Integral Control via Bias Estimation

Consider the system

\[ \dot{x} = Ax + Bu + Ed, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p, \quad d \in \mathbb{R}^m \]
\[ y = Cx +Fd, \quad y \in \mathbb{R}^q \]

where \(d\) is an unknown constant vector. It is possible to view \(d\) as a step disturbance: \(d(t) = d_0 \cdot 1(t)\). (If in fact \(d(t)\) varies with time, but more slowly than the other system dynamics, then modelling the disturbance as a step may be a good approximation.)

Let us consider \(d\) as the state of a dynamical system with unknown initial conditions:

\[ \dot{d} = 0, \quad d_0 \text{ unknown} \]

(It is also possible to treat other dynamics in this way; I have asked you to do so in Problems 2 and 3 of Problem Set 3.)

Suppose that we wish to force \(y(t)\) to asymptotically track a constant reference signal despite the unknown disturbance. We have seen two approaches to this problem. In each case, we used state feedback to stabilize the system; this entails no loss of generality because we can always use an observer to reconstruct the state. (In Problem 1 of Problem Set 3, you are asked to verify that this procedure works.)

Our two approaches are, first, to use a constant gain precompensator:
If the closed loop system is stable, then we can calculate that the response to a step command \( r(t) = r_0 \) and step disturbance \( d(t) = d_0 \) satisfies \( y(t) \to y_{ss} \), where

\[
y_{ss} := C(-A + BK)^{-1} BNr_0 + [C(-A + BK)^{-1} E + F]d_0
\]

If \( C(-A + BK)^{-1} B \) is right invertible, then setting \( N = [C(-A + BK)^{-1} B]^* \) yields

\[
y_{ss} := r_0 + [C(-A + BK)^{-1} E + F]d_0
\]

The disadvantages of this scheme are that it cannot for the effects of disturbances and small parameter variations.

An alternate approach that does account for disturbances and parameter variations is to use integral control:

We have seen in class that if the closed loop system is stable, then the steady state response to a step command and a step disturbance satisfies \( y(t) \to y_{ss} \), where \( y_{ss} := r_0 \). Hence, perfect command tracking and disturbance rejection are possible; moreover, the system is insensitive to small modelling errors.

Of course, in practice we won't usually be able to measure the states of the plant. Hence, we must use an observer to estimate these states,
as depicted in the following diagram. In Problem 1 of Problem Set 3, you are asked to show that this system is stable if the state feedback system in the previous diagram is stable, and if the observer is stable.

Note that the control input in this configuration does not depend explicitly upon the command signal $r$. It is often possible to measure $r$, and hence we can consider using this measurement in a feedforward control scheme as we did in Problem 3 of Problem Set 2. The advantage of the resulting Two Degree of Freedom (2DOF) control topology is that it affords greater freedom in achieving design tradeoffs.
Alternately, let us suppose for the sake of argument that we can measure the state of the *disturbance*. We can then consider using this measurement in a feedforward/feedback control law

\[ u = -K\dot{x} - K_I w + Gd \]

as shown below:

Presumably, by using feedforward from the disturbance, we can obtain a faster response to the disturbance than otherwise. There is one problem: we will generally not be able to measure the disturbance!

Recall, however, that a step disturbance may be viewed as the output of a dynamical system consisting solely of integrators:

\[ d = 0, \quad d_0 \text{ unknown} \]
Since we need to use an observer anyway, suppose that we design the observer, if possible, to estimate the state of the disturbance as well as that of the plant, and use a control law

\[ u = -K\hat{x} - K_I w + G\hat{d} \]

Because a step disturbance may be viewed as an unknown bias, the observer in the above diagram is sometimes termed an "unknown bias estimator". A similar procedure may be used to estimate the state of other sorts of disturbances.

We shall now consider the problem of designing an observer to estimate the disturbance. Because the concepts we need for this do not depend upon the use of integral control in the above diagram, we will instead explain the procedure based upon the simpler control scheme with constant gain precompensator. Once we know how to design an observer for the disturbance, the result can be applied to the problem posed above.

