)(1 - \delta)]/\varphi \), that is, \( \bar{w}_i \) is the loan demand of the well capitalized firm when the cost of intermediated finance is \( R'_i = \beta^{-1} \). Note that \( R'_i < \beta^{-1} \) on \((\bar{w}_i, w^*_i) \) since the intermediary has more than
enough net worth to accommodate the loan demand of the well capitalized firm (and thus any constrained firm) at $R_i' = \beta^{-1}$. Thus, the firm’s collateral constraint binds, that is, $w_i = (R_i')^{-1}(\theta_i - \theta)k(1 - \delta)$. If the firm is poorly capitalized, $d = 0$ and (31) implies $w + w_i = \varphi k$, and $R_i' = (\theta_i - \theta)(1 - \delta)(w/w_i + 1)$. If the firm is well capitalized, $\mu = 1$ and $\bar{k}(w_i)$ solves $1 = \beta[A'f_k(\bar{k}(w_i)) + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta_i^{-1})$ and $\bar{w}(w_i) = \varphi k^{-1} - w_i = \varphi_i(\beta_i^{-1})k^* = w_i^*$, that is, the two boundaries coincide at $w_i^*$. In contrast, at $w_i$ we have $w(\bar{w}_i) = \varphi_i(\beta_i^{-1})/(\beta_i(\theta_i - \theta)(1 - \delta))\bar{w}_i = \varphi_i(\beta_i^{-1})\beta/\beta_i - \bar{w}_i < \bar{w}(w_i)$ and $R_i'(\bar{w}_i) = \beta^{-1}$.

Finally, consider $w_i \in (0, \bar{w}_i)$ and $w > \underline{w}(w_i)$ as defined above. If the firm is well capitalized (39) implies $\lambda_i' = (R_i'\beta_i^{-1})^{-1} - 1 \geq 0$. Moreover, since $w_i < \bar{w}_i$ the intermediary cannot meet the well capitalized firm’s loan demand at $R_i' = \beta^{-1}$ and thus the cost of intermediated finance is in fact $\beta^{-1}$ and $\lambda_i = 0$, that is, the collateral constraint for intermediated finance does not bind. Thus, the firm’s investment Euler equation (50) simplifies to $1 = \beta[A'f_k(\bar{k}) + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta_i^{-1})$ which is solved by $\bar{k}$ as defined earlier in the proof. Define $\bar{w}(w_i) = \varphi \bar{k} - w_i$; the firm is well capitalized for $w \geq \bar{w}(w_i)$. Suppose $w < \bar{w}(w_i)$ and hence $\mu > 1$. If the collateral constraint for intermediated finance does not bind, then (39) implies $R_i' = \beta^{-1}\mu > \beta^{-1}$ and (50) implies $R_i' = [A'f_k(k) + (1 - \theta_i)(1 - \delta)]/\varphi$, while (31) yields $w + w_i = \varphi k$. Observe that $k < \bar{k}$ and $R_i'$ decreases in $w$. If instead the collateral constraint binds, then $R_i' = (\theta_i - \theta)k(1 - \delta)/w_i$ and $w + w_i = \varphi k$ (so long as $w > \underline{w}(w_i)$). Note that $k$ and $R_i'$ increase in $w$ in this range. The collateral constraint is just binding at $\bar{w}(w_i) = \varphi \bar{k} - w_i$, where $[A'f_k(\bar{k}(w_i)) + (1 - \theta_i)(1 - \delta)]/\varphi = (\theta_i - \theta)(\bar{k}(w_i) - 1)/w_i$.

We now show that if the collateral constraint for intermediated finance binds at some $w < \bar{w}(w_i)$ then it binds for all $w < \bar{w}(w_i)$. Note that $d = 0$ in this range and $w + w_i = \varphi k$. At $w^-$, either $b_i^- < w_i$ and $R_i' = \beta_i^{-1}$ and hence $\lambda_i^- = (\beta_i^{-1})^{-1}\mu^ - 1 > 0$ or $b_i^- = w_i$ and $w^+ + w_i = \varphi k^-$, implying $k^- < k$. Suppose the collateral constraint for intermediated finance is slack at $w^-$. Then $R_i^-b_i^- < (\theta_i - \theta)k(1 - \delta) < (\theta_i - \theta)k(1 - \delta) = R_i'b_i'$ and since $b_i^- = w_i$ and $b_i' \leq w_i$ by above $R_i^-w_i < R_i'b_i' \leq R_i'w_i$, which implies $R_i^- < R_i'$.

$$
R_i^- \beta = \mu^- = \beta A'f_k(k^-) + (1 - \theta_i)(1 - \delta)]/\varphi - (R_i^-)^{-1}(\theta_i - \theta)(1 - \delta) > \beta A'f_k(k^-) + (1 - \theta_i)(1 - \delta)]/\varphi - (R_i')^{-1}(\theta_i - \theta)(1 - \delta) = \mu > R_i'\beta
$$

or $R_i^- > R_i'$, a contradiction.

Moreover, $\underline{w}(w_i) < \bar{w}(w_i) < \bar{w}(w_i)$ on $w_i \in (0, \bar{w}_i)$. Suppose, by contradiction, that $\bar{w}(w_i) \leq \underline{w}(w_i)$ and recall that $\underline{w}(w_i) + w_i = \varphi k$ and $\bar{w}(w_i) + w_i = \varphi \bar{k}(w_i)$, so $\bar{k}(w_i) \leq k$. But $R_i'(w_i) = (\theta_i - \theta)\bar{k}(w_i)(1 - \delta)/w_i \leq (\theta_i - \theta)k(1 - \delta)/w_i = \beta_i^{-1}$.
But if \( \hat{R}_i'(w_i) \leq \beta^{-1} \), then at \( \hat{w}(w_i) \) we have \( \mu = \hat{R}_i'(w_i)\beta < 1 \) (since the collateral constraint is slack), a contradiction. Thus, \( \hat{w}(w_i) < \hat{w}(w_i) \). Suppose, again by contradiction, that \( \hat{w}(w_i) \leq \hat{w}(w_i) \) and hence \( \hat{k} \leq \hat{k}(w_i) \). Recall that \( \hat{k}(w_i) \) solves \([A'f_k(\hat{k}(w_i)) + (1 - \theta)(1 - \delta)]/\varphi = (\theta_i - \theta)\hat{k}(w_i)(1 - \delta)/w_i \). At \( \hat{w} \) this equation is solved by \( \hat{k} \) (and \( \hat{R}_i'(\hat{w}) = \beta^{-1} \)), but since \( w_i < \hat{w}, \hat{k}(w_i) < \hat{k}, \) a contradiction. Moreover, as \( w_i \to 0, \hat{k}(w_i) \to 0 \) and \( \hat{w}(w_i) = \varphi \hat{k}(w_i) - w_i \to 0 \). □

**Proof of Proposition 3.** Consider a deterministic economy. Suppose intermediaries pay out their entire net worth at some point. From that point on, the firm’s problem is as if there is no intermediary. We first characterize the solution to this problem and then show that the solution implies shadow interest rates on intermediated finance at which it would not be optimal for intermediaries to exit.

