Internal Model Principle

If \( y_{\text{des}}(t) \) is modelled as the output of an unstable linear system then, as we have seen, it is generally impossible to achieve tracking with finite cost\(^1\). One exception occurs when the unstable dynamics of the reference model are present in the system we wish to control. This is another manifestation of the \textit{Internal Model Principle}.

Suppose that the desired trajectory has the form

\[
\dot{z} = A_{11} z \\
y_{\text{des}} = C_1 z
\]

where \( A_{11} \) has unstable eigenvalues.

Assume that there is a set of coordinates for the state space of the plant in which the state equations have the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\
0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\
 x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\
B_2 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\
 x_2 \end{bmatrix}
\]

Hence, the plant contains a copy of the unstable reference dynamics. Define an augmented state vector \( \hat{x} = \begin{bmatrix} x_1 - z \\
x_2 \end{bmatrix} \). Note that this vector depends upon the \textit{difference} between the states of the unstable reference system and the corresponding states of the plant. The augmented system and cost function have the form

\[
\begin{bmatrix}
\dot{\hat{x}}_1 \\
\dot{\hat{x}}_2
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\
0 & A_{22} \end{bmatrix} \hat{x} + \begin{bmatrix} B_1 \\
B_2 \end{bmatrix} u
\]

\[
y - y_{\text{des}} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \hat{x}
\]

\[
J(\hat{x}_0, u) = \int_{0}^{\infty} \left( (y - y_{\text{des}})^T Q_1 (y - y_{\text{des}}) + u^T R u \right) d\tau
\]

\(^1\) It may not matter that the cost is infinite -- in general the LQ cost is merely a part of the mathematics and may or may not be related to actual \textit{engineering considerations}. \]
Note that the optimal closed loop system will be stable and achieve finite cost \textit{precisely} when the system we wish to control is stabilizable and detectable. Furthermore, if $Q_1 > 0$, then $y(t) \to y_{des}(t)$, as $t \to \infty$.

Consider the block diagram of the resulting feedback system ($T$ is the change of basis matrix taking the system into the required form):

As an example, consider the system $P(s) = \frac{1}{s(s+2)}$, and suppose that we wish this system to track a step input. Consider the resulting block diagram:

It is clear how this system can simultaneously achieve zero steady state tracking of a step input \textit{and} a zero steady state control signal: the integrator in the plant assumes the desired steady state value of the output.