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Large deviations of convex polyominoes*

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Abstract

Many open problems in machine learning, pattern recognition, and geometric analysis require enumeration of different types of lattice polygons, and in particular convex polyominoes. In this work, we develop a large deviation principle for convex polyominoes under different restrictions, such as fixed area and/or perimeter.

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1 Introduction

The dramatic growth in practical applications for machine learning and pattern recognition over the last ten years has been accompanied by many important developments in the underlying algorithms and techniques. One of the most prominent tools allowing to asses the quality of these algorithms consists in derivation of lower information-theoretic bounds serving as benchmarks in comparative studies. Such bounds may quantify the best achievable performance (e.g. Cramer-Rao bounds), sample or computational complexity, etc. Derivation of many of such bounds essentially boils down into estimation of the number of admissible models one of which is to be chosen as the outcome of the learning process. For example, in pattern recognition the goal is to select a pattern from a family of models best suiting the input data under some performance criteria. The sample complexity of such model selection is controlled by the richness of the set of eligible models or patterns, e.g. [1, 14, 15]. In its turn the problem of counting the cardinalities of model classes is equivalent to the questions of enumeration of geometric shapes.

Enumeration of geometric shapes has been one of the central combinatorial problems for a long time [10, 18]. Recently discovered connections to the theory of random partitions and concentration of measure have drawn attention of many scientists to this

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area. Let us start by mentioning the fundamental pioneering works of Vershik, Blinovskii, Dembo and Zeitouni [2, 7, 16], who developed the large deviation principle for integer partitions. A partition of a positive integer n is a finite non-increasing sequence of natural numbers

$$\iota = (\iota_1, \dots, \iota_b), \tag{1.1}$$

with $\sum_j \iota_j = n$. A convenient and useful way of viewing and analyzing partitions is through their Young diagrams. The Young diagram of ι is the union of b rectangles of height 1 and length ι_b each aligned with the integer lattice over the $\mathbb{R}^2$ plane and stacked upon each other in such a way that their left sides form a straight vertical line and are ordered as in ι starting from the bottom, as in Figure 1. Because of this ordering, the upper right part of the diagram's boundary – also refereed to as the interface of the diagram – shall form a non-increasing piece-wise constant function. It was shown in [2, 7, 16] that for an integer n, the interfaces of the $\frac{1}{\sqrt{n}}$-scaled Young diagrams endowed with the uniform measure concentrate around a non-random limiting curve when n grows to infinity. In [16] Vershik calculated the exact shape of the curve. Given an arbitrary curve satisfying some natural regularity conditions, the authors of [2, 7] derived the exact speed and rate function controlling the number of scaled Young diagrams in a small vicinity of the curve. This line of research was further extended by other mathematicians to different setups and conditions. In [7] a large deviation principle for strict partitions was derived, in [17, 19] – for convex polygons on an integer lattice, etc. In some cases only the limiting curve was obtained without the large deviation principle, e.g. the case of restricted and boxed partitions [13].

Consider the integer (square) lattice on the $\mathbb{R}^2$ plane. A lattice *polyomino* is a union of elementary lattice cells which must be joined at their sides. A polyomino is said to be *column-convex* in a given lattice direction if all the cells along any line in that direction are connected through cells in the same line. A polyomino on the integer lattice is *convex* if it is column-convex in both horizontal and vertical directions, as in Figure 2. Note that Young diagrams are convex polyominoes. One of the main problems in the field of convex polyominoes is their enumeration [10]. There exists a large body of literature addressing the problem of polyomino counting according to their perimeter and/or area [3–6]. However, in all these works the desired numbers are given implicitly as coefficients of the corresponding terms in the series expansions of the generating functions derived therein. These series are usually too complicated and bulky to be directly analyzed and the sought for coefficients cannot be easily extracted. Moreover, even the asymptotic behavior of these coefficients is by no means obvious to derive. Using general ideas on counting of Young diagrams from [2], in this work we develop a large deviation principle for convex polyominoes with different constraints, such as perimeter, area or both. Interestingly, our findings generalize some of the results in the works devoted to the study of equilibrium shapes of convex polyominoes of fixed perimeter under different pressure [11, 12].

The rest of the text is organized as follows. First we introduce the concept of a large deviation principle and discuss the main geometric properties of convex polyominoes in Section 2. We formulate the main results in Section 3 and provide their proofs in Section 4.

2 Definitions

2.1 Large deviation principle

In this section we introduce the notion of a Large Deviation Principle (LDP) [8]. Our main result concerning the enumeration of convex polyominoes will be formulated in terms of an LDP. Let $\mathcal{P}$ be a Polish space meaning that it is a separable topological space

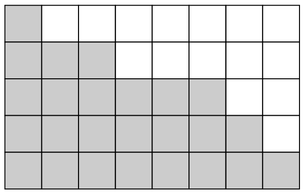

Figure 1: A Young diagram of the partition $\iota = (8, 7, 6, 3, 1)$ of $n = 25$.

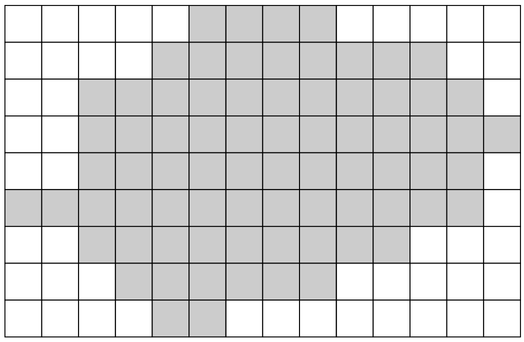

Figure 2: A convex polyomino.

such that there exists a metric on it that generates the topology of the space and $\mathcal{P}$ equipped with that metric is complete. We shall also equip our Polish space with its Borel σ-algebra turning it into a measurable space. For a measurable $B \subset \mathcal{P}$, denote by B^0 the interior of B and by $\bar{B}$ its closure (which are necessarily measurable, too).

Definition 2.1. *A sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of probability measures on $\mathcal{P}$ satisfies a Large Deviation Principle with speed a_n and rate function I if*

$$-\inf_{b \in B^0} I(b) \leqslant \liminf_{n\to\infty} \frac{\log\, \mathbb{P}_n(B)}{a_n} \leqslant \limsup_{n\to\infty} \frac{\log\, \mathbb{P}_n(B)}{a_n} \leqslant -\inf_{b \in \bar{B}} I(b), \quad \forall B \subset \mathcal{P},$$

where $I : \mathcal{P} \to \overline{\mathbb{R}}_+$ is lower semi-continuous (its level sets $L(M) = \{b \in \mathcal{P} | I(b) \leqslant M\}$ are closed for any $M \geqslant 0$). If all the level sets in $L(M)$ are compact, we refer to I as a good rate function.

Given an element $\gamma \in \mathcal{P}$, let $U_\varepsilon(\gamma)$ be its ε-vicinity. In addition to the LDP, we also formulate the so-called local LDP.

Definition 2.2. *Assume that for all $\gamma \in \mathcal{P}$,*

$$\liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{\log\, \mathbb{P}_n(U_\varepsilon(\gamma))}{a_n} = \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log\, \mathbb{P}_n(U_\varepsilon(\gamma))}{a_n} = -I(\gamma),$$

then we say that $\mathbb{P}_n$ satisfies the local LDP.

The last definition can be roughly interpreted as

$$\mathbb{P}_n(U_\varepsilon(\gamma)) \sim e^{-a_n I(\gamma)}.$$

2.2 Convex polyominoes

We start with the following trivial property of convex polyominoes.

Lemma 2.3 ([10]). *A polyomino is convex if and only if its perimeter coincides with the perimeter of its circumscribed rectangle.*

Proof. Immediately follows from the convexity. □

Figure 2 shows an example of a convex polyomino on a square lattice and its circumscribed rectangle. In the discrete scenario we have the following analog of the isoperimetric inequality.