To characterize the solution in the absence of intermediaries, consider a steady state at which \( \mu = \mu' \equiv \bar{\mu} \) and note that (38) implies \( \bar{\lambda}' = ((R\beta)^{-1} - 1)\bar{\mu} > 0 \). The investment Euler equation (50) simplifies to \( 1 = \beta[A'f_k(\hat{k}) + (1 - \theta)(1 - \delta)/\varphi \) which defines \( \bar{k} \). The firm’s steady state net worth is \( \varphi \hat{k} = A'f(\bar{k}) + (1 - \theta)\bar{k}(1 - \delta) \) and the firm pays out
\[
\bar{d} = \bar{w}' - \varphi \bar{k} = A'f(\bar{k}) - \bar{k}[1 - (R^{-1}\theta + (1 - \theta))(1 - \delta)]
\]
\[
> A'f(\bar{k}) - \beta^{-1}\bar{k}[1 - (R^{-1}\theta + \beta(1 - \theta))(1 - \delta)]
\]
\[
= \int_0^{\bar{k}} [A'f_k(k) - \beta^{-1}(1 - (R^{-1}\theta + \beta(1 - \theta))(1 - \delta))]dk > 0.
\]

Therefore, \( \bar{\mu} = 1 \). Investment \( \bar{k} \) is feasible as long as \( w \geq \hat{w} = \bar{w}' - \bar{d} \). Whenever \( w < \hat{w} \), \( k < \bar{k} \) and hence using (50) we have \( \mu/\mu' = \beta[A'f_k(\hat{k}) + (1 - \theta)(1 - \delta)]/\varphi > 1 \). The shadow interest rate on intermediated finance is \( R'_i = \beta^{-1}\mu/\mu' \geq \beta^{-1} \) for all values of \( w \). But then it cannot be optimal for intermediaries to pay out all their net worth in a deterministic economy as keeping \( \varepsilon > 0 \) net worth for one more period improves the objective by \( (\beta_i R'_i - 1)\varepsilon > 0 \).

Consider now an eventually deterministic economy. From time \( T \) onward, the economy is deterministic and the conclusion obtains by above as long as the intermediary has positive net worth in all states at time \( T \). Suppose not, that is, suppose intermediary net worth is zero for some state. As before the discounted marginal value on an infinitesimal amount of intermediary net worth at time \( T \) lent out for one period is at least \( \beta_i R'_i \geq \beta_i \beta^{-1} > 1 \) since \( R'_i \geq \beta^{-1} \). Lending for \( \tau \) periods thus guarantees a discounted marginal value of \( (\beta_i \beta)^\tau \). As \( \tau \to \infty \), the marginal value grows without bound. (Note that since we consider an infinitesimal amount, the collateral constraint cannot be biding for any finite \( \tau \).) The expected marginal value of this lending policy at time \( 0 \) is at least \( (\beta_i R)^T \) times the marginal value at time \( T \) and hence grows without bound as \( \tau \to \infty \).
But the marginal value of intermediary net worth at time 0 is finite as either the intermediary pays dividends and the marginal value is one, or the intermediary saves into at least one state at $R_i'$ and thus $\mu_i = R_i' \beta \mu_i'$ and $R_i'$ is bounded above by (33) and otherwise $R_i' = R$. Furthermore, $\mu_i'$ is bounded by a similar argument going forward until dividends are paid at which point the marginal value is one. But then it cannot be an equilibrium for intermediaries to pay out all their net worth. □

**Proof of Lemma 3.** Part (i): If $R_i' < R$, then using (38) and (39) we have $0 < (R - R_i') \beta \mu' \leq R_i' \beta \lambda_i'$ and thus $b_i' > 0$. But (46) and (47) imply that $0 < (R - R_i') \beta \mu' \leq R_i' \beta \eta_i'$ and thus $l_i' = 0$, which is not an equilibrium.

Part (ii): Given $\nu_i' = 0$ (see part (iii)), (38) and (39) imply that $\lambda_i' = (R_i' / R - 1) \mu' + R_i' / R \lambda_i' \geq \lambda_i'$.

Part (iii): First, suppose to the contrary that $\nu_i' > 0$. Then $\lambda_i' = 0$ as $b_i' = 0 < (R_i')^{-1} (\theta - \theta) k (1 - \delta)$ implies that (33) is slack. Using (39) and (38) we have $\beta \mu_i R_i' \beta \mu_i R_i' > R_i' > R$. Equations (46) and (47) imply that $R \eta_i' - R_i' \eta_i' = (R_i' - R_i) \mu_i' > 0$ and thus $\eta_i' > 0$ and $\nu_i' = 0$. But if $w_i' > 0$, which is always true under the conditions of Proposition 3, we have $l_i' = (R_i')^{-1} w_i' > 0 = b_i'$, which is not an equilibrium. If instead $w_i' = 0$, then $\nu_i' = 0$ and we can set $R_i' = (\beta \mu_i' / \mu_i)$ and $R > R_i'$ contradicting the result of part (i). Thus, $\eta_i' = 0$ and $\mu_i' = (\beta_i R_i')^{-1} \mu_i$.

Part (iv): Suppose to the contrary that $\eta_i' > 0$ (and hence $l_i' = 0$). Since intermediaries never pay out all their net worth in a deterministic economy, equation (43) implies $0 < w_i' \leq R \mu_i'$ and hence $\eta_i' = 0$. But then (46) and (47) imply $\beta_i \mu_i' / \mu_i R = 1 > \beta_i \mu_i' / \mu_i R_i'$ or $R > R_i'$ contradicting the result of part (i). Thus, $\eta_i' = 0$ and $\mu_i' = (\beta_i R_i')^{-1} \mu_i$.

Part (v): Suppose $\lambda_i' = 0$. Then (38) reduces to $1 = \beta \mu_i' / \mu_i R$ and thus $1 \leq \mu_i = \beta R \mu_i' < \mu_i'$ and $d_i' = 0$. By part (ii), $\lambda_i' = 0$ and using (39) we have $R_i' = R$, $\mu_i' = (\beta R)^{-1} \mu_i > 1$, and $d_i' = 0$. The investment $k^{**}$ solves $R = [A' f_k(k^{**}) + (1 - \theta_i)(1 - \delta)] / \varphi_i(R)$ or $R - 1 + \delta = A' f_k(k^{**})$; this is the first best investment when dividends are discounted at $R$ and it can never be optimal to invest more than that. To see this use (50) and note $[A' f_k(k) + (1 - \theta_i)(1 - \delta)] / \varphi_i(R_i') = \mu_i / (\beta \mu_i') \geq R = [A' f_k(k^{**}) + (1 - \theta_i)(1 - \delta)] / \varphi_i(R)$, that is, $f_k(k) \geq f_k(k^{**})$. Note that the firm’s net worth next period, using (32) and (50), is

\[
\begin{align*}
w_i' &= A' f(k^{**}) + (1 - \theta_i)(1 - \delta) k^{**} - [R b_i' - \theta(1 - \delta) k^{**}] - [R b_i' - (\theta_i - \theta)(1 - \delta) k^{**}] \\
&> R \varphi_i(R) k^{**} - [R b_i' - \theta(1 - \delta) k^{**}] - [R b_i' - (\theta_i - \theta)(1 - \delta) k^{**}] = R[k^{**} - b_i' - b_i'] \\
&= R w_{ex}.
\end{align*}
\]

Note that $d_i' = 0$, $d_i' = 0$, $k_i' \leq k^{**}$, and $w_i' > w_{ex}$, and from (31) next period, $k_i' = w_i' + b_i'' + b_i'$. If $R_i'' > R$, then $b_i'' = w_i'$ and $b_i'' = R^{-1} \theta(1 - \delta) k_i''$. Therefore, $\varphi k_i' = w_i' + w_i'$,
but using (31) we have $\phi k^* - b' = w_{ex} + b'_i < w' + w'_i = \phi k'$, a contradiction. If $R''_i = R$, then $b''_i + w''_i = k' - w' < k^* - w_{ex} = b' + b'_i$, that is, the firm is paying down debt, and $w'' > w'$ and $w''_i > w'_i$. But then $w$ and $w_i$ grow without bound unless the firm or the intermediary eventually pay a dividend. But since $\mu$ and $\mu_i$ are strictly increasing as long as $R'_i = R$, if either pays a dividend at some future date, then $\mu < 1$ or $\mu_i < 1$ currently, a contradiction. □

