Tracking Problems as Regulator Problems

**Review:** Consider the system

\[ \dot{x} = Ax + Bu \quad (A,B) \text{ stabilizable} \]

Since the system is time invariant, we shall henceforth choose \( t_0 = 0 \).

We have now developed a procedure to choose a stabilizing state feedback, \( u = -Kx \), for this system. Namely, minimize

\[
J(x_0,u) = \int_0^\infty (x^TQx + u^TRu)dt
\]

where \((A,Q^{1/2})\) is assumed detectable.

**Note:** If \( E = Q^{1/2} \), then we may view \( y := Ex \) as a vector of controlled outputs. (In general, the signals we wish to control in a system may differ from those signals we can measure.)

The obvious question is how to choose the weighting matrices \( Q \) and \( R \)? This is a nontrivial problem, in general requiring considerable design insight and iteration. We shall discuss many aspects of it during the remainder of the term.

One approach to weight selection: Recall that \( x_i \) and \( u_i \) each represent deviations from nominal values. These deviations are generally required to be sufficiently small that the linear system model remains valid. One approach to select initial choices for the weighting matrices is to determine the maximum allowable deviations from nominal, and define

\[
Q = \text{diag} \left[ \left( \frac{1}{x_{i,\text{max}}} \right)^2 \right] \quad \text{and} \quad R = \text{diag} \left[ \left( \frac{1}{u_{i,\text{max}}} \right)^2 \right]
\]

These weightings penalize each variable in inverse proportion to the allowable deviation from its nominal value.

The control problem discussed above is termed the optimal regulator problem, whose goal is to drive \( x(t) \to 0 \) as \( t \to \infty \). Other design goals may require that a set of selected outputs track a desired reference signal: \( y(t) \equiv y_{\text{des}}(t) \), where \( y = Cx \) is the vector of selected outputs.
(which may or may not be identical with the measured outputs). Problems such as this are termed *tracking problems*.

When $y_{des}(t) = r(t)$, a vector of step inputs, we have already seen two solutions to the tracking problem:

(i) State feedback $u = -Kx$ and a constant gain precompensator


device_diagram_1

where $N = [C(-A+BK)^{-1}B]^{-1}$

(ii) State feedback $u = -Kx - K_I w$ with augmented integrator states


device_diagram_2

In case (i), we may use the LQ regulator methodology to choose the state feedback gain, $K$, and then define the constant gain precompensator, $N$. Barring disturbances and uncertainty, perfect steady state tracking will be achieved.

In case (ii), we may use the LQ regulator methodology to choose both gains, $K$ and $K_I$. As long as the closed loop is stable, perfect steady state tracking will be achieved despite the effects of constant disturbances and small parameter variations. We have also seen how
this methodology may be modified to track other types of signals, such as sinusoids.

In both cases cited above, the solution to a regulator problem is modified, \textit{a posteriori}, to provide a solution to the tracking problem. This is the simplest approach to the problem, and is probably the one to try first. (Always try the simplest approach first...) We shall now discuss techniques for using the LQ regulator methodology to solve tracking problems wherein the tracking goal is incorporated into the problem statement \textit{a priori}.

Consider again the linear system \( \dot{x} = Ax + Bu \) \((A,B)\) stabilizable, and \( x \in \mathbb{R}^n \). Suppose we want certain linear combinations of states, \( y = Cx \), \( y \in \mathbb{R}^m \), \( m \leq n \), to track desired trajectories: \( y(t) \equiv y_{des}(t) \). (We assume that the outputs \( y \) are linearly independent, and thus that \( \text{rank}C = m \)).

We can formulate this problem as a finite horizon linear regulator problem by finding a control to minimize the cost function:

\[
J(x_0, y_{des}, u, T) = \int_0^T ((y - y_{des})^T Q_1 (y - y_{des}) + u^T Ru) dt
\]

where \( Q_1 \geq 0 \) and \( R > 0 \). Our next step is to translate the desired output trajectory into a desired state trajectory. Define

\[
L := C^T (C C^T)^{-1} \quad \text{and} \quad x_{des} := Ly_{des}.
\]

Then the above cost function is equivalent to

\[
J(x_0, x_{des}, u, T) = \int_0^T ((x - x_{des})^T Q (x - x_{des}) + u^T Ru) dt ,
\]

where \( Q := C^T Q_1 C \). Note that \( Cx_{des} = y_{des} \). It follows that if \( x \equiv x_{des} \), then \( y \equiv y_{des} \).