Lemma 2.4 (Isoperimeteric inequality on the square lattice). *For a polyomino of area A and perimeter L on the square lattice,*

$$A \leqslant \frac{L^2}{16}, \tag{2.1}$$

the equality is reached when the polyomino is a square.

Proof. We must only prove (2.1) for convex polyominoes. Due to Lemma 2.3, the perimeter of the circumscribed rectangle of a convex polyomino of perimeter L is also L. Clearly, the area of such a polyomino is maximized when it coincides with its circumscribed rectangle. Among the rectangles of perimeter L, the area is maximal for the square, which completes the proof. □

3 Large deviation principle for convex polyominoes

Consider the plane $\mathbb{R}^2$ with the standard basis and fixed origin. Assume that we are given a closed piece-wise differentiable curve $\Gamma \subset \mathbb{R}^2$ which is unimodal in both vertical and horizontal directions. In other words, every horizontal and vertical line intersects the curve in at most two points. Denote the region embraced by Γ by G and its area by

$$\text{area}(G) = A.$$

For convenience, let us assume that the barycenter (center of mass, assuming uniform density) of G coincides with the coordinate origin.

Given two curves Γ_1 and Γ_2, the distance between them is defined as

$$d(\Gamma_1, \Gamma_2) = \text{area}(G_1 \Delta G_2). \tag{3.1}$$

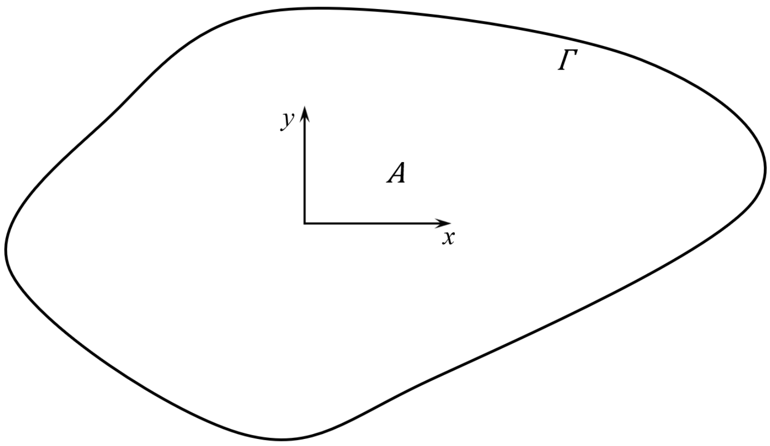

Figure 3: The original unimodal curve $\Gamma \subset \mathbb{R}^2$.

For every $n \in \mathbb{N}$, we construct the integer lattice centered at the origin and scale it by $\frac{1}{\sqrt{n}}$ so that the area of every elementary cell becomes $\frac{1}{n}$. Consider the set of convex polyominoes in the ε-vicinity of Γ, which we denote by

$$\mathbb{Q}_n = \mathbb{M}_n \cap U_\varepsilon(\Gamma),$$

where $\mathbb{M}_n$ is the set of all convex polyominoes on the $\frac{1}{\sqrt{n}}$-grid and the vicinity of Γ is determined with respect to the distance $d(\cdot,\cdot)$ defined in (3.1). Our goal will be to count

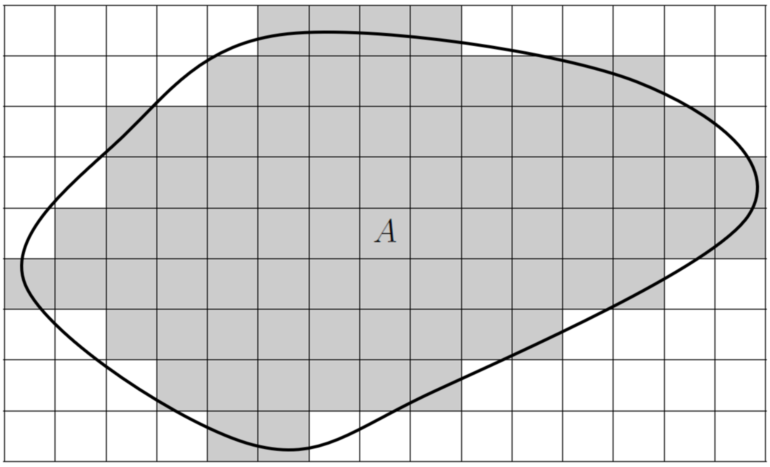

Figure 4: The curve and the approximating convex polyomino.

the polyominoes in $\mathbb{Q}_n$ satisfying different conditions, for example, the polyominoes in $\mathbb{Q}_n$ having fixed area $\mathbb{Q}_A$[1], fixed perimeter $\mathbb{Q}_L$, or both fixed area and perimeter $\mathbb{Q}_{A,L}$, etc. We start from enumerating the polyominoes of fixed area and later show that the other cases are treated analogously. We write[2]

$$Q_X = |\mathbb{Q}_X|, \quad X = A, L, \{A, L\}.$$

Also, following our convention for the curve Γ that assumes its barycenter to coincide with the origin, we identify all shifted copies of the same polyomino and count them as one.

Remark 1. By convention, below we write

$$\int_\Gamma f(\Gamma)(|dx| + |dy|) = \int_\Gamma f(\Gamma(s))(|\sin\theta| + |\cos\theta|)ds,$$

where $\Gamma(s)$ is the natural parameterization of the curve by its arc length and $\theta = \arctan(y')$ is the angle between the tangent line at a point and the horizontal axis.

Let

$$\mathbb{P}_{X,n}(U_\varepsilon(\Gamma)) = \frac{Q_X(U_\varepsilon(\Gamma))}{V_X}, \quad X = A, L, \{A, L\}.$$

[1] We suppress the n index to simply the notation.

[2] To be precise, we note that if for some specific n, A is not divisible by $\frac{1}{n}$ which is the area of the unit $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$ grid cell, we can round it to the closest integer multiple of $\frac{1}{n}$ and denote by $\widetilde{A}_n$. If necessary, we do the same for L and round it to a multiple of $\frac{2}{\sqrt{n}}$, where the coefficient 2 is due to Lemma 2.3 and the fact that lattice rectangles have even perimeters. We also note that below we only count the number of polyominoes in some ε-vicinity of the target curve and this vicinity always contains a range or parameters including $\widetilde{A}_n$ and A, therefore the distinction between the latter is of minor importance. Therefore, to simplify the notation we shall denote the sequences of areas $\{\widetilde{A}_n\}$ and lengths $\{\widetilde{L}_n\}$ by A and L, respectively.

where V_X is the total number of convex polyominoes satisfying condition X.

As mentioned earlier, we first consider convex polyominoes of fixed area, $X = A$, and denote $\mathbb{P}_n = \mathbb{P}_{A,n}$. Below we explain how to treat the general case.

Theorem 3.1 (LDP for Convex Polyominoes)**.** *Let Γ be a unimodal in the vertical and horizontal directions piece-wise differentiable curve embracing a region of area A. Then the sequence $\{\mathbb{P}_{A,n}\}_n$ satisfies the local LDP with speed $\sqrt{n}$ and good rate function*

$$I(\Gamma) = C_A - \int_\Gamma H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|) = C_A - \int_\Gamma (1+|\tan\theta|)H\left(\frac{1}{1+|\cot\theta|}\right)|dx|,$$

where $H(u) = -u\ln u - (1-u)\ln(1-u)$ is the binary entropy, $y = y(x)$ is the local parametrization of the curve, and C_A is the normalization constant (log-partition function in the statistical mechanics terminology).

As an immediate corollary, we obtain the following statement where we count the actual number of the polyominoes and not the (normalized) probability. This allows us to get rid of the constant C_A.