**Proof of Proposition 5.** First, note that $k^* > 0$ due to the Inada condition and hence $w^* \geq A'f(k^*) + k^*(1 - \theta_i)(1 - \delta) > 0$. Moreover, $d^* > 0$ since otherwise the value would be zero which would be dominated by paying out all net worth. Hence, $\mu^* = \mu^* = 1$. By Proposition 3 intermediary net worth is positive and hence $d^*_i > 0$ (arguing as above), which implies $\mu^*_i = \mu^*_i = 1$. But then $\eta^* = (R\beta_i)^{-1} - 1 > 0$ and $l^*_i > 0$ (and $\eta^*_i = 0$), since otherwise intermediary net worth would be 0 next period. Therefore, $R^*_i = \beta_i^{-1}$, and thus $\lambda^*_i = (\beta_i^{-1})^{-1} - 1 > 0$, that is, the firm’s collateral constraint for intermediated finance binds. Moreover, $k^*$ solves $1 = \beta[A'_f k^* + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta_i^{-1})$ and $d^*_i$, $b^*_i$, $b'^*_i$, and $w^*$ are determined by (31)-(33) at equality. Specifically, $d^*_i = A' f(k^*) + k^*(1 - \theta_i)(1 - \delta) - \varphi_i(\beta_i^{-1})k^* > 0$ and $b'^*_i = \beta_i(\theta_i - \theta)k^*(1 - \delta)$. The net worth of the firm after dividends is $w^* = \varphi_i(\beta_i^{-1})k^*$. Finally, $l^*_i = b'^*_i = w^*_i$ and $d^*_i = (\beta_i^{-1} - 1)w^*_i$. □

**Proof of Proposition 6.** Consider first region D and take $w \geq \bar{w}(w_i)$ (to be defined below) and $d > 0$ forever ($\mu = \mu^* = 1$). The investment Euler equation then implies $1 = \beta[A'_f k + (1 - \theta_i)(1 - \delta)]/\varphi_i(R_i)$. If the collateral constraint for intermediated finance (33) does not bind, then $\mu = R'_i\beta\mu'$, that is, $R'_i = \beta^{-1}$, and investment is constant at $\bar{k}$ which solves $1 = \beta[A'_f \bar{k} + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta^{-1})$ or, equivalently, $1 = \beta[A'_f \bar{k} + (1 - \theta_i)(1 - \delta)]/\varphi_i(\beta^{-1})$. Define $\bar{w}(w_i) \equiv \varphi \bar{k} - w_i$ and $\bar{w}_i = \beta(\theta_i - \theta)\bar{k}k(1 - \delta)$. At $\bar{w}_i$, (33) is just binding. For $w_i \in (0, \bar{w}_i)$, (33) is slack. Moreover, $w'_i = \beta^{-1}w_i$ and, if $w'_i \in (0, \bar{w}_i)$, the ex dividend net worth is $w_{ex} = \bar{w}(w_i)$ both in the current and next period, and we have immediately $w'_{ex}/w'_i > w_{ex}/w_i$. Further, using (32) and (50) we have

$$w' = A' f(\bar{k}) + (1 - \theta)\bar{k}(1 - \delta) = R'_i w_i > [A'_f(\bar{k}) + (1 - \theta)(1 - \delta)]\bar{k} - R'_i w_i = R'_i \bar{w}(w_i).$$

But $w'_{ex} = \bar{w}(w'_i) < \bar{w}(w_i)w'_i/w_i = R'_i w_{ex}$, so $d' = w' - w'_{ex} > 0$. For $w_i \in [\tilde{w}_i, w^*_i]$, (33) binds and $k(w_i)$ solves $1 = \beta[A'_f k(w_i)] + (1 - \theta_i)(1 - \delta)]/[\varphi - w_i/k(w_i)]$ and $R'_i = (\theta_i - \theta)k(w_i)/w_i(1 - \delta)$. Note that the last two equations imply that $k(w_i) \geq \bar{k}$, $w_i/k(w_i) \geq \tilde{w}_i/\bar{k}$, and $R'_i \leq \beta^{-1}$ in this region. As before, define $\bar{w}(w_i) = \phi k(w_i) - w_i$ and note that the ex dividend net worth is $w_{ex} = \bar{w}(w_i)$. Suppose $w^*_i > w_i$ then $k(w^*_i) > k(w_i)$, $k(w^*_i)/w^*_i < k(w_i)/w_i$, and $w^*_i/w^*_i = \phi k(w^*_i)/w^*_i - 1 < w_{ex}/w_i$. Moreover, $w'_i = R'_i w_i > w_i$ and hence $k$ (strictly) increases and $R'_i$ (strictly) decreases in this region.
Proceeding as before,

\[ w' = A'f(k(w_i)) + (1 - \theta_i)k(w_i)(1 - \delta) > [A'f_k(k(w_i)) + (1 - \theta_i)(1 - \delta)]k(w_i) \]

\[ \geq R'_i[\beta[A'f_k(k(w_i)) + (1 - \theta_i)(1 - \delta)]k(w_i) = R'_i\bar{w}(w_i). \]

But \( w'_{ex} = \bar{w}(w'_i) < \bar{w}(w_i)w'_i/w_i = R'_i w_{ex}, \) so \( d' = w' - w'_{ex} > 0. \) Finally, if \( w_i \geq w_i^* \) and \( w \geq \bar{w}(w_i) = w^* \), the steady state of Proposition 5 is reached.

We now show that the above policies are optimal for both the firm and the intermediary given the interest rate process in region D and hence constitute an equilibrium. Since \( R'_i > \beta_i^{-1} \) before the steady state is reached, the intermediary lends its entire net worth to the firm, \( b'_i = w_i \), and does not pay dividends until the steady state is reached. Hence, the intermediary’s policy is optimal. To see that the firm’s policy is optimal in region D, suppose that the firm follows the optimal policy from the next period onward but sets \( d = 0 \) in the current period. If the firm invests the additional amount, then \( \tilde{k} = (w_i + w)/\varphi > k \) and \( \tilde{w}' > w' \) (and therefore \( \tilde{\mu}' = 1 \)). The investment Euler equation requires \( 1 = \beta/\tilde{w} [A'f_k(\tilde{k}) + (1 - \theta_i)(1 - \delta)]/\varphi_i(R'_i) \), but since \( f_k(\tilde{k}) < f_k(k) \) and \( k \) satisfies the investment Euler equation at \( \mu = \mu' = 1 \), this implies \( \tilde{\mu} < 1 \), a contradiction. Suppose the firm instead invests the same amount \( \tilde{k} = k \) but borrows less \( \tilde{b}'_i < b'_i \). Then \( \tilde{w}' > w', \tilde{\mu}' = 1 \), and from (50) \( \tilde{\mu}_i = 1 \). If \( R'_i < \beta^{-1} \), then (33) is binding, a contradiction. If \( R'_i = \beta^{-1} \), then the firm is indifferent between paying dividends in the current period or in the next period. But in equilibrium \( b'_i = w_i \) and hence \( d = d > 0 \) for the representative firm. By induction starting at the steady state and working backwards, the firm’s policy is optimal in region D. Further, we show in Lemmata 5 and 6 that the equilibrium in region D is the unique equilibrium converging to the steady state.

Consider now region ND with \( w_i \leq w_i^* \) (as Lemma 4 shows) and \( w < \bar{w}(w_i) \) as defined in the characterization of region D above and \( d = 0 \). Denote the firm’s ex dividend net worth by \( w_{ex} \leq w \). There are 3 cases to consider: \( w_{ex}/w_i > \bar{w}/\bar{w}_i, \ w_{ex}/w_i \in [w^*/w_i^*, \bar{w}/\bar{w}_i], \) and \( w_{ex}/w_i < w^*/w_i^* \).