We now distinguish between two classes of tracking problems, depending on whether we wish to track one member of a \textit{class} of reference signals (such as steps), or a single \textit{known} signal.
Finite Horizon Servo Problem Suppose that \( y_{\text{des}}(t) \) is a member of a class of reference signals that is generated from the zero input response of a linear time invariant system:

\[
\dot{z} = Fz \quad y_{\text{des}} = Hz \quad (F,H) \text{ observable}
\]

Examples would be when \( y_{\text{des}} \) is a step or a sinusoid, although asymptotically stable trajectories may also be considered.

Form a new system by augmenting the states of the reference generator to those of the system we wish to control: \( \hat{x} = \begin{bmatrix} x \vline z \end{bmatrix} \),

\[
\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u
\]

where

\[
\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u
\]

Then, with \( L = C^T(CC^T)^{-1} \) and \( \hat{Q} = \begin{bmatrix} Q & -QLH \\ -H^T L^T Q & H^T L^T QLH \end{bmatrix} \), the cost function reduces to

\[
J(\hat{x}_0,u,T) = \int_0^T (\hat{x}^T \hat{Q}\hat{x} + u^T Ru) dt
\]

with the constraint \( \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \), \( \hat{x}(0) = \hat{x}_0 \).

Exercise: Verify that all three expressions for the cost index are equivalent.

The solution to this problem is given by

\[
u(t) = -R^{-1}\hat{B}^T \hat{P}(t)\hat{x}(t),
\]

where \( \hat{P}(t) \) is the solution to the Riccati differential equation

\[
-\dot{\hat{P}}(t) = \hat{P}(t)\hat{A} + \hat{A}^T \hat{P}(t) + \hat{Q} - \hat{P}(t)\hat{B}\hat{R}^{-1}\hat{B}^T \hat{P}(t), \quad \hat{P}(T) = 0
\]

If we partition the cost matrix as \( \hat{P}(t) = \begin{bmatrix} P(t) & P_{12}(t) \\ P_{12}^T(t) & P_{22}(t) \end{bmatrix} \), then the optimal control has the form
\[ u(t) = -K(t)x(t) - K_z(t)z(t) \]
\[ = -R^{-1}B^TP(t)x(t) - R^{-1}B^TP_{12}(t)z(t) \]

Furthermore, the various blocks of the cost matrix may be found from the differential equations

(i) \[ -\dot{P}(t) = P(t)A + A^TP(t) + Q - P(t)BR^{-1}B^TP(t) , \quad P(T) = 0 \]
(ii) \[ -\dot{P}_{12}(t) = P_{12}(t)F + A^TP_{12}(t) - QLH - P(t)BR^{-1}B^TP(t) , \quad P_{12}(T) = 0 \]
(iii) \[ -\dot{P}_{22}(t) = P_{22}(t)F + F^TP_{22}(t) + H^T L^T QLH - P_{12}(t)BR^{-1}B^TP_{12}(t) , \quad P_{22}(T) = 0 \]

**Exercise:** Verify that the Riccati equation decomposes into three subequations as stated above.

The optimal cost is

\[ J^*(x_0,z_0,T) = x^T(0)P(0)x(0) + 2x^T(0)P_{12}(0)z(0) + z^T(0)P_{22}(0)z(0). \]

**Notes:**

(1) Equation (i) is a standard Riccati differential equation that may be solved for \( P(t) \). *Note that this equation is independent of the reference signal.*

(2) Once (i) is solved, equation (ii) becomes a *linear* differential equation which may be solved for \( P_{12}(t) \).

(3) Equation (iii) only needs to be solved if we are interested in the value of the optimal cost.

(4) The optimal control has a feedback/forward structure, with feedback from the states of the plant we wish to control, and feedforward from the states of the reference system:
Finite Horizon Trajectory Following Problem

Suppose that the reference signal is an *a priori* known trajectory. Examples include problems in which an aerospace vehicle must follow a desired path over known terrain, path following robotics problems, and aerospace/automotive problems when a vehicle is equipped with "look ahead" radar or other sensors.