Corollary 3.2. *The number of convex polyominoes of area A inside $U_\varepsilon(\Gamma)$ satisfies*

$$\begin{aligned}\liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{\log Q_A}{\sqrt{n}} = \limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log Q_A}{\sqrt{n}} &= \int_\Gamma H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|)\\ &= \int_\Gamma (1+|\tan\theta|)H\left(\frac{1}{1+|\cot\theta|}\right)|dx|.\end{aligned}$$

Remark 2. Similar results can be obtained for convex polyominoes with fixed perimeter and with both fixed area and perimeter. The only difference will be in the value of constant C_A. Given a specific family X of polyominoes, this constant can be calculated as

$$C_X = \max_{\Gamma\in X}\int_\Gamma (1+|\tan\theta|)H\left(\frac{1}{1+|\cot\theta|}\right)|dx|. \tag{3.2}$$

It is easy to see that the curves on which the extremum is reached [13] are concatenations of the properly scaled segments of Vershik's limiting shape [16] given by the equation

$$e^{\frac{-\pi x}{\sqrt{6}}} + e^{\frac{-\pi y}{\sqrt{6}}} = 1. \tag{3.3}$$

In order to find the segments of this curve that maximize (3.2) for the family X under consideration, we need to find such parts of Vershik's curve (3.3) that satisfy the required relations between the perimeter (coinciding with the perimeter of the circumscribed rectangle) and area. As shown in [13] for any admissible combination of L and A we can always find the necessary segments on the curve (3.3).

4 Proofs

This section is devoted to the proof of the main result and contains a number of auxiliary lemmas.

Proof of Theorem 3.1. According to the definition of the LDP, the proof will be complete if we demonstrate that

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log \mathbb{P}_n(U_\varepsilon(\Gamma))}{\sqrt{n}} \leqslant -I(\Gamma), \tag{4.1}$$

and

$$\liminf_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{\log \mathbb{P}_n(U_\varepsilon(\Gamma))}{\sqrt{n}} \geqslant -I(\Gamma). \tag{4.2}$$

Let us start with (4.1). By the very definition of Γ, it can be partitioned into four segments each of which is a graph of a strictly monotonic function. In our example in Figure 5, the four segments are the curve arcs $T_N T_E$, $T_E T_S$, $T_S T_W$, and $T_W T_N$ connecting the points of intersection of Γ with its two horizontal and two vertical tangent lines.

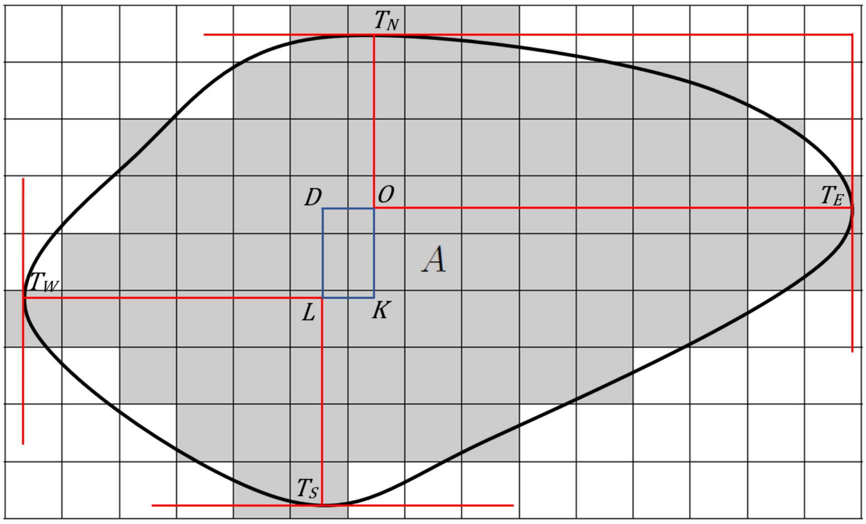

Figure 5: Partitioning of the curve.

Given a polyomino $Z \in \mathbb{Q}_A$, consider its top row of cells and choose the center of one of these cells. We call the obtained point the north extreme point of the polyomino and denote it by N (see Figure 6). Analogously, we define the other extreme points E, S, and W[3].

Now let us consider all the polyominoes from $\mathbb{Q}_A$ whose N and S extreme points have the same x coordinate and whose E and W extreme points have the same y coordinate. Denote this set by $\mathbb{Q}_A^e$ and let us bound its cardinality from above. Indeed,

$$Q_A^e \leqslant Q_A^{T_N N} Q_A^{N T_E} Q_A^{T_E E} Q_A^{ES} Q_A^{S T_S} Q_A^{T_S T_W} Q_A^{T_W W} Q_A^{W T_N}, \tag{4.3}$$

where $Q_A^{T_N N}$ is the number of decreasing diagrams in the ε-vicinity of $T_N R$, where R is the point of intersection of the vertical line through N with Γ, that fit in between the vertical lines through T_N and N, $Q_A^{N T_E}$ is the number of decreasing diagrams in the ε-vicinity of $R T_E$ belonging to the quadrant to the north-east from the vertical line through N and the horizontal line through T_E and so on in an analogous manner. Note that our definition (3.1) of the distance concerns closed curves and here we are talking about monotonic curve segments. Let the two curve segments be defined by the monotonic functions $f : [x_l^f, x_r^f] \to \mathbb{R}$ and $g : [x_l^g, x_r^g] \to \mathbb{R}$. To make our argument precise, by the distance between these two curves we will understand the integral $\int_{[x_l, x_r]} |f(x) - g(x)| dx$, where $x_l = \min[x_l^f, x_l^g]$ and $x_r = \max[x_r^f, x_r^g]$, and the functions are extended as constants by continuity, that is f over $[x_b, x_b^f]$ is defined to be $f(x_b^f)$, g over $[x_b, x_b^g]$ is defined to be $g(x_b^g)$, and similarly for the right end of the interval. Note also that in our setup, we always have $x_b^f = x_b^g$ or $x_e^f = x_e^g$.

[3] As shown below, the obtained extreme points will naturally split the polyomino into Young diagrams. There is some ambiguity in the choice of our extreme points when the edge rows or columns of the polyomino consist of more than one cell, like in Figure 6. Obviously, this will only affect the counts of the Young diagrams but not the total polyomino enumeration.

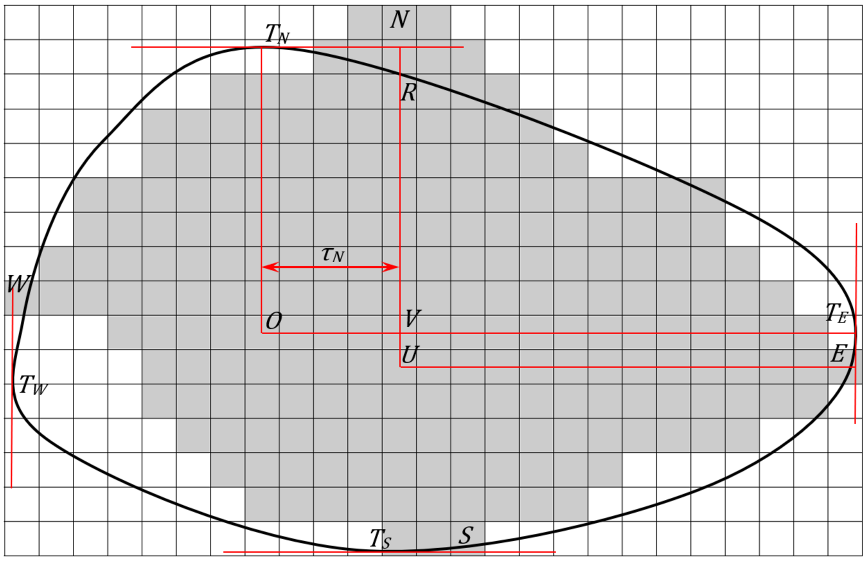

Figure 6: Deviation of the polyomino from the middle curve.