First, if \( w_{ex}/w_i > \bar{w}/\bar{w}_i \), then \( w_{ex} + w_i < \bar{w}(w_i) + w_i = \bar{w} + \bar{w}_i \) and \( k \leq (w_{ex} + w_i)/\varphi < (\bar{w} + \bar{w}_i)/\varphi = \bar{k} \). Note that since \( b'_i \leq w_i - d_i \leq w_i \), we have \( w_{ex}/b'_i \geq w_{ex}/w_i > \bar{w}/\bar{w}_i \). If (33) binds, then \( R'_i = (\theta_i - \theta)(1 - \delta)(w_{ex}/b'_i + 1)/\varphi > (\theta_i - \theta)(1 - \delta)(\bar{w}/\bar{w}_i + 1)/\varphi = \beta^{-1} \). If (33) does not bind, then \( R'_i = [A'f_k(\bar{k}) + (1 - \theta_i)(1 - \delta)]/\varphi > [A'f_k(\bar{k}) + (1 - \theta_i)(1 - \delta)]/\varphi = \beta^{-1} \). In either case, \( R'_i > \beta^{-1} \), and hence \( d = 0, d_i = 0 \), and \( b'_i = w_i \).

Second, consider \( w_{ex}/w_i < [w^*/w_i^*, \bar{w}/\bar{w}_i] \). If \( w_{ex}/b'_i > \bar{w}/\bar{w}_i \) we are in the first region and hence \( d_i = 0 \) and \( b'_i = w_i \), a contradiction. Hence, w.l.o.g. \( w_{ex}/b'_i \in [w^*/w_i^*, \bar{w}/\bar{w}_i] \). Take \( \tilde{w}_i \) such that \( w_{ex}/b'_i = \tilde{w}(\tilde{w}_i)/\bar{w}_i \). Note that (33) binds at \( w_i \) and \( \bar{w}(\tilde{w}_i) \), and thus
\( b_i' + w_{ex} < \bar{w}_i + \bar{w}(\bar{w}_i) \) and moreover \( k < \bar{k}(\bar{w}_i) \). If (33) does not bind, then

\[
\tilde{R}_i'(\bar{w}_i) = (\theta_i - \theta)(1 - \delta)(\bar{w}(\bar{w}_i)/\bar{w}_i + 1)/\theta > (\theta_i - \theta)(1 - \delta)(w_{ex}/b_i' + 1)/\theta > R_i' \]

\[
= [A'f_k(k) + (1 - \theta)(1 - \delta)]/\theta > [A'f_k(\bar{k}(\bar{w})) + (1 - \theta)(1 - \delta)]/\theta.
\]

But since (33) binds at \( \bar{w}_i \) and \( \bar{w}(\bar{w}_i) \), \( \tilde{R}_i'(\bar{w}_i) < [A'f_k(\bar{k}(\bar{w})) + (1 - \theta)(1 - \delta)]/\theta \), a contradiction. Therefore, (33) binds and \( R_i' = \tilde{R}_i'(\bar{w}_i) \). From (50), \( \beta \mu'/\mu[A'f_k(k) + (1 - \theta)(1 - \delta)]/\varphi_i(R_i') = 1 = \beta[A'f_k(\bar{k}(\bar{w}_i)) + (1 - \theta)(1 - \delta)]/\varphi_i(\tilde{R}_i'(\bar{w}_i)) \) and, since \( k < \bar{k}(\bar{w}_i) \), \( \mu > \mu' \geq 1 \), that is, \( d = 0 \). Further, if \( w_{ex}/w_i \in (w^*/w_i, w/w_i) \), then \( R_i' \in (\beta_i^{-1}, \beta_i^{-1}) \), and thus \( d_i = 0 \) and \( b_i' = w_i \). If \( w_{ex}/w_i = w^*/w_i^* \), then either \( d_i > 0 \) or \( b_i' < w_i \) yields \( R_i' > \beta_i^{-1} \) and therefore \( d_i = 0 \) and \( b_i' = w_i \) at such \( w_{ex} \) and \( w_i \) as well.

Third, consider \( w_{ex}/w_i < w^*/w_i^* \). As before, w.l.o.g. \( w_{ex}/b_i' < w^*/w_i^* \). Then from (33), \( R_i' \leq (\theta_i - \theta)(1 - \delta)(w_{ex}/b_i' + 1)/\theta < (\theta_i - \theta)(1 - \delta)(w^*/w_i^* + 1)/\theta = \beta_i^{-1} \), that is, \( R_i' < \beta_i^{-1} \).

From (50), \( \beta \mu'/\mu[A'f_k(k) + (1 - \theta)(1 - \delta)]/\varphi_i(R_i') = 1 = \beta[A'f_k(k^*) + (1 - \theta)(1 - \delta)]/\varphi_i(\beta_i^{-1}) \) and, since \( k < k^* \) and \( R_i' < \beta_i^{-1} \), \( \mu > \mu' \geq 1 \), that is, \( d = 0 \). Moreover, (33) binds, since otherwise \( \beta_i^{-1} > R_i' = [A'f_k(k) + (1 - \theta)(1 - \delta)]/\varphi > [A'f_k(k^*) + (1 - \theta)(1 - \delta)]/\varphi \), a contradiction.

Thus, we conclude that \( d = 0 \) (property (i) in the statement of the proposition), \( d_i = 0 \) (except possibly in the first period (see Lemma 4), that \( R_i' \) satisfies the equation in property (ii) of the proposition), and that \( b_i' = w_i \) and \( k = (w + w_i)/\theta \) if \( R_i' > R \) and \( k = w/\varphi_i(R) \) if \( R_i' = R \) (property (iii)). Moreover, using (32) and (50) we have

\[
w' = A'f(k) + (1 - \theta)(1 - \delta)k - [R_i'b_i' - (\theta_i - \theta)(1 - \delta)k] > R_i'\varphi_i(R_i')k - [R_i'b_i' - (\theta_i - \theta)(1 - \delta)k] \geq R_i'\varphi k - R_i'b_i' = R_i'w,
\]

which, together with the fact that \( w' = R_i'w_i \), implies that \( w'/w_i' > w/w_i \) (property (iv)). Note that the equilibrium is thus unique in region ND as well. \( \square \)

**Proof of Lemma 4.** We first show that \( d_i > 0 \) when \( w_i > w_i^* \). If \( w \geq w^* \), the stationary state is reached and the result is immediate. Suppose hence that \( w < w^* \). Suppose instead that \( d_i = 0 \). We claim that \( R_i' < \beta_i^{-1} \) for such \( w_i \) and \( w \). Either \( R_i' = R \) and hence the claim is obviously true or \( R_i' > R \), but then \( b_i' = w_i \). Using (33) and (31) we have \( R_i' \leq (\theta_i - \theta)(1 - \delta)k/b_i' \leq (\theta_i - \theta)(1 - \delta)(w/w_i + 1) < (\theta_i - \theta)(1 - \delta)(w^*/w_i^* + 1) = \beta_i^{-1} \), that is, \( R \leq R_i' < \beta_i^{-1} \). But as long as \( d_i = 0 \), \( w_i' = R_i'w_i \geq Rw_i > w_i \), that is, intermediary net worth keeps rising. If eventually firm net worth exceeds \( w^* \), then the steady state is reached and \( \mu_i' = 1 \) from then onward. But then \( \mu_i = \beta_i R_i' \mu_i' = \beta_i R_i' < 1 \), which is not possible. The intermediary must pay a dividend in the first period, because if it pays a dividend at any point after that, an analogous argument would again imply that \( \mu_i < 1 \).
in the first period, which is not possible. Similarly, if \( w < w^* \) forever, then \( w > w_i^* \) forever and the firm must eventually pay a dividend in this region, as never paying a dividend cannot be optimal. But by the same argument again then the dividend must be paid in the first period.

