Linear Quadratic Regulator

State Feedback/ Observer Control

Given a controllable/observable linear system, we can always design a stabilizing controller using state feedback and observer techniques.

Furthermore, by augmenting additional dynamics, we can also achieve steady state tracking and disturbance rejection goals.

**Question:** How do we choose the state feedback and observer gains?

**One Problem:** Too many degrees of freedom. If there are n states and p inputs, then the state feedback gain matrix has np elements. If p = 1, then there is a 1-1 correspondence between closed loop poles and elements of K. If p > 1, then there are degrees of freedom beyond pole placement.

**Idea:** Choose state feedback and observer gains to optimize some measure of system quality, such as

(i) dynamic response
(ii) disturbance/noise rejection
(iii) robustness/sensitivity

Several different optimal control strategies have been developed for designing linear feedback systems.

• These all suffer from the inability of a mathematical cost function to adequately describe the complexities of a feedback design problem.

• The trick is to use the optimization routine, not as an end in itself, but rather as a means of twiddling until you end up with a result that is satisfactory. For this strategy to be effective, we must understand how to twiddle intelligently.

**Linear Quadratic Gaussian Optimal Control Problem:**

• First stated and solved in the early '60s (Kalman).

• Much information is available about its use as a design tool, independent from its origins in optimal control.
Doesn't solve all problems, but an intelligent user can solve many problems, and avoid potential difficulties.

**Problem Statement:**

Consider: \[ \dot{x} = Ax + Bu \quad x(t_0) = x_0 , \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^p \]

**Problem:** Find an input \( u(t) \) to drive \( x(t) \to 0 \) rapidly and without incurring excessively large transients in the state and/or control signals.

**Notes:**
1. In practice, the linear system represents the behavior of a nonlinear system in the neighborhood of an equilibrium point. If the states or control signal become too large, then the system may be driven into a nonlinear operating regime.

2. If \((A,B)\) were controllable, and the size of the control signal were not an issue, we could choose \( u = -Kx \), where the eigenvalues of \( A-BK \) are such that \( x(t) = e^{(A-BK)t}x_0 \to 0 \) arbitrarily rapidly.

**Definitions:** Consider \( M \in \mathbb{C}^{n \times n} \). We say that \( M \) is

- **positive semidefinite** (denoted \( M \geq 0 \)) if
  \[ M = M^H \text{ and } x^H M x \geq 0 , \quad \forall \ x \in \mathbb{C}^n \]

- **positive definite** (denoted \( M > 0 \)) if
  \[ M = M^H , \quad x^H M x \geq 0 , \quad \forall \ x \in \mathbb{C}^n , \text{ and } x^H M x = 0 \text{ only for } x = 0 \]

There are analogous definitions of negative semidefinite and negative definite.

**Finite Horizon Optimal Regulator Problem:**

Minimize (over \( u \)) the cost function

\[
J(t_0,x_0,u,T) := \int_{t_0}^{T} (x^TQx + u^TRu)dt + x^T(T)P_f x(T)
\]
subject to the constraint \( \dot{x} = Ax + Bu \), \( x(t_0) = x_0 \).

The weighting matrices \( Q \), \( R \), and \( P_f \) are assumed to satisfy

- \( Q \) is positive semidefinite: \( Q = Q^T \geq 0 \)
- \( R \) is positive definite: \( R = R^T > 0 \)
- \( P_f \) is positive semidefinite: \( P_f^T = P_f \geq 0 \)

**Note:**

- \( P_f \) penalizes deviations of the final state from zero
- \( Q \) penalizes deviations of the transient state from zero
- \( R \) penalizes deviations of the transient control from zero

**Program:**

There are many ways to derive the solution to this problem, using techniques from optimal control theory (the Hamilton Jacobi Bellman equation, calculus of variations, dynamic programming, etc.)

We shall present a simpler derivation that avoids higher mathematics at the expense of looking unmotivated. This technique is termed "completing the square".

For technical reasons, we first solve the finite horizon problem, defined above, and then solve the infinite horizon problem by considering the limit of the solution as \( T \to \infty \).

There are thousands of papers and dozens of books that discuss various aspects of the LQG problem. A particularly complete reference is


The "completing the square" technique is discussed in

Theorem:  (1) The optimal control minimizing $J(t_0,x_0,u,T)$ is given by

$$u(t) = - R^{-1}B^TP(t)x(t),$$

where $P(t)$ is the unique solution to the Riccati differential equation:

$$\dot{P}(t) = P(t)A + A^TP(t) + Q - P(t)BR^{-1}B^TP(t), \quad P(T) = P_f$$

(2) The optimal value of the cost index is

$$J^*(t_0,x_0,T) = x^T(t_0)P(t_0)x(t_0)$$

(3) $P(t) \geq 0$ for all values of $t$

(4) $P(t)$ is bounded for all values of $t$.