For convenience, take the logarithm of both sides of (4.3) to obtain

$$\log\, Q_A^e \leqslant \log\, Q_A^{NT_E} + \log\, Q_A^{T_S T_W} + \log\, Q_A^{ES} + \log\, Q_A^{WT_N} + \log\, Q_A^{T_N N} + \log\, Q_A^{T_E E} + \log\, Q_A^{ST_S} + \log\, Q_A^{T_W W}. \quad (4.4)$$

Our goal will be to show that the main contribution to (4.3) is made by the diagrams inside the large quadrants (the first row in the righ-hand side of (4.4)) and those parts of the boundary that correspond to the segments of the form $Q_A^{XT_X}$ or $Q_A^{T_X X}$ (the second row) tend to zero as ε approaches zero.

Let us start from bounding the value of $\log\, Q_A^{T_N N}$. Indeed, the horizontal distance τ_N between the points T_N and N must shrink with ε because the curve is strictly monotonic,

$$\tau_N(\varepsilon) \to 0, \ \text{ as } \ \varepsilon \to 0.$$

Analogous relations hold for the other τ_X as well,

$$\tau_X(\varepsilon) \to 0, \ \text{ as } \ \varepsilon \to 0, \quad X = N, E, S, W. \quad (4.5)$$

Below we use the following simple result.

Lemma 4.1 (Diagrams With Fixed Endpoints)**.** *The number of monotonic diagrams connecting points (a_1, b_1) and (a_2, b_2) of the integer square lattice with $a_1 \neq a_2$ and $b_1 \neq b_2$ which are also right-continuous at (a_1, b_1) is given by*

$$N((a_1, b_1), (a_2, b_2)) = \binom{|a_1 - a_2| + |b_1 - b_2| - 1}{|b_1 - b_2|}. \quad (4.6)$$

Proof. Let us associate 1 with every horizontal edge and 0 with every vertical edge. Then the desired number of diagrams coincides with the amount of ways $|a_1 - a_2| - 1$ ones and $|b_1 - b_2|$ zeros can be written into a binary codeword of length $|a_1 - a_2| + |b_1 - b_2| - 1$, which is given by the binomial coefficient in (4.6). □

Lemma 4.2 (Binomial Coefficient Bound, [9])**.** *For all natural $a > b$,*

$$aH\left(\frac{b}{a}\right) - \log\left(\sqrt{8\pi b(1 - b/a)}\right) \leqslant \log \binom{a}{b} \leqslant aH\left(\frac{b}{a}\right). \quad (4.7)$$

Proof. Using Stirling's approximation

$$a! = \sqrt{2\pi a}\left(\frac{a}{e}\right)^a e^{r(a)}, \quad \frac{1}{12a} < r(a) < \frac{1}{12a+1},$$

we obtain

$$\begin{aligned}\binom{a}{\beta a} &= \frac{\sqrt{2\pi a}}{\sqrt{2\pi\beta a}\sqrt{2\pi(1-\beta)a}}\frac{(a/e)^a}{(\beta a/e)^{\beta a}((1-\beta)a/e)^{(1-\beta)a}}\exp^{r(a)-r(\beta a)-r((1-\beta)a)}\\ &= \frac{1}{\sqrt{2\pi\beta(1-\beta)a}}\frac{1}{\beta^{\beta a}(1-\beta)^{(1-\beta)a}}\exp^{r(a)-r(\beta a)-r((1-\beta)a)},\end{aligned}$$

where $\beta = \frac{b}{a}$. Taking logarithms of both sides and recalling that

$$H(\beta) = -\beta\log\beta - (1-\beta)\log(1-\beta),$$

we write

$$\log\binom{a}{\beta a} = aH(\beta) - \log\left(\sqrt{2\pi\beta(1-\beta)a}\right) + r(a) - r(\beta a) - r((1-\beta)a).$$

Note in addition that

$$\begin{aligned}\frac{1}{12a} - \frac{1}{12\beta a+1} - \frac{1}{12(1-\beta)a+1} &< r(a) - r(\beta a) - r((1-\beta)a)\\ &< \frac{1}{12a+1} - \frac{1}{12\beta a} - \frac{1}{12(1-\beta)a}.\end{aligned}$$

Further bounding the left- and right- hand sides of the last inequality, we obtain

$$-\log 2 < r(a) - r(\beta a) - r((1-\beta)a) < \log\left(\sqrt{2\pi\beta(1-\beta)a}\right),$$

which necessarily implies the desired inequality (4.7). □

Consider the segment RT_E of the curve Γ and represent it as a monotonic function

$$y = f(x),$$

supported over the interval $[V, T_E]$, see Figure 6 for reference. Let the part of the boundary of the polyomino Z supported on the same interval be $\kappa_n(x)$. Below we show that condition

$$d(\Gamma, \partial Z) \leqslant \varepsilon,$$

where ∂Z is the polyomino boundary curve, implies that at the points O and V,

$$|y(x) - \kappa_n(x)| \leqslant \gamma(\varepsilon), \quad x \in \{O, V\},$$

for some function $\gamma(\varepsilon) \to 0$, as $\varepsilon \to 0$. Using this fact and the last two lemmas, we can write

$$\log Q_A^{T_N N} \leqslant \log\binom{(2\gamma(\varepsilon)+\tau_N)\sqrt{n}}{2\gamma(\varepsilon)\sqrt{n}} \leqslant \sqrt{n}(2\gamma(\varepsilon)+\tau_N)H\left(\frac{\tau_N}{2\gamma(\varepsilon)+\tau_N}\right),$$

for all n large enough. Here and below to keep the notation short we omit the rounding square brackets in $2\gamma(\varepsilon)\sqrt{n}$ and all similar expressions and assume the corresponding numbers to be integers. Similarly,

$$\log Q_A^{T_X X} \leqslant \sqrt{n}(2\gamma(\varepsilon)+\tau_X)H\left(\frac{\tau_X}{2\gamma(\varepsilon)+\tau_X}\right), \quad X = N, E, S, W, \tag{4.8}$$

where the index $T_X X$ is used to denote both $T_X X$ and XT_X interchangeably without loss of rigor. Since $H(x)$ is bounded and $2\gamma(\varepsilon)+\tau_X \to 0$, when $\varepsilon \to 0$, inequality (4.8) immediately implies that

$$\lim_{\varepsilon\to 0} \frac{\log\, Q_A^{T_X X}}{\sqrt{n}} = 0, \quad X = N, E, S, W. \tag{4.9}$$

Next we focus on bounding the value of $\log\, Q_A^{NT_E}$. The rest of the terms in the first line of (4.4) are treated analogously to the way we treated $\log\, Q_A^{NT_E}$.

Lemma 4.3. *Under the assumptions of Theorem 3.1,*

$$\limsup_{n\to\infty} \frac{\log\, Q_A^{NT_E}}{\sqrt{n}} \leqslant \int_{NT_E} H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|) + \phi^{NT_E}(\varepsilon), \tag{4.10}$$

where NT_E is the segment of Γ in the north-eastern quadrant and $\phi^{NT_E}(\varepsilon) \to 0$, when $\varepsilon \to 0$.

Proof. The proof can be found below in this section. □

Remark 3. Using exactly the same reasoning as in Lemma 4.3, we can obtain similar bounds for the rest of the quadrant segments $ES, T_S T_W$ and WT_N of Γ.