To see that at most an initial dividend is paid and no further dividends are paid until the steady state is reached, note that in equilibrium once \( R_i' > \beta_i^{-1} \), then this is the case until the steady state is reached. But as long as \( R_i' > \beta_i^{-1} \), the intermediary does not pay a dividend (and this is true w.l.o.g. also at a point where \( R_i' = \beta_i^{-1} \) before the steady state is reached). Before this region is reached, \( R_i' < \beta_i^{-1} \), but then the intermediary would not postpone a dividend in this region, as otherwise again \( \mu_i = \beta_i R_i' \mu_i' = \beta_i R_i' < 1 \), which is not possible. □

Lemma 5. Consider an equilibrium with \( R_i' \in [\beta_i^{-1}, \beta_i^{-1}] \) and \( \mu = \mu' = 1 \) and assume the equilibrium is unique from the next period onward. Consider another equilibrium interest rate \( \tilde{R}_i' \), then \( \tilde{k} \leq k \) and \( \tilde{R}_i' \leq R_i' \) is impossible.

Proof of Lemma 5. Using (50) at the two different equilibria, if \( \tilde{k} \leq k \) and \( \tilde{R}_i' \leq R_i' \), then
\[
\frac{\tilde{\mu}}{\mu'} = \beta \frac{A' f_k(\tilde{k}) + (1 - \theta_i)(1 - \delta)}{\varphi_i(\tilde{R}_i')} \geq \beta \frac{A' f_k(k) + (1 - \theta_i)(1 - \delta)}{\varphi_i(R_i')} = 1
\]

If \( \tilde{k} < k \) and \( \tilde{R}_i' < R_i' = \beta_i^{-1} \), then by (57) \( \tilde{\mu} > \mu' \). Thus, \( \tilde{\mu} > \mu' \tilde{R}_i' \beta_i \) implying that (33) must be binding. But then the firm must pay a dividend and \( 1 = \tilde{\mu} > \mu' \), a contradiction.

If \( \tilde{k} > k \) and \( \tilde{R}_i' > R_i' \) and the collateral constraint binds at the original equilibrium, then \( \tilde{w}' \geq A' f(\tilde{k}) + (1 - \theta_i)(1 - \delta) \tilde{k} > A' f(k) + (1 - \theta_i)(1 - \delta)k = w' \). Since \( \tilde{w}' > w' \), \( \mu' = 1 \), and the equilibrium is unique , \( \mu' = 1 \). By (57), \( \tilde{\mu} < \mu' = 1 \), a contradiction.

If \( \tilde{k} > k \) and \( \tilde{R}_i' > R_i' \) and the collateral constraint does not bind at the original equilibrium, the \( R_i' = \beta_i^{-1} \) (using (39)). But then \( \tilde{\mu}/\mu' \geq \tilde{R}_i' \beta_i > 1 \) while (57) implies \( \tilde{\mu}/\mu' < 1 \), a contradiction. □

Lemma 6. The equilibrium in region D is the unique equilibrium converging to the steady state.

Proof of Lemma 6. The proof is by induction. First, note that if \( w \geq w^* \) and \( w_i \geq w_i^* \), then the unique steady state is reached. Consider an equilibrium interest rate \( R_i' \) in region D and suppose the equilibrium is unique from the next period on. Suppose \( R_i' \in [\beta_i^{-1}, \beta_i^{-1}] \) and consider another equilibrium with \( \tilde{R}_i' \). If the collateral constraint (33) binds at this equilibrium, then \( \tilde{R}_i' = (\theta_i - \theta)(1 - \delta) \tilde{k}/w_i \geq (\theta_i - \theta)(1 - \delta)k/w_i = R_i' \), which is impossible by Lemma (5). If the collateral constraint (33) does not bind at this
equilibrium and \( \tilde{k} < k \), then \( \tilde{R}'_i < (\theta_i - \theta)(1 - \delta)\tilde{k}/w_i < (\theta_i - \theta)(1 - \delta)k/w_i = R'_i \), which is also impossible by Lemma (5). If the collateral constraint (33) does not bind at this equilibrium and \( \tilde{k} > k \), by Lemma (5) \( \tilde{R}'_i < R_i \). But then by (39) \( \tilde{\mu}/\tilde{\mu}' = \beta\tilde{R}'_i < \beta R'_i < 1 \). Since \( \tilde{k} > k \) and the collateral constraint binds at \( R'_i \), \( \tilde{w}' > w' \) implying \( \tilde{\mu}' = 1 \) and by above inequality \( \tilde{\mu} < 1 \), a contradiction. Thus for \( R'_i \in [\beta_i^{-1}, \beta^{-1}] \) the equilibrium is unique. Suppose \( R'_i = \beta^{-1} \). By Lemma (5), we need only consider the two cases \( \tilde{k} \geq k \) and \( \tilde{R}'_i \leq R'_i = \beta^{-1} \). If \( \tilde{k} < k \) and \( \tilde{R}'_i > \beta^{-1} \), (39) implies that \( \tilde{\mu} > 1 \) and hence the firm does not pay a dividend. But then the firm must be borrowing less from intermediaries, which cannot be an equilibrium as \( I'_i = w_i \) at this interest rate. If \( \tilde{k} > k \) and \( \tilde{R}'_i < R'_i = \beta^{-1} \), and if (33) binds at \( \tilde{R}'_i, \tilde{R}'_i = (\theta_i - \theta)(1 - \delta)\tilde{k}/w_i > (\theta_i - \theta)(1 - \delta)k/w_i \geq R'_i \), a contradiction; if (33) instead does not bind at \( \tilde{R}'_i, \tilde{\mu}/\tilde{\mu}' = \beta\tilde{R}'_i < 1 \). Since \( \tilde{k} > k \) and \( \tilde{R}'_i \leq R'_i w_i, \tilde{w}' > w' \) implying \( \tilde{\mu}' = 1 \) and by above inequality \( \tilde{\mu} < 1 \), a contradiction. Therefore the equilibrium in region D is unique. \( \Box \)

**Proof of Proposition 7.** Part (i): By assumption the expected productivity in the first period equals the deterministic productivity from time 1 onward (denoted \( \tilde{A} \) here), that is, \( E[A'] = \tilde{A} \). Define the first best level of capital \( k_{fb} \) by \( r + \delta = \tilde{A}'f_k(k_{fb}) \). Using the definition of the user cost of capital the investment Euler equation (50) for the deterministic case can be written as

\[
r + \delta + \frac{\rho}{R + \rho}(1 - \theta_i)(1 - \delta) + \frac{\rho_i}{R + \rho_i}(\theta_i - \theta)(1 - \delta) = R\beta\tilde{A}'f_k(k^*) < \tilde{A}'f_k(k^*)
\]

and thus \( k^* < k_{fb} \). Now suppose that \( \lambda(s') = 0, \forall s' \in S \). Part (ii) of Lemma 3 then implies that \( \lambda_i(s') = 0, \forall s' \in S \), and (38) and (39) simplify to \( \mu = R\beta\mu' \) and \( \mu = R'_i\beta\mu' \), implying that \( R'_i = R_i, \forall s' \in S \), and that \( d'_i = 0, \forall s' \in S \), as otherwise \( \mu < 1 \). Moreover, (47) simplifies to \( \mu_i = R\beta\mu'_i \) and thus \( d'_i = 0, \forall s' \in S \), as well since otherwise \( \mu_i < 1 \). Investment Euler equation (50) reduces to \( r + \delta = \tilde{A}'f_k(k_{fb}) \), that is, investment must be \( k_{fb} \). We now show that this implies that the sum of the net worth of the intermediary and the firm exceeds their steady state (cum dividend) net worth in at least one state, which in turn implies that at least one of them pays a dividend, a contradiction. To see this note that \( w' = A'f_k(k_{fb}) + k_{fb}(1 - \delta) - Rb' - R'_ib'_i \) and \( w'_i = Rl' + R'_il'_i \geq R'_il'_i = R'_ib'_i \) and thus

\[
w' + w'_i \geq A'f_k(k_{fb}) + k_{fb}(1 - \delta) - Rb' \geq A'f_k(k_{fb}) + (1 - \theta)k_{fb}(1 - \delta) > A'f_k(k^*) + (1 - \theta)k^*(1 - \delta)
\]

whereas \( w'^* + w'_i^* = \tilde{A}'f_k(k^*) + (1 - \theta)k^*(1 - \delta) \). For \( \tilde{A} > \tilde{A}' \), \( w' + w'_i > w'^* + w'_i^* \), and either the intermediary or the firm (or both) must pay a dividend, a contradiction.