Proof:

$$J(t_0,x_0,u) :=$$

$$\int_{t_0}^{T} (x^TQx + u^TRu)dt + x^T(T)P_f x(T) - x^T(t_0)P(t_0)x(t_0) + x^T(t_0)P(t_0)x(t_0)$$

$$= \int_{t_0}^{T} (x^TQx + u^TRu + \frac{d}{dt}[x^TPx])dt + x^T(t_0)P(t_0)x(t_0)$$

(because $P(T) = P_f$)

$$= \int_{t_0}^{T} (x^TQx + u^TRu + (Ax+Bu)^TPx + x^TP(Ax+Bu) + x^T\dot{P}x)dt + x^T(t_0)P(t_0)x(t_0)$$
If \( P(t) \) satisfies the Riccati equation, then substituting and rearranging (these algebraic manipulations are termed "completing the square"):

\[
T = \int_{t_0}^{T}(u+R^{-1}B^TPx)R(u+R^{-1}B^TPx)dt + x^T(t_0)P(t_0)x(t_0)
\]

Since only the first term depends on the control signal, it follows that minimizing this term will yield the minimum value of the cost function. Since the term in question is nonnegative, it follows that the control \( u(t) = -R^{-1}B^TP(t)x(t) \) minimizes the cost index. This proves (1) and (2).

To prove (3), we suppose that \( P(\hat{t}) \) is NOT \( \geq 0 \). Then there exists \( \hat{x} \) such that \( \hat{x}^T P(\hat{t}) \hat{x} < 0 \). Let \( t_0 = \hat{t} \), and \( x_0 = \hat{x} \). Then the optimal cost satisfies \( J^*(t_0, x_0, T) < 0 \). But this contradicts the fact that the right hand side of the cost function is \( \geq 0 \) by the assumptions that \( Q \geq 0, R > 0, \) and \( P_f \geq 0 \).

To prove (4), we note that the cost associated with the optimal control must be less than the cost associated with any control, including \( u(t) \equiv 0 \). Hence if we can show that the zero control incurs finite cost, then so does the optimal control. Setting \( u(t) \equiv 0 \) yields

\[
J(t_0, x_0, 0, T) := \int_{t_0}^{T}(x^TQx)dt + x^T(T)P_f x(T)
\]

\[
= \int_{t_0}^{T}(e^{A(t-t_0)}x_0)^TQ(e^{A(t-t_0)}x_0)dt + x^T(T)P_f x(T)
\]

\[< \infty\]

because the integrand is bounded over finite time intervals. It follows that the optimal cost \( J^*(t_0, x_0, T) \) must be bounded. Since \( x_0 \) is arbitrary, it is a straightforward exercise to show that \( P(t) \) must also be bounded. (See A&M, p. 24.)
Notes:

(1) The optimal control law is state feedback \( u(t) = -K(t)x(t) \) with a time varying feedback gain, \( K(t) = R^{-1}B^TP(t) \).

(2) The differential equation for \( P(t) \) must be solved backwards in time from the terminal condition \( P(T) = P_f \). This fact implies that the control gain must be calculated off-line for a given \( T \).

Example 1:

\[
\dot{x} = u \quad J(t_0, x_0, u, T) := \int_{t_0}^{T} (x^2 + u^2) \, dt
\]

\[- \dot{P}(t) = 1 - P(t)^2, \quad P(T) = 0\]

Integrating by separation of variables yields

\[
\Rightarrow \quad P(t) = \frac{e^{2(T-t)_-1}}{e^{2(T-t)_+1}}
\]

It follows that the optimal cost associated with an initial state \( x(t_0) = x_0 \) is given by

\[
J^*(t_0, x_0, T) = P(t_0)x_0^2 = \frac{e^{2(T-t_0)_-1}}{e^{2(T-t_0)_+1}} x_0^2.
\]

Note that the optimal cost depends only upon \( T-t_0 \), the difference between initial and final times. This is to be expected since the system and cost function are both time-invariant.

Infinite Horizon Case

The solution to the finite horizon regulator problem has the disadvantage that it is a time-varying gain defined only over a finite time interval. We now assume that the terminal cost is zero, and study the solution to the finite horizon problem in the limit as \( T \to \infty \):
Unlike the finite horizon problem, the optimal cost is not always finite.

**Example 2:**

\[ \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

with \[ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \] and \( R = 1 \).

Since \( x_1(t) = e^{(t-t_0)} \), it is easy to show that

\[ J(t_0, x_0, u, T) := \lim_{T \to \infty} \int_{t_0}^{T} (e^{2(t-t_0)} + u^2) \, dt \]

which \( \to \infty \) as \( t \to \infty \) regardless of how the control \( u \) is chosen.