Let us get back to the upper bound on Q_A. Note that Q_A is a sum of Q_A^e-s for all possible choices of the extreme points. We know that the point N can move around T_N such that its abscissa belongs to the range $N_x \in [T_{N,x} - \tau_N, T_{N,x} + \tau_N]$. Similarly for the rest of the extreme points. Overall,

$$\begin{aligned} Q_A &= \sum_{N\in\Delta_N, E\in\Delta_E, S\in\Delta_S, W\in\Delta_W} Q_A^e \\ &= \sum_{N\in\Delta_N, E\in\Delta_E, S\in\Delta_S, W\in\Delta_W} Q_A^{T_N N} Q_A^{NT_E} Q_A^{T_E E} Q_A^{ES} Q_A^{ST_S} Q_A^{T_S T_W} Q_A^{T_W W} Q_A^{WT_N}, \end{aligned} \tag{4.11}$$

where $\Delta_N = [T_{N,x} - \tau_N, T_{N,x} + \tau_N]$, $\Delta_E = [T_{E,y} - \tau_E, T_{E,y} + \tau_E]$, $\Delta_S = [T_{S,x} - \tau_S, T_{S,x} + \tau_S]$, and $\Delta_W = [T_{W,y} - \tau_W, T_{W,y} + \tau_W]$. Note that the specific sequence of nodes in the superscripts of the right-hand side of (4.11) is chosen according to Figure 6, and can alter for a different set Q_A^e, but we will always have four multipliers corresponding to the curve segments of the form $T_X X$ and four corresponding to the segments in quadrants, so it is only a matter of notation.

Let us bound the logarithm of the left-hand side of (4.11) from above,

$$\frac{1}{\sqrt{n}} \log\left(\sum_{N\in\Delta_N, E\in\Delta_E, S\in\Delta_S, W\in\Delta_W} Q_A^{T_N N} Q_A^{NT_E} Q_A^{T_E E} Q_A^{ES} Q_A^{ST_S} Q_A^{T_S T_W} Q_A^{T_W W} Q_A^{WT_N}\right) \tag{4.12}$$

$$\begin{aligned} &\leqslant \frac{1}{\sqrt{n}} \log\left(16n^2 \prod_X (2\tau_X) \max_{N,E,S,W} Q_A^{T_N N} Q_A^{NT_E} Q_A^{T_E E} Q_A^{ES} Q_A^{ST_S} Q_A^{T_S T_W} Q_A^{T_W W} Q_A^{WT_N}\right) \\ &\leqslant \frac{1}{\sqrt{n}} \log\Bigl(\max_N Q_A^{T_N N} \max_N Q_A^{NT_E} \max_E Q_A^{T_E E} \max_{E,S} Q_A^{ES} \max_S Q_A^{ST_S} Q_A^{T_S T_W} \max_W Q_A^{T_W W} \max_W Q_A^{WT_N}\Bigr) \\ &\qquad + \frac{\log\left(16n^2 \prod_X (2\tau_X)\right)}{\sqrt{n}} \\ &\overset{(i)}{\leqslant} \left(\int_{T_N}^{T_E} + \int_{T_E}^{T_S} + \int_{T_S}^{T_W} + \int_{T_W}^{T_N}\right) H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|) + \phi(\varepsilon) + \frac{\log\left(16n^2 \prod_X (2\tau_X)\right)}{\sqrt{n}} \end{aligned}$$

$$= \int_\Gamma H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|) + \phi(\varepsilon) + \frac{\log\left(16n^2\prod_X(2\tau_X)\right)}{\sqrt{n}}, \qquad (4.13)$$

where in (i) we used (4.9), Lemma 4.3 and Remark 3, and therefore $\phi(\varepsilon)\to 0$ when $\varepsilon\to 0$. Take the $\limsup_{\varepsilon\to\infty}\limsup_{n\to\infty}$ of (4.12) and (4.13) to get the required bound,

$$\limsup_{\varepsilon\to\infty}\limsup_{n\to\infty}\frac{\log\, Q_A}{\sqrt{n}} \leqslant \int_\Gamma H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|). \qquad (4.14)$$

Let us now turn to the proof of the lower bound (4.2). Similarly to (4.4), it is easy to note that

$$\log\, Q^e_A \geqslant \log\, Q_A^{NT_E} + \log\, Q_A^{T_ST_W} + \log\, Q_A^{ES} + \log\, Q_A^{WT_N}. \qquad (4.15)$$

To treat this bound we use the following result.

Lemma 4.4. *Under the assumptions of Theorem 3.1,*

$$\liminf_{n\to\infty}\frac{\log\, Q_A^{NT_E}}{\sqrt{n}} \geqslant \int_{NT_E} H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|) + \psi^{NT_E}(\varepsilon), \qquad (4.16)$$

where NT_E is the segment of Γ in the north-eastern quadrant and $\psi^{NT_E}(\varepsilon)\to 0$, as $\varepsilon\to 0$.

Proof. The proof can be found below in this section. □

Remark 4. Here again, through the same reasoning as in Lemma 4.4, we can obtain similar bounds for the segments ES, T_ST_W and WT_N of Γ.

In manner similar to (4.14), we get the lower bound from Lemma 4.4 and Remark 4,

$$\liminf_{\varepsilon\to\infty}\liminf_{n\to\infty}\frac{\log\, Q_A}{\sqrt{n}} \geqslant \int_\Gamma H\left(\frac{|y'|}{1+|y'|}\right)(|dx|+|dy|).$$

Note that we can express the asymptotic behavior of V_A as

$$\lim_{n\to\infty}\frac{\log\, V_A}{\sqrt{n}} = \max_{\Gamma\in X} I(\Gamma),$$

and the statement of Theorem 3.1 follows. □

Proof of Lemma 4.3. As we have already mentioned earlier, the curve segment RT_E can be parametrized as a monotonically decreasing function $y(x)$ supported on the horizontal projection $[\alpha,\beta]=[V,T_E]$ of RT_E onto the x axis, where we assume $y(x)$ to be positive and for convenience denote $\alpha=V,\ \beta=T_E$. By the Lebesgue decomposition theorem, there exists a unique way the function y can be represented as

$$y(x)=y_1(x)+y_2(x),$$

where the monotonic functions $y_1(x)$ and $y_2(x)$ are absolutely continuous and singular correspondingly. Without loss of generality assume that $y_2(x)$ is continuous from the right. Note that the monotonically decreasing part of the polyomino boundary $y=\kappa_n(x)$ considered here is also continuous on the right.

Due to the monotonocity and singularity of y_2, for some set $A\subset[\alpha,\beta]$ of measure 0,

$$y_2'(x)=0,\ \ x\notin A.$$

Fix $\delta>0$ (it will later be made arbitrarily small). Since the Lebesgue measure μ is regular, we conclude the existence of an open set B measurable with respect to the Borel measure on the real line such that $A\subset B$ and $\mu(B)<\delta$. The function y_2 naturally defines a measure ν on $[\alpha,\beta]$ supported on A and such that $\nu((a,b])=y_2(b)-y_2(a)$. Since B is at most a countable union of open intervals $B=\bigcup_{i=1}^{\infty}B_i$, from the continuity of ν (w.r.t. the sequence of sets) it follows that

$$\lim_{m\to\infty}\nu\left(\bigcup_{i=m}^{\infty}B_i\right)=0.$$

Choose m such that

$$\nu\left(\bigcup_{i=m+1}^{\infty}B_i\right)<\delta.$$

Now let us expand each interval $B_i,\ i\leqslant m$ by incorporating the endpoints and denote the obtained closed intervals by $\widetilde{B}_i,\ i\leqslant m$. The set $\bigcup_{i=1}^{m}\widetilde{B}_i$ can be viewed as a union of a finite number of closed intervals intersecting only on their boundaries. Let $[d_1,g_1],\ [d_2,g_2],\dots,[d_s,g_s]$ be the intervals $\widetilde{B}_i$ where s is minimal possible. The set $[\alpha,\beta]\setminus\bigcup_{i=1}^{s}[d_i,g_i]$ consists of a finite number of disjoint intervals, by adding to them their end points we get a set of closed intervals $[a_j,b_j],\ j=1,\dots,p$ such that $\mu\left([\alpha,\beta]\setminus\bigcup_j[d_j,g_j]\right)<\delta$. In fact $\{a_j,b_j\}$ and $\{d_j,g_j\}$ are equal as sets and we denote them differently just for the sake of convenience. Partition every interval $[a_j,b_j]$ into s_j closed intervals

$$[a_j,b_j]=\bigcup_{k=1}^{s_j}[c_j^k,q_j^k],$$

intersecting only on their boundaries. Now move the constructed intervals slightly, such that all the conditions above are satisfied and for any x which is an end point of one of the considered intervals and

$$y'(x)<c, \tag{4.17}$$

for some constant[4] c.