Part (ii): If \( \lambda_i(s') = 0, \forall s' \in S \), then \( (\beta\mu'/\mu)^{-1} = R'_i = (\beta_i\mu'_i/\mu_i)^{-1} \) where the first
equality uses (39) and the second equality uses (47) and the fact that part (iv) of Lemma 3 holds for an eventually deterministic economy.

Part (iii): Since \( \lambda(s') = 0 \), \( \lambda_i(s') = 0 \) by part (ii) of Lemma 3 and \( R_i(s') = R \). From (38), \( \mu(s') = \mu(s') + \lambda(s') > \mu(s') \). Using (47), \( (\beta_i \mu_i(s')/\mu_i)^{-1} = R \leq R_i(s') = (\beta_i \mu_i(s')/\mu_i)^{-1} \) and thus \( \mu_i(s') \geq \mu_i(s') \). □

**Appendix B: Equivalence of limited enforcement and collateral constraints in static environment**

This appendix shows the equivalence of the economy with limited enforcement and with collateral constraints in the static case. We start by stating the firm’s and the intermediary’s problem with limited enforcement.

The *firm’s problem* is to choose dividends \( \{d, d'\} \), investment \( k \), and payments to the household \( \{p, p'\} \) and intermediary \( \{p_a, p'_m, p'_a\} \) to maximize

\[
d + \beta d'
\]

subject to the budget constraints for time 0 and time 1

\[
w \geq d + k + p + p_a,
\]

\[
A'f(k) + k(1 - \delta) \geq d' + p' + p'_m + p'_a,
\]

the participation constraints for the intermediary and household

\[
p_a + q'_m p'_m + q'_a p'_a \geq 0,
\]

\[
p + R^{-1} p' \geq 0,
\]

the limited enforcement constraints for the morning and afternoon

\[
d' \geq A'f(k) + (1 - \theta_i) k (1 - \delta),
\]

\[
d' \geq A'f(k) + (1 - \theta) k (1 - \delta) - p'_m,
\]

and the non-negativity constraints

\[
d, d', p'_m \geq 0.
\]

The firm can abscond with a fraction \( 1 - \theta_i \) of capital in the morning, whereas in the afternoon it can abscond with fraction \( 1 - \theta \) but not payments \( p'_m \) already made.

The *intermediary’s problem* is to choose dividends \( \{d_i, d'_i\} \) and payments from the household \( \{p_h, p'_h\} \) and from the firm \( \{\bar{p}_a, \bar{p}'_m, \bar{p}'_a\} \) to maximize

\[
d_i + \beta_i d'_i
\]

47
subject to the budget constraints for time 0 and time 1

\[ w_i \geq d_i - p_h - \bar{p}_a, \quad (67) \]

\[ 0 \geq d'_i - p'_h - \bar{p}_m - \bar{p}_a', \quad (68) \]

the participation constraints for the firm and household

\[ -(\bar{p}_a + q'_m \bar{p}_m + q'_a \bar{p}_a') \geq 0, \quad (69) \]

\[ -(p_h + R^{-1} p'_h) \geq 0, \quad (70) \]

the limited enforcement constraint for the afternoon

\[ d'_i \geq \bar{p}_m, \quad (71) \]

and the non-negativity constraints

\[ d_i, d'_i, \bar{p}_m \geq 0. \quad (72) \]

The intermediary can abscond in the afternoon with payments received in the morning. We emphasize that the intermediary’s limited enforcement constraint in the morning is redundant, because the intermediary would abscond without anything in the morning.

Using the firm’s time 1 budget constraint (60) which holds with equality and the limited enforcement constraint for the morning (63), we have

\[ A' f(k) + k(1 - \delta) - (p' + p'_m + p'_a) = d' \geq A' f(k) + (1 - \theta_i) k(1 - \delta) \]

which is equivalent to the following collateral constraint

\[ \theta_i k(1 - \delta) \geq p' + p'_m + p'_a, \quad (73) \]

and similarly the limited enforcement constraint for the afternoon is equivalent to the following collateral constraint

\[ \theta k(1 - \delta) \geq p' + p'_a. \quad (74) \]

We will further simplify the collateral constraint for the morning below, after a few intermediate steps.

Using the intermediary’s time 1 budget constraint (68) which holds with equality and the limited enforcement constraint for the afternoon (71), we have

\[ p'_h + \bar{p}_m + \bar{p}_a = d'_i \geq \bar{p}_m' \]

which is equivalent to the following collateral constraint

\[ \bar{p}_a' \geq -p'_h. \quad (75) \]
The intermediary can borrow from the household only against claims it has on the firm.

Next we show that the interest rate on intermediated loans repaid in the afternoon must equal $R$ and that the interest rate on intermediated loans repaid in the morning must (weakly) exceed $R$. The intuition is that if the afternoon interest rate would differ from the interest rates charged by the household, the intermediary could arbitrage this spread. Moreover, since the intermediary can always lend at $R$ to the household, loans repaid in the morning must yield at least $R$.

**Lemma 7.** Equilibrium state prices satisfy (i) $q'_a = R^{-1}$ and (ii) $q' \leq R^{-1}$ without loss of generality.

**Proof of Lemma 7.** Part (i): Suppose not and assume $q'_a < R^{-1}$ without loss of generality. Take $\varepsilon > 0$ and consider the alternative payments $\hat{p}_h = p_h + \varepsilon$ and $\hat{p}_a = \tilde{p}_a - \varepsilon$ at time 0 and $\hat{p}'_h = p'_h - R\varepsilon$ and $\hat{p}'_a = \tilde{p}'_a + (q'_a)^{-1}\varepsilon$ in the afternoon. These payments satisfy the intermediary’s time 0 budget constraint (67) and the firm’s and household’s participation constraints, (69) and (70), by construction. Moreover, the intermediary’s afternoon collateral constraint (75) is satisfied as

$$\hat{p}_h + \hat{p}_a = p'_h + \tilde{p}'_a + ((q'_a)^{-1} - R)\varepsilon > p'_h + \tilde{p}'_a \geq 0,$$

and using the intermediary’s time 1 budget constraint (68) we can choose $\hat{d}'_i = d'_i + ((q'_a)^{-1} - R)\varepsilon > d'_i$, which is an improvement, contradicting the optimality of the original solution.

Part (ii): Suppose not, i.e., $q' > R^{-1}$ and $\tilde{p}'_i > 0$. (If $\tilde{p}'_i = 0$, then we can set $q' = R^{-1}$ without loss of generality.) Then consider the alternative choice $\hat{p}'_i = \tilde{p}_i - \varepsilon$ and $\hat{p}'_a = \tilde{p}'_a + q'/R^{-1}\varepsilon$, where $\varepsilon > 0$, which satisfies the firm’s participation constraint (69) by construction. Moreover, we can choose $\hat{d}'_i = d'_i + (q'/R^{-1} - 1)\varepsilon > d'_i$, which is an improvement and hence again impossible. \(\square\)

Observe that the firm’s problem only determines the sum of $p + p_a$ and $p' + p'_a$. Similarly, the intermediary’s problem only determines the sum of $p_h + \tilde{p}_a$ and $p'_h + \tilde{p}'_a$.