The following discussion is adapted from

Consider the system depicted below:

The state equations for this configuration are:

**Plant:**
\[
\dot{x} = Ax + Bu + Ed \\
y = Cx + Fd
\]

**Observer:**
\[
\begin{bmatrix}
\dot{x} \\
\dot{d}
\end{bmatrix} =
\begin{bmatrix}
A & E \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{d}
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} u + L (y - \hat{y}) , \\
L =
\begin{bmatrix}
L_1 \\
L_2
\end{bmatrix}
\]
\[
\hat{y} =
\begin{bmatrix}
C & F
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{d}
\end{bmatrix}
\]

**Control:**
\[
u = -K\dot{x} + G\dot{d} + Nr
\]

We must answer several questions:
(1) When can a stable observer for \(x\) and \(w\) be designed?
(2) When is the closed loop system stable?
(3) Does \(y(t) \to r_0\) as desired?

We also need to find values for \(K\), \(L\), \(N\), and \(G\).
**Theorem:** Given the system described above, define

\[ P_{yu}(s) = C(sI - A)^{-1}B \quad \text{and} \quad P_{yd}(s) = C(sI - A)^{-1}E + F \]

Assume that

(i) \((A,B)\) is controllable,
(ii) \((A,C)\) is observable,
(iii) normal rank \(P_{yu}(s) = q\)
(iv) \((A,B,C)\) has no zeros at \(s = 0\)
and

(v) \((A,E)\) is controllable,
(vi) normal rank \(P_{yd}(s) = m\)
(vii) \((A,E,C,F)\) has no zeros at \(s = 0\)

Then \(K\) and \(L\) may be chosen to stabilize the resulting closed loop system, and \(N\) and \(G\) may be chosen so that the steady state response to a step command \(r(t) = r_0\mathbf{1}(t)\) and a step disturbance \(d(t) = d_0\mathbf{1}(t)\) satisfies \(y(t) \rightarrow y_{ss} = r_0\).

**Notes:**

(1) If \(A\) has no eigenvalues at \(s = 0\), then conditions (iv) and (vi) may be replaced by rank \(P_{yu}(0) = q\) and rank \(P_{yw}(0) = m\), respectively.

(2) Condition (iii) implies that there are at least as many control inputs as controlled outputs, and that these outputs are independent.

(3) Conditions (v)-(vii) essentially insure that all the disturbance states affect the steady state outputs of the system. These conditions will guarantee that we can construct an observer for the disturbance states.
Proof: First substitute the equation for \( \hat{y} \) into the observer; this yields:

\[
\begin{bmatrix}
\dot{x} \\
\dot{d}
\end{bmatrix} = \begin{bmatrix} \tilde{A} - L\tilde{C} \\ -L_2C \end{bmatrix} \begin{bmatrix} \hat{x} \\
\hat{d}
\end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + Ly
\]

where \( \tilde{A} = \begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C & F \end{bmatrix} \).

Define the estimation error states \( \Delta x := x - \hat{x} \) and \( \Delta d := d - \hat{d} \). Then the estimation error dynamics are given by:

\[
\begin{bmatrix}
\Delta\dot{x} \\
\Delta\dot{d}
\end{bmatrix} = \begin{bmatrix} A - L_1C & E - L_1F \\ -L_2C & L_2F \end{bmatrix} \begin{bmatrix} \Delta x \\
\Delta d
\end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y
\]

It follows that we can design a stable observer if the matrices \( \tilde{A},\tilde{C} \) constitute an observable pair. Using the feedback control

\[ u = -K\hat{x} + G\hat{d} + Nr \]

yields the following state space description for the closed loop system:

\[
\begin{bmatrix}
\dot{x} \\
\dot{d}
\end{bmatrix} = \begin{bmatrix} A - L_1C & E - L_1F \\ -L_2C & L_2F \end{bmatrix} \begin{bmatrix} \hat{x} \\
\hat{d}
\end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} (-K\hat{x} + G\hat{d} + Nr) + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (C\hat{x} + F\hat{d})
\]

Combining these equations yields:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}} \\
\dot{\hat{d}}
\end{bmatrix} = \begin{bmatrix} A & -BK & BG \\ L_4C & A - BK - L_4C & E + BG - L_1FL_2 \\ L_2C & -L_2