Below, in the course of proving (4.1) we replace the requirement $\kappa_n\in U_\varepsilon(y)$, by

$$|\kappa_n(x)-y(x)|<\gamma(\varepsilon), \tag{4.18}$$

where x runs through the end points of the intervals and $\gamma(\varepsilon)>0$. Later we explain that (4.17) and the condition $\kappa_n\in U_\varepsilon(y)$ imply that $\gamma(\varepsilon)$ can be chosen in such a way that $\gamma(\varepsilon)\to 0$, as $\varepsilon\to 0$.

Define a function

$$L(z)=(1-z)H\left(\frac{-z}{1-z}\right),\quad z\leqslant 0 \tag{4.19}$$

which is continuous and

$$0\overset{(i)}{\leqslant}L(z+\xi)-L(z)\overset{(ii)}{\leqslant}L(\xi)\to 0,\ \text{ as }\ \xi\to 0, \tag{4.20}$$

where (i) follows from the monotonicity of L and (ii) from the relation

$$L_z'(z+\xi)-L_z'(z)=\log\frac{-z-\xi}{1-z-\xi}-\log\frac{-z}{1-z}\geqslant 0,\quad(\xi\leqslant 0).$$

Let $n_1,\ n_2,\dots$ be the sequence on which the $\limsup$ is reached in (4.10). For any x which is an endpoint of an interval $[c_j^k,q_j^k]$ there exist at most $[2\gamma(\varepsilon)\sqrt{n_i}]$ values of $\kappa_{n_i}(x)$ for

[4] If this condition does not hold for some segment of Γ, we can always consider the other local parametrization $x=x(y)$, for which it will hold.

which (4.18) holds true. Due to Lemma 4.1, given the values $\kappa_{n_i}(c_j^k) \geqslant \kappa_{n_i}(q_j^k)$ of κ_n at points c_j^k and q_j^k respectively, we have

$$\binom{\sqrt{n_i}\left(\kappa_{n_i}(c_j^k) - \kappa_{n_i}(q_j^k) + q_j^k - c_j^k\right) - 1}{\sqrt{n_i}\left(q_j^k - c_j^k\right)}$$

possibilities for the restrictions of κ_n onto the interval $[c_j^k, q_j^k]$. Let us now bound the number Y_{n_i} of the possibilities of restricting κ_n onto the set of intervals $[c_j^k, q_j^k],\ [d_l, g_l]$ from above as

$$\begin{aligned} Y_{n_i} \leqslant \prod_{j=1}^{p}\prod_{k=1}^{s_j} (2\gamma(\varepsilon))^2\, n_i \binom{\sqrt{n_i}\left(\kappa_{n_i}(c_j^k) - \kappa_{n_i}(q_j^k) + q_j^k - c_j^k\right) - 1}{\sqrt{n_i}\left(q_j^k - c_j^k\right)} \\ \times \prod_{l=1}^{s} (2\gamma(\varepsilon))^2\, n_i \binom{\sqrt{n_i}\left(\kappa_{n_i}(d_j) - \kappa_{n_i}(g_j) + g_j - d_j\right) - 1}{\sqrt{n_i}\left(q_j - c_j\right)}. \end{aligned} \tag{4.21}$$

Taking the logarithms of the both sides we get,

$$\begin{aligned} \log Y_{n_i} &\leqslant pm \log\left((2\gamma(\varepsilon))^2\, n_i\right) \\ &+ \sqrt{n_i}\sum_{j=1}^{p}\sum_{k=1}^{s_j}(q_j^k - c_j^k)\left(1 - \frac{\kappa_{n_i}(q_j^k) - \kappa_{n_i}(c_j^k)}{q_j^k - c_j^k}\right) H\left(\frac{-(\kappa_{n_i}(q_j^k) - \kappa_{n_i}(c_j^k))}{q_j^k - c_j^k - (\kappa_{n_i}(q_j^k) - \kappa_{n_i}(c_j^k))}\right) \\ &+ \sqrt{n_i}\sum_{l=1}^{s}(g_l - d_l)\left(1 - \frac{\kappa_{n_i}(g_l) - \kappa_{n_i}(d_l)}{g_l - d_l}\right) H\left(\frac{-(\kappa_{n_i}(g_l) - \kappa_{n_i}(d_l))}{g_l - d_l - (\kappa_{n_i}(g_l) - \kappa_{n_i}(d_l))}\right), \end{aligned} \tag{4.22}$$

where we have utilized the bound (4.7). Let $\gamma(\varepsilon, x)$ be a monotonically decreasing function of x for any fixed ε satisfying the condition

$$\kappa_{n_i}(x) = y(x) + \gamma(\varepsilon, x),$$

for x running through the values of c_j^k, q_j^k, d_l, and g_l and such that

$$|\gamma(\varepsilon, x)| < \gamma(\varepsilon),$$

to comply with (4.18). Divide (4.21) by $\sqrt{n_i}$ and let $i \to \infty$. Let us show that the contribution of the term $\sum_l$ in (4.21) can be made arbitrarily small through the choice of δ. Recall that

$$\sum_l (g_l - d_l) \leqslant \mu\left(\bigcup_{i=1}^{m} \widetilde{B}_i\right) < \delta. \tag{4.23}$$

Since the entropy function $H(z)$ is convex, we can use Jensen's inequality to obtain

$$\sum_{l=1}^{s}(\xi_l - z_l) H\left(\frac{-z_l}{\xi_l - z_l}\right) \leqslant \left(\sum_{l=1}^{s}\xi_l - \sum_{l=1}^{s} z_l\right) H\left(\frac{-\sum_{l=1}^{s} z_l}{\sum_{l=1}^{s}\xi_l - \sum_{l=1}^{s} z_l}\right), \quad \xi_l - z_l \geqslant 0. \tag{4.24}$$

Overall, we get that the term $\sum_l$ to in (4.21) is bounded from above by

$$\left(\sum_l (g_l - d_l) - D\right) H\left(\frac{-D}{\sum_l (g_l - d_l) - D}\right), \tag{4.25}$$

where

$$D = \sum_l y(g_l) - y(d_l) + \gamma(\varepsilon, g_l) - \gamma(\varepsilon, d_l).$$

From (4.23) and the fact that

$$(\delta - D)\, H\left(\frac{-D}{\delta - D}\right) \to 0, \text{ as } \delta \to 0,$$

we conclude that (4.25) can be made arbitrarily small by a proper choice of δ. Equation (4.20) implies that

$$L\left(\frac{y_1(q_j^k) - y_1(c_j^k) + y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k)}{q_j^k - c_j^k}\right) \\ \leqslant L\left(\frac{y_1(q_j^k) - y_1(c_j^k)}{q_j^k - c_j^k}\right) + L\left(\frac{y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k)}{q_j^k - c_j^k}\right). \quad (4.26)$$

Using (4.24), let us bound the contribution of the last summand (4.26) to (4.22),

$$\sum_{j=1}^{p} \sum_{k=1}^{s_j} (q_j^k - c_j^k) L\left(\frac{y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k)}{q_j^k - c_j^k}\right) \\ \leqslant \sum_{j,k} (q_j^k - c_j^k) L\left(\frac{\sum_{j,k} y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k)}{\sum_{j,k} q_j^k - c_j^k + \sum_{j,k} y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k)}\right). \quad (4.27)$$

Since

$$\sum_{j,k} q_j^k - c_j^k \geqslant (\beta - \alpha) - \delta,$$

and

$$\sum_{j,k} y_2(q_j^k) - y_2(c_j^k) + \gamma(\varepsilon, q_j^k) - \gamma(\varepsilon, c_j^k) < \delta + f(\varepsilon),$$

where $f(\varepsilon)$ can be chosen in such a way that

$$f(\varepsilon) \to 0, \text{ as } \varepsilon \to 0,$$

we conclude that the right-hand side of (4.27) is of the order of

$$(\beta - \alpha) L\left(\frac{\delta + f(\varepsilon)}{(\alpha - \beta) - \delta}\right) \to 0, \text{ as } \delta,\ \varepsilon \to 0.$$

Therefore, the contribution of the second summand from (4.26) into the right-hand side of (4.22) tends to zero together with ε.