We can now show that the firm’s limited enforcement constraints are equivalent to the following collateral constraints

\[
(\theta_i - \theta)k(1 - \delta) \geq p'_m
\]

\[
\theta k(1 - \delta) \geq p' + p'_a.
\]

We need to show that (76) and (77) are equivalent to (73) and (74). First, note that (77) and (74) are identical. Second, observe that adding (76) and (77) yields (73) which
establishes the first direction. To establish the other direction, suppose that (76) does not hold, i.e., \((\theta_i - \theta)k(1 - \delta) < p'_m\). Then (77) must be slack as otherwise adding (76) and (77) would imply that (73) is violated. Thus, \(\theta k(1 - \delta) > p'_m + p'_a\). Consider the alternative payments \(\hat{p'}_m = p'_m - \epsilon\) and \(\hat{p'}_a = p'_a + q'/R^{-1}\epsilon\) which satisfy (61), (76), and (77) by construction, and \(\hat{d}' = d' + (1 - q'/R^{-1})\epsilon \geq d\), which is a (weak) improvement (and a strict improvement and hence contradiction whenever \(q' < R^{-1}\)). Therefore, (76) holds without loss of generality. This establishes the equivalence of the economies with limited enforcement and collateral constraints.

To recover the formulation in the text set \(p'_a = \bar{p}'_a = 0\), that is, the firm makes no payments to the intermediary in the afternoon, and change notation by letting \(R'_i \equiv (q')^{-1}, b \equiv -p, b_i \equiv (R'_i)^{-1}p'_m, l \equiv -p_h, and l_i \equiv (R'_i)^{-1}\bar{p}'_m,\) where \(\{b, b_i\}\) are the amounts the firm borrows from the household and intermediary and \(\{l, l_i\}\) are the amounts the intermediary lends to the household and firm. Using the fact that the participation constraints for the intermediary and the household, (69) and (70), bind, we can rewrite the firm’s problem as maximizing (58) by choosing \(\{d, d', k, b, b_i\}\) subject to the constraints

\[
\begin{align*}
    w &\geq d + k - b - b_i \\
    A'f(k) + k(1 - \delta) &\geq d' + Rb + R'_i b_i \\
    (\theta_i - \theta)k(1 - \delta) &\geq R'_i b_i \\
    \theta k(1 - \delta) &\geq Rb \\
    d, d', b_i &\geq 0.
\end{align*}
\]

Similarly, using the fact that the participation constraints for the firm and the household, (69) and (70), bind, we can rewrite the intermediary’s problem as maximizing (66) by choosing \(\{d_i, d'_i, l, l_i\}\) subject to the constraints

\[
\begin{align*}
    w_i &\geq d_i + l + l_i \\
    0 &\geq d'_i - Rl - R'_i l_i \\
    l &\geq 0 \\
    d_i, d'_i, l_i &\geq 0.
\end{align*}
\]

We refer to this implementation as the direct implementation as all afternoon loans to the firm are extended by the household directly. The intermediary has hence no income from collateralized loans in the afternoon and thus cannot make pledges to the household and can lend to but not borrow from the household. This can be seen from the collateral constraint (85) which reduces to a non-negativity constraint on lending to the household.

Alternatively, let, as before, \(R'_i \equiv (q')^{-1}, b_i \equiv (R'_i)^{-1}p'_m,\) and \(l_i \equiv (R'_i)^{-1}\bar{p}'_m,\) but now let \(b \equiv R^{-1}p'_a, l \equiv R^{-1}p'_h, and l'_a \equiv R^{-1}\bar{p}'_a,\) and set \(p = p' = 0,\) that is, the firm does not
borrow from the household directly. Then (61), which holds with equality, implies that 
\( p_a = -(b_i + b) \), and substituting into (59), (60), (76), and (77), we obtain the constraints of the firm’s problem which are identical to equations (78) through (82). However, now we interpret \( b \) as loans extended by the intermediary to be repaid in the afternoon. Similarly, for the intermediary, (70) at equality implies that \( p_h = -l \) and (69) at equality implies that \( p_a = -(l_i + l_a) \) which yields the following set of constraints:

\[
\begin{align*}
  w_i & \geq d_i + l + l_i + l_a \quad (87) \\
  0 & \geq d_i' - Rl - R_i' l_i - Rl_a \quad (88) \\
  l_a & \geq -l \quad (89)
\end{align*}
\]

and (86). This is the indirect implementation in which the intermediary extends morning and all afternoon loans to the firm and in turn borrows from the household against its collateralized loans. The afternoon collateral constraint (89), similar to equation (75), implies that the intermediary can borrow from the household up to the amount that the firm is due to repay in the afternoon.

Finally, we emphasize that the firm needs to repay morning loans \((b_i)\) in the morning and, in the dynamic model, these loans cannot be simply rolled over. Instead, even in the steady state, the firm has to take out the morning loans every afternoon and then repay them the next morning, only to take them out again in the afternoon. There is, thus, an explicit and novel sense in which these loans are short term in our model.

Appendix C: Intermediated vs. direct finance in the cross section

This appendix considers the static environment without uncertainty of Section 3 taking the spread on intermediated finance as given to show that our model has plausible implications for the choice between intermediated and direct finance in the cross section of firms. Consider the firm’s problem taking the interest rate on intermediated finance \( R_i' \) as given. Each firm maximizes (52) subject to (31) through (33) given its net worth \( w \). Severely constrained firms borrow as much as possible from intermediaries while less constrained firms borrow less from intermediaries and dividend paying firms do not borrow from intermediaries at all, consistent with the cross sectional stylized facts. These cross-sectional results are similar to the ones in Holmström and Tirole (1997).

Proposition 8 (Intermediated vs. direct finance across firms). Suppose \( R_i' > \beta^{-1} \).16

\[\text{We consider the case in which } R_i' > \beta^{-1} \text{ since, proceeding analogously as in the first part of the proof, one can show that } R_i' < \beta^{-1} \text{ would imply that } \lambda_i' > 0 \text{ and thus the cross sectional financing implications would be trivial as all firms would borrow the maximal amount from intermediaries. When } R_i' = \beta^{-1}, \text{ this would also be true without loss of generality.}\]
(i) Firms with net worth \( w \leq w_i \) borrow as much as possible from intermediaries, firms with net worth \( w_i < w < w_u \) borrow a positive amount from intermediaries but less than the maximal amount, and firms with net worth exceeding \( w_u \) do not borrow from intermediaries, where \( 0 < w_i < w_u \). (ii) Only firms with net worth exceeding \( \bar{w} \) pay dividends at time 0, where \( w_u < \bar{w} < \infty \). (iii) Investment is increasing in \( w \) and strictly increasing for \( w \leq w_i \) and \( w_u < w < \bar{w} \).

**Proof of Proposition 8.** The first order conditions are (36)-(39) and \( \mu' = 1 + \nu_d' \) where \( \beta \nu_d' \) is the multiplier on the constraint \( w' \geq 0 \). By the Inada condition, (37) implies that \( k > 0 \) and using (32) at equality and (34) and (33) we have \( d' \geq A' f(k) + k (1 - \theta_i) (1 - \delta) > 0 \) and \( \mu' = 1 \). But (36) and (38) imply \( 1 \leq \mu = R \beta + R \beta \lambda' \) and thus \( \lambda' > 0 \) since \( R \beta < 1 \) by assumption; that is, all firms raise as much financing as possible from households.