The reason for the unbounded cost is that the eigenvalue at +1 is simultaneously (i) unstable, (ii) uncontrollable, (iii) observable in the cost.

**Theorem:** Assume that \((A,B)\) is stabilizable, and consider the optimal control problem of minimizing over \( u \) the cost function

\[ J(t_0, x_0, u) := \lim_{T \to \infty} \int_{t_0}^{T} (x^T Q x + u^T R u) \, dt \]

subject to the constraint \( \dot{x} = Ax + Bu, \quad x(t_0) = x_0 \). For a given \( T \), let \( P(t,T) \) denote the solution to the corresponding finite horizon optimal control problem.

(1) The optimal control minimizing \( J(t_0, x_0, u, T) \) is given by
u(t) = -R^{-1}B^TPx(t),

where \( \bar{P} = \lim_{T \to \infty} P(t,T) \) exists and is a constant matrix (independent of \( t \)).

(2) The matrix \( \bar{P} \) is a positive semidefinite solution to the Algebraic Riccati Equation (ARE):

\[
\bar{P}A + A^T\bar{P} + Q - \bar{P}BR^{-1}B^T\bar{P} = 0.
\]

(3) The optimal value of the cost index is

\[
J^*(t_0,x_0) = x^T(t_0)\bar{P}x(t_0)
\]

Proof: (1) Note first that \( J^*(t_0,x_0,T) \) is a monotonically increasing function of \( T \) (because \( Q \geq 0 \) and \( R > 0 \)). Next, we show that the optimal cost is bounded above. By the assumption that \( (A,B) \) is stabilizable, there exists \( K \) such that \( A-BK \) has stable eigenvalues. Using the control \( u(t) = -Kx \) yields \( x(t) = e^{(A-BK)(t-t_0)}x_0 \) and \( u(t) = -Ke^{(A-BK)(t-t_0)}x_0 \). Since both these functions converge exponentially to zero, it follows that

\[
J(t_0,x_0,-Kx) = \lim_{T \to \infty} \int_{t_0}^{T} x(t)^TQx(t) + u(t)^TRu(t))dt < \infty
\]

because the integrand converges to zero exponentially. Hence, the optimal cost is bounded above by the cost associated with an arbitrary stabilizing control.

It follows that for any \( (t_0,x_0) \) the optimal cost is a monotonically increasing function of \( T \) that satisfies an upper bound which is independent of \( T \). Results from real analysis show that \( J^*(t_0,x_0,T) \) is a convergent sequence. It is straightforward to use this fact, together
with the assumption that $x_0$ is arbitrary, to show that the elements of $P(t,T)$ are also bounded and convergent (cf. A&M).

The fact that $\bar{P}$ is constant (i.e., independent of $t$) follows since the system is time invariant and the integrand of the cost function does not depend specifically upon $t$.

(2) Since $\bar{P}$ is the limit of a sequence of positive semidefinite functions, it follows that $\bar{P}$ must be positive semidefinite also. Since $\bar{P}$ is constant, it follows that $\bar{P}$ must be a solution of the Riccati equation with the derivative term set equal to zero.

Notes:

(1) The optimal control is linear state feedback.

(2) Calculating the optimal control is awkward, because we must first solve the finite horizon problem, and then take the limit as $T \to \infty$.

(3) It isn't clear whether the optimal control will be stabilizing!

Example 1 (continued):

$$\dot{x} = u \quad J(t_0, x_0, u, T) := \int_{t_0}^{T} (x^2 + u^2) \, dt$$

$$P(t,T) = \frac{e^{2(T-t)} - 1}{e^{2(T-t)} + 1}$$

$$\Rightarrow \lim_{T \to \infty} P(t,T) = 1.$$
The ARE \( 0 = 1 - P^2 \) has two solutions, \( P = \pm 1 \). Only one of these corresponds to the optimal cost in the limit as \( T \to \infty \).

The optimal control, \( u = -x \) yields a stable closed loop system:

\[
\dot{x} = -x
\]

**Example 3:**

\[
\dot{x} = x + u \quad J(t_0, x_0, u, T) := \int_{t_0}^{T} u^2 \, dt
\]

It is easy to see that the optimal control is \( u(t) \equiv 0 \). But the closed loop system is given by \( \dot{x} = x \), which is unstable.

**Notes:**

(i) The problem here is that there is an unstable mode that does not appear in the cost function. Since the unstable mode doesn't cause the cost to increase, why should we spend control energy to stabilize it?

(ii) Consider the ARE for this example: \( 0 = 2P - P^2 \). This equation has two positive semidefinite solutions, \( P = 0 \) and \( P = 2 \).