Next, let us demonstrate that $\gamma(\varepsilon, x) \to 0$, for $x \in \{c_j^k,\ q_j^k\}$ when $\varepsilon \to 0$ under the condition (4.17). Indeed, choose $x_0 \in \{c_j^k,\ q_j^k\}$. For a fixed $\omega > 0$ let $h > 0$ be such small that

$$y(x) - y(x_0) < (c + \omega)(x - x_0), \quad 0 < x - x_0 < h.$$

Let

$$\kappa_{n_i}(x_0) - y(x_0) = \gamma_1(\varepsilon) > 0.$$

Since κ_{n_i} is a monotonic function and

$$\kappa_{n_i}(x) - y(x) \geqslant 0,$$

for $\max[0, x'] \leqslant x \leqslant x_0$, where

$$x' = \max\left[x_0 - h,\ \frac{(c + \omega)x_0 - \gamma_1(\varepsilon)}{c + \omega}\right],$$

we obtain

$$\varepsilon > \int_{\max[0,x']}^{x_0} |\kappa_{n_i}(x) - y(x)|dx = \int_{\max[0,x']}^{x_0} \kappa_{n_i}(x)dx - \int_{\max[0,x']}^{x_0} y(x)dx$$
$$\geqslant \frac{\gamma_1(\varepsilon)}{2}\left(x_0 - \max[0,x']\right). \quad (4.28)$$

The last inequality basically says that the leftmost integral is bounded from below by the area of the triangle determined by the lines

$$f_1(x) = y(x_0) + \gamma_1(\varepsilon), \quad f_2(y) = y(x_0), \quad f_3(x) = y(x_0) + \frac{x_0 - x}{x_0 - x'}\gamma_1(\varepsilon).$$

When $\varepsilon \to 0$, (4.28) implies that $\gamma_1(\varepsilon) \to 0$. Similar reasoning applies if

$$\kappa_{n_i}(x_0) - y(x_0) = \gamma_1(\varepsilon) < 0.$$

Taking into consideration the obtained bounds and applying $\lim_{n_i\to\infty}$ to the both sides of (4.22), we get

$$\lim_{n_i\to\infty} \frac{\log Y_{n_i}}{\sqrt{n_i}} \leqslant \sum_{j,k} (q_j^k - c_j^k) L\left(\frac{y_1(q_j^k) - y_1(c_j^k)}{q_j^k - c_j^k}\right) + \phi(\varepsilon) + \zeta(\delta), \quad (4.29)$$

where $\zeta(\delta)$ is the contribution of $\sum_l$ from (4.22) into the bound (4.29) and $\phi(\varepsilon) \to 0$, as $\varepsilon \to 0$. As we already know the term $\zeta(\delta)$ can be made arbitrarily small for a small enough δ, therefore, below we omit it from the upper bound.

Next we increase each s_j in such a way that

$$w = \max_{j,k}(q_j^k - c_j^k) \to 0.$$

Then the first summand in the right-hand side of (4.29) becomes

$$\sum_{j=1}^{p}\sum_{k=1}^{s_j}(q_j^k - c_j^k)L\left(\frac{1}{q_j^k - c_j^k}\int_{c_j^k}^{q_j^k} y_1'(x)dx\right) = \int_{\bigcup_j [a_j,b_j]} L\left(y_c(x)\right)dx,$$

where $y_c(x)$ is a step function such that for the given partition $\{[c_j^k, q_j^k]\}$ of the set $\bigcup_j [a_j, b_j]$,

$$y_c(x) = \int_{c_j^k}^{q_j^k} y_1'(x)dx, \quad x \in [c_j^k, q_j^k).$$

Taking into consideration the last two equations and applying $\liminf_{w\to 0}$ to the both sides of (4.29), we obtain

$$\limsup_{n\to\infty} \frac{\log Y_n}{\sqrt{n}} - \phi(\varepsilon) \leqslant \liminf_{w\to 0} \int_{\bigcup_j [a_j,b_j]} L\left(y_c(x)\right)dx \overset{(i)}{\leqslant} \int_{\bigcup_j [a_j,b_j]} \liminf_{w\to 0} L\left(y_c(x)\right)dx$$
$$\overset{(ii)}{=} \int_{\bigcup_j [a_j,b_j]} L\left(\liminf_{w\to 0} y_c(x)\right)dx \overset{(iii)}{=} \int_{\bigcup_j [a_j,b_j]} L\left(y_1'(x)\right)dx = \int_{\bigcup_j [a_j,b_j]} L\left(y'(x)\right)dx, \quad (4.30)$$

where (i) follows from Fatou's lemma, (ii) follows form the continuity of L, and (iii) is a consequence of the fact that if $z(x) \in L^1([a,b])$, then for a.e. $x_0 \in [a,b]$,

$$\lim_{q\to\infty} \frac{1}{|D_q|}\int_{D_q} z(x)dx = z(x_0),$$

where D_q is any sequence of intervals such that $x \in D_q$ and $|D_q| \to 0$, as $q \to \infty$. The last equality in (4.30) follows from the fact that $y_1' = y'$ a.s.

Since $\mu\left(\bigcup_j [a_j, b_j]\right) \geqslant (\beta - \alpha - \delta)$ and δ can be chosen arbitrarily small, from the absolute continuity of the integrals in (4.30), we have

$$\limsup_{n\to\infty} \frac{\log Y_n}{\sqrt{n}} \leqslant \int_{[\alpha,\beta]} L\left(y'(x)\right) dx + \phi(\varepsilon),$$

and (4.10) follows. □

Proof of Lemma 4.4. Consider a subsequence n_i on which the $\liminf$ is attained in (4.16). Define the interval $[\alpha, \beta]$ exactly as in the proof of Lemma 4.3, partition it into s equal intervals $[a_j, b_j], j = 1, \dots, s$, and denote their lengths by

$$\Delta = b_j - a_j = \frac{\beta - \alpha}{s}.$$

Above we focused on the upper bound and considered an excessive number of functions κ_n. Indeed, some of κ_n did not belong to $U_\varepsilon(y)$, moreover, some of them could not serve as boundaries of the polyominoes under consideration because since their areas could be larger than the area A^{NT_E} of the quadrant of G at hand. Now we treat the lower bound and must only count those κ_n that are the boundaries of convex polyominoes of area A^{NT_E} belonging to $U_\varepsilon(y)$.

Consider those κ_n which for every $x_0 \in \{a_j, b_j\}$ take the same value $\kappa_n(x_0)$ and satisfy the condition

$$|\kappa_n(x_0) - y(x_0)| \leqslant \frac{1}{\sqrt{n}}. \tag{4.31}$$

Assume we build our diagram from left to right. Two issues can happen during the course of such construction under the condition (4.31):

1) we can exhaust the area A^{NT_E} before we reach the rightmost point of y,

2) we can reach the rightmost point of y having diagram of a smaller area than required.