Suppose the firm pays dividends at time 0. Then \( \mu = \mu' = 1 \) and (39) implies \( 0 > 1 - R'_i \beta = R'_i \beta \lambda'_i - R'_i \beta \nu_i' \) and thus \( \nu_i' = 1 - (R'_i \beta)^{-1} > 0 \), \( b'_i = 0 \), and \( \lambda'_i = 0 \); thus, the firm does not use intermediated finance. Note that the problem of maximizing (52) subject to (31) through (33) has a (weakly) concave objective and a convex constraint set and hence induces a (weakly) concave value function. Thus, \( \mu \) is (weakly) decreasing in \( w \) and let \( \bar{w} \) be the lowest value of net worth for which \( \mu = 1 \); by the Inada condition, such a \( \bar{w} < +\infty \) exists. At \( \bar{w} \), \( d = 0 \), \( \bar{w} = k \phi \) (using (31)), and \( \bar{k} \) solves \( 1 = \beta [A' f(k) + (1 - \theta) (1 - \delta)] / \phi \) (using (37)). For \( w \geq \bar{w}, d = w - \bar{w} \) while the rest of the optimal policy is unchanged.

Suppose \( \lambda'_i = 0 \) and \( \nu'_i = 0 \). Then \( \mu = R'_i \beta > 1 \). Moreover, rearranging (37) we have \( 1 = \beta / (R'_i \beta) [A' f(k) + (1 - \theta) (1 - \delta)] / \phi \) which defines \( k < \bar{k} \). Define \( w_u \) such that investment is \( \bar{k} \) and \( b'_i = 0 \); then \( w_u = k \phi \). Similarly, define \( w_i \) such that investment is \( \bar{k} \) and \( b'_i = (R'_i)^{-1} (\theta_i - \theta) \bar{k} (1 - \delta) \); then \( w_i = k [\phi - (R'_i)^{-1} (\theta_i - \theta) (1 - \delta)] \). Note that \( w_i < w_u < \bar{w} \). So firms below \( w_i \) raise as much financing as possible from intermediaries (since \( \mu > R'_i \beta \) by concavity and hence \( \lambda'_i > 0 \)). Firms with net worth between \( w_i \) and \( w_u \) pay down intermediary financing linearly. Firms with net worth above \( w_u \) do not borrow from intermediaries and scale up until \( \bar{k} \) is reached at \( \bar{w} \), at which point firms initiate dividends. □

Intermediated finance is costlier than direct finance. Indeed, under the conditions of the proposition, intermediated finance is costlier than the shadow cost of internal finance of well capitalized firms. Thus, well capitalized firms, which pay dividends, do not borrow from financial intermediaries. In contrast, firms with net worth below some threshold \( (w_u) \) have a shadow cost of internal finance which is sufficiently high that they choose to borrow a positive amount from intermediaries. For severely constrained firms, with net worth below \( w_i \), the shadow cost of internal funds is so high that they borrow as much as
they can from intermediaries, that is, their collateral constraint for intermediated finance binds. Moreover, more constrained firms have lower investment and are hence smaller.

The cross-sectional capital structure implications are plausible: smaller (and more constrained) firms borrow more from financial intermediaries and have higher costs of financing, while larger (and less constrained firms) borrow from households, for example in bond markets, and have lower financing costs.
References


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Figure 1: Capital, Collateral Value, and Financing

This figure shows, on the left, the extent to which one unit of capital can be collateralized by households (fraction $\theta$, interpreted as structures) and intermediaries (fraction $\theta_i$, interpreted to include equipment), in the middle, the collateral value next period after depreciation, and on the right, the maximal amount that households and intermediaries can finance, as well as the minimum amount of internal funds required.

Figure 2: Role of Firm and Financial Intermediary Net Worth

Interest rate on intermediated finance $R^i_t - 1$ (percent) as a function of firm ($w$) and intermediary net worth ($w_i$). The parameter values are: $\beta = 0.90$, $R = 1.05$, $\beta_i = 0.94$, $\delta = 0.10$, $\theta = 0.80$, $\theta_i = 0.90$, $A' = 0.20$, and $f(k) = k^{\alpha}$ with $\alpha = 0.333$. 
Figure 3: Role of Firm and Financial Intermediary Net Worth

Contour of area where spread exceeds $\beta_i^{-1} - R$: $w_i^*$ (solid) and $\bar{w}(w_i)$ (solid); $\hat{w}(w_i)$ (dashed); contour of area where spread equals $\beta_i^{-1} - R$: $\bar{w}_i$ (dash dotted) and $\bar{\hat{w}}(w_i)$ (dash dotted). The parameter values are as in Figure 2.

Figure 4: Role of Firm and Financial Intermediary Net Worth

Interest rate on intermediated finance $R_i' - 1$ (percent) as a function of firm ($w$) for different levels of intermediary net worth ($w_i$). The parameter values are as in Figure 2.
Figure 5: Dynamics of Firm and Financial Intermediary Net Worth

Contours of the regions describing the deterministic dynamics of firm and financial intermediary net worth (see Proposition 6). Region ND, in which firms pay no dividends, is to the left of the solid line and Region D, in which firms pay positive dividends, is to the right of (and including) the solid line. The point where the solid line reaches the dotted line is the deterministic steady state \((w^*, w_i^*)\). The kink in the solid line is the point \((\bar{w}, \bar{w}_i)\) where \(R'_i = \beta^{-1}\) and the collateral constraint just binds. The solid line segment between these two points is \(\bar{w}(w_i) = \phi k(w_i) - w_i\) (with \(R'_i \in (\beta^{-1}, \beta^{-1})\)). The solid line segment sloping down is \(\bar{w}(w_i) = \phi k - w_i\) (with \(R'_i = \beta^{-1}\)). Region ND is dividend by two dash dotted lines: below the dash dotted line through \((\bar{w}, \bar{w}_i)\) \(R'_i > \beta^{-1}\); between the two dash dotted lines \(R'_i \in (\beta^{-1}, \beta^{-1})\); and above the dash dotted line through \((w^*, w_i^*)\) \(R'_i < \beta^{-1}\). The parameter values are as in Figure 2.
Figure 6: Dynamics of Firm and Financial Intermediary Net Worth

This figure illustrates the deterministic dynamics starting from initial values of net worth \( w = 0.0222 \) and \( w_i = w_i^* \). Panel A traces out the path of firm and intermediary net worth in \( w \) vs. \( w_i \) space with the contours as in Figure 5. Panel B shows the evolution of the interest rate on intermediated finance (Panel B1), firm net worth (dashed) and intermediary net worth (solid) (cum dividends (higher) and ex dividend (lower)) (Panel B2), intermediated lending to firms (solid) and households (dashed) (Panel B3), and investment (Panel B4). The parameter values are as in Figure 2 except that \( \alpha = 0.8 \).

Panel A: Joint evolution of firm and intermediary net worth

Panel B: Interest rates, net worth, lending, and investment over time
Figure 7: Dynamics of a Credit Crunch

This figure illustrates the deterministic dynamics after a credit crunch starting from initial values of net worth \( w = w^* \) and \( w = 0.01 \). Panel A traces out the path of firm and intermediary net worth in \( w \) vs. \( w_i \) space with the contours as in Figure 5. Panel B shows the evolution of the interest rate on intermediated finance (Panel B1), firm net worth (dashed) and intermediary net worth (solid) (cum dividends (higher) and ex dividend (lower)) (Panel B2), intermediated lending to firms (solid) and households (dashed) (Panel B3), and investment (Panel B4). The parameter values are as in Figure 6 except that \( \theta = 0.65 \).

Panel A: Joint evolution of firm and intermediary net worth

Panel B: Interest rates, net worth, lending, and investment over time