We now show that if the unstable modes are penalized in the cost, then the optimal state feedback must stabilize the closed loop system.

**Preliminary Facts:** Consider \( M = M^H \in \mathbb{C}^{n \times n} \).

(i) If \( M \geq 0 \), and \( \text{rank}(M) = m \), then \( \exists \ N \in \mathbb{C}^{n \times n} \) with \( \text{rank}(N) = m \) such that \( M = N^H N \).

(ii) For any \( N \in \mathbb{C}^{m \times n} \), the matrix \( N^H N \geq 0 \). If \( \text{rank}(N) = m \), then \( \text{rank}(N^H N) = m \). If \( \text{rank}(N) = n \), then \( N^H N > 0 \).
Given a matrix $M$, any matrix $N$ satisfying (i) is termed a "square root" of $M$, and is denoted $M^{1/2}$.

**Theorem:** Consider the infinite horizon regulator problem. Assume that $(A,B)$ is stabilizable. Let $E$ be any matrix such that $Q = E^TE$. Assume that $(A,E)$ is detectable.

1. the optimal control $u = -Kx$, where $K = R^{-1}B^TP$, stabilizes the closed loop system; i.e., the eigenvalues of $A-BK$ are stable

2. $P$ is the *unique positive semidefinite* solution of the ARE.

3. if $(A,E)$ is observable, then $P$ is the *unique positive definite* solution of the ARE.

**Proof:** (1) (by contradiction) Assume $(A,E)$ is detectable, and suppose that $A-BK$ has an eigenvalue, $\lambda$, with $\text{Re}(\lambda) \geq 0$. Then there exists $v \neq 0$ such that $(A-BK)v = \lambda v$. Rearrange the ARE as follows:

$$
P(A-BK) + (A-BK)^TP = - Q - PBR^{-1}B^TP - PBR^{-1}B^TP = 0
$$

$$
\Rightarrow P(A-BK) + (A-BK)^TP = - Q - PBR^{-1}B^TP
$$

Pre and post multiply by $v^H$ and $v$, respectively:

$$
v^H P v(\lambda + \bar{\lambda}) = -(v^H Q v + v^H PBR^{-1}B^TP v)
$$

Together, the fact that $v^H P v \geq 0$ and the hypothesis that $\text{Re}(\lambda) \geq 0$ imply that the left hand side of this equation is $\geq 0$. Furthermore, since $v^H Q v \geq 0$ and $v^H PBR^{-1}B^TP v \geq 0$, the right hand side must be $\leq 0$. Since the left hand side must be $\geq 0$. Hence

$$
0 = v^H Q v + v^H PBR^{-1}B^TP v.
$$

But this implies that

$$
0 = v^H Q v \text{ and } 0 = v^H PBR^{-1}B^TP v
$$
Since $v^H Q v = v^H E^T E v = \|E v\|_2^2$. $v^H Q v = 0 \Rightarrow Ev = 0$.

Similarly, $v^H \tilde{P} B R^{-1} B^T \tilde{P} v = 0 \Rightarrow R^{-1} B^T \tilde{P} v = 0$.

The latter condition implies that $(A-BK)v = (A-BR^{-1}B^T\tilde{P})v = Av = \lambda v$.

Hence $\lambda$ is an eigenvalue of $A$; since $Ev = 0$, it follows that $\lambda$ is an undetectable eigenvalue of $A$, which contradicts the assumption that $(A,E)$ is detectable.

Notes:

(1) Proofs of (2)-(3) can be found in Anderson and Moore.

(2) There are, in general many matrices $E$ that satisfy $Q = E^T E$. It is straightforward to show that the properties of detectability and observability don't depend upon which of these we choose. Indeed, we may show that $(A,E)$ is detectable $\iff (A,Q)$ is detectable.

(3) Given $A$ and $Q$, we say that an eigenvalue of $A$ is "observable in the cost" if $\text{rank} \left[ \begin{array}{c} \lambda I - A \\ E \end{array} \right] = n$, where $E$ is any square root of $Q$.

(4) We shall henceforth the "-" notation and denote the unique positive solution to the ARE by $P$.

Example 1: Recall that the ARE $0 = 1 - P^2$ has two solutions $\pm 1$. The solution corresponding to the optimal control is $+ 1$.

Example 2: This example yields unbounded cost because there is an unstable mode that cannot be stabilized, but is observable in the cost. Specifically, let $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and verify that $\text{rank} \left[ \begin{array}{c} \lambda I - A \\ E \end{array} \right] = 2$, for $\lambda = 1$.

Example 3: This example fails to yield a stabilizing state feedback because the unstable mode is unobservable in the cost.

There is a vast literature on properties of the ARE and techniques for this solution. See, for example, Appendix E of Anderson and Moore.