Later we show that in the case 1) the remaining area is small and can be spread above the constructed diagram without pushing it beyond $U_\varepsilon(y)$, and in the case 2) the total length of the remaining not covered intervals $[a_j, b_j]$ can be made arbitrarily small. Roughly speaking, we need to show that the areas under the curves $\kappa_n(x)$ and $y(x)$ for $x \in [0, \eta],\ \eta < \beta$ are close, where η is the point where κ_n becomes zero for the first time. During the construction of a diagram from left to right as described above, if we reached β, then we spread the remaining cells above the already constructed diagram. Let us show that the total area of these extra cells can be made arbitrarily small. Indeed, since $\kappa_n \in U_\varepsilon(y)$ the extra area can be represented as

$$A^{NT_E} - \int_\alpha^\beta \kappa_n(x)dx = \int_\alpha^\beta y(x) - \kappa_n(x)dx \leqslant \xi, \tag{4.32}$$

where $\xi \to 0$ when Δ shrinks and n grows.

Next we recycle the ideas used for the proof of (4.28), but this time we will also upper bound the L^1-distance between the curves. For two monotonically non-increasing functions $z_1(x)$ and $z_2(x)$, such that $|z_1(x) - z_2(x)| \leqslant 1/\sqrt{n}$ for $x = a,\ b$ where $a < b$ are arbitrary reals, we clearly have

$$\int_a^b |z_1(x) - z_2(x)|dx \leqslant (b-a)\left(z_1(a) - z_2(b) + \frac{2}{\sqrt{n}}\right). \tag{4.33}$$

Assume that (4.31) holds for all $x \in \{a_j, b_j\}$, then from (4.33) we get

$$\int_{[\alpha,\beta]} |\kappa_n(x) - y(x)|dx = \sum_{j=1}^{s} \int_{[a_j,b_j]} |\kappa_n(x) - y(x)|dx$$
$$\leqslant \sum_{j=1}^{s} (b_j - a_j)\left(y(a_j) - y(b_j) + \frac{2}{\sqrt{n}}\right) \leqslant \Delta\left(y(\alpha) - y(\beta) + \frac{2}{\sqrt{n}}\right). \quad (4.34)$$

Let now $\eta < \beta$ so that $\kappa_n(x) = 0,\ |y(x) - \kappa_n(x)| > \frac{1}{\sqrt{n}}$ for $x > \eta$ and $\kappa_n(x) > 0$ for $x < \eta$. This implies that

$$\int_0^{\eta} \kappa_n(x)dx = A^{NT_E},$$

and

$$\int_0^{\eta} y(x)dx = A^{NT_E} - \rho < \int_0^{\beta} y(x)dx.$$

Now

$$\int_0^{\eta} |\kappa_n(x) - y(x)|dx > \left|\int_0^{\eta} \kappa_n(x) - y(x)dx\right| > \rho.$$

On the other hand, the left-hand side of the last inequality is bounded from above by the expression in the right-hand side of (4.34). As a consequence, for small enough Δ, the value of ρ must be also small,

$$\rho \to 0, \text{ as } \Delta \to 0.$$

This is only possible if η is large enough. Let

$$\theta = \beta - \eta$$

then

$$\int_{\eta}^{\beta} y(x)dx \to 0, \text{ as } \Delta \to 0.$$

As we have already mentioned above, for Δ small enough, condition (4.32) will hold with ξ small. Overall, (4.32) and (4.34) imply that the constructed $\kappa_n(x)$ will belong to $U_\varepsilon(y)$ under the appropriate choice of Δ.

Let Y_{n_i} be the number of admissible diagrams we want to count, then it is lower bounded by the number of κ_{n_i} satisfying the conditions (4.32) and (4.34). By Lemma 4.1, the number of possible restrictions of $\kappa_{n_i}(x)$ onto an interval $[a_j, b_j]$ for given $\kappa_{n_i}(a_j)$ and $\kappa_{n_i}(b_j)$ is

$$\binom{\sqrt{n_i}\,(\kappa_{n_i}(a_j) - \kappa_{n_i}(b_j) + b_j - a_j) - 1}{\sqrt{n_i}\,(b_j - a_j)}.$$

The number of such restrictions on $[\alpha, \beta]$, for the given $\kappa_{n_i}(a_j)$ and $\kappa_{n_i}(b_j)$, is lower bounded by the product

$$\prod_{j=1}^{l} \binom{\sqrt{n_i}\,(\kappa_{n_i}(a_j) - \kappa_{n_i}(b_j) + b_j - a_j) - 1}{\sqrt{n_i}\,(b_j - a_j)},$$

where l is found from the following conditions. Let $\eta < \beta$ and r be the largest number such that $\mu([a_r, b_r] \cap [\eta, \beta]) = 0$, then set $l = s$. Otherwise set $l = r+1$ and $a_l = a_{r+1},\ b_l = \eta$. Clearly,

$$\mu\left(\bigcup_{j=l}^{s} [a_j, b_j]\right) < \beta - \eta + b_l - a_l < \theta + \Delta \to 0, \text{ as } \Delta \to 0.$$

Using the bound

$$\log \binom{m}{s} \geqslant mH\left(\frac{s}{m}\right) + o(m), \text{ as } m \to \infty,$$

following from (4.7) and taking into account that

$$|\kappa_{n_1}(x) - y(x_0)| \leqslant \frac{1}{\sqrt{n_i}}, \quad x = a_j,\ b_j,\ j \leqslant l,$$

we get

$$\frac{\log Y_{n_i}}{\sqrt{n_i}} \geqslant \sum_{j=1}^{l}\left(1 - \frac{y(b_j)-y(a_j)}{b_j-a_j} + O\left(\frac{1}{\sqrt{n_i}}\right)\right) H\left(\frac{-\frac{y(b_j)-y(a_j)}{b_j-a_j} + O\left(\frac{1}{\sqrt{n_i}}\right)}{1 - \frac{y(b_j)-y(a_j)}{b_j-a_j} + O\left(\frac{1}{\sqrt{n_i}}\right)}\right).$$

Let $n_i \to \infty$ and recall the definition of L from (4.19) to obtain,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log Y_{n_i}}{\sqrt{n_i}} &\geqslant \sum_{j=1}^{l}(b_j-a_j)L\left(\frac{y_1(b_j)-y_1(a_j)}{b_j-a_j} + \frac{y_2(b_j)-y_2(a_j)}{b_j-a_j}\right) \\
&\overset{(i)}{\geqslant} \sum_{j=1}^{l}(b_j-a_j)L\left(\frac{y_1(b_j)-y_1(a_j)}{b_j-a_j}\right) = \sum_{j=1}^{l}(b_j-a_j)L\left(\frac{1}{b_j-a_j}\int_{a_j}^{b_j} y_1'(x)dx\right) \\
&\overset{(ii)}{\geqslant} \sum_{j=1}^{l}\int_{a_j}^{b_j} L\left(y_1'(x)\right)dx = \int_{\alpha}^{\min[\eta,\beta]} L\left(y'(x)\right)dx,
\end{aligned}$$

where in (i) we utilized the monotonicity of L and in (ii) its convexity together with Jensen's inequality. Now let $\Delta \to 0$ to obtain

$$\liminf_{n\to\infty} \frac{\log Y_n}{\sqrt{n}} \geqslant \int_{T_N T_E} L\left(y'(x)\right)dx + \psi(\varepsilon),$$

which completes the proof. □

Remark 5. Assume now that instead of fixed area we deal with convex polyominoes of fixed perimeter. This case is even simpler since for most of the polyominoes the perimeter constraint will never be active. Indeed, the perimeter constraint only plays role if the diagram $\kappa_n(x)$ hits the boundary of the circumscribing rectangle. By appropriate choice of the extreme points of the polyomino we can easily satisfy this requirement, thus the bulk of the diagram will not be affected by it. The same applies to the polyominoes with both perimeter and area fixed (unless $\int_\Gamma (1+|\tan\theta|)H\left(\frac{1}{1+|\cot\theta|}\right)dx = 0$, which is not the case we consider).

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