We shall approach this problem by *temporarily* assuming that the desired trajectory $y_{des}(t)$ is the output of a linear system

$$\dot{z} = Fz \quad y_{des} = H z,$$

and applying the results derived above. As we shall see, the optimal control and cost may be manipulated so that they depend only upon $y_{des}(t)$.

Note that the optimal control

$$u(t) = -R^{-1}B^TP(t)x(t) - R^{-1}B^TP_{12}(t)z(t)$$

depends *only* upon the product

$$\beta(t) := P_{12}(t)z(t).$$

The optimal cost also depends upon the product

$$\gamma(t) := z^T(t)P_{22}(t)z(t).$$

By differentiating $\beta(t)$ and $\gamma(t)$ it is straightforward to show that

$$-\dot{\beta}(t) = (A - BR^{-1}B^TP(t))\beta(t) - QLy_{des}(t), \quad \beta(T) = 0$$

and

$$\dot{\gamma}(t) = \beta^T(t)BR^{-1}B^T\beta(t) - y_{des}(t)^TQLy_{des}(t), \quad \gamma(T) = 0.$$
Hence, the optimal control and cost may be calculated solely from a knowledge of the desired trajectory; the augmented reference system is not needed. The system has the form:

\[ \beta(t) \xrightarrow{\Sigma} \mathbf{B} \xrightarrow{(sI-A)^{-1}} \mathbf{C} \]

Note that \( \beta(t) \) must be calculated offline.
Infinite Horizon Problems

We first consider the trajectory following problem. The optimal control is
\[ u(t) = -R^{-1}B^T P(t)x(t) - R^{-1}B^T \beta(t) \]

Recall that \( P(t) \) is the solution to a standard Riccati equation. Hence, under appropriate stabilizability/detectability assumptions,
\[ \lim_{T \to \infty} P(t) = \overline{P} \]

where \( \overline{P} \) is the unique positive semidefinite solution to the ARE. Furthermore, the eigenvalues of \( A - BK = A - BR^{-1}B^T \overline{P} \) are all stable. To calculate the optimal control, we must also find \( \overline{\beta}(t) = \lim_{T \to \infty} \beta(t) \).

Some calculations yield
\[ \overline{\beta}(t) = -\lim_{T \to \infty} \int_t^T e^{(A-BK)^T (\tau-t)} QL_y \hat{y}_{des}(\tau) d\tau \]

Since \( A-BK \) has stable eigenvalues, it follows that \( \overline{\beta}(t) \) will be bounded whenever \( y_{des}(t) \) is bounded.

Except in special cases the optimal cost will be infinite. This is because the integrand of
\[ \gamma(t) = \int_t^\infty (\beta^T (\tau)BR^{-1}B^T \beta(\tau) - y_{des}(\tau)^T L^T QL_y \hat{y}_{des}(\tau)) d\tau \]

will not, in general, converge to zero as \( \tau \to \infty \).

Example: Consider the system
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]
and the design goal \( y(t) \to y_{des} = r_0 \mathbf{1}(t) \), a vector of step functions. The corresponding cost function is
\[
J(x_0, u) = \int_0^\infty ((y - y_{des})^T Q (y - y_{des}) + u^T R u) dt, \quad Q \geq 0, \ R > 0
\]

In this case, \( \beta(t) \) is constant:

\[
\beta(t) = -\int_t^\infty e^{(A-BK)^T (\tau-t)} Q L r_0 \ dt
\]

\[
= (A-BK)^T Q L r_0
\]

\[
\Rightarrow \quad u(t) = -K x(t) - K_r r_0, \quad \text{ where } K_r := R^{-1} B^T (A-BK)^T Q L
\]

and \( \dot{x} = (A-BK)x - BK_r r_0 \)

It follows that

\[
x_{ss} = (A-BK)^{-1} BK_r r_0
\]

\[
y_{ss} = C (A-BK)^{-1} BK_r r_0 \quad (\neq r_0, \text{in general})
\]

\[
u_{ss} = -K x_{ss} - K_r r_0 \quad (\neq 0, \text{in general})
\]

The optimal cost will generally be infinite.

The block diagram of the system is as follows:

To illustrate, consider the harmonic oscillator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad x(0) = 0
\]

\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} x
\]

If we wish to track a unit step reference, then we can minimize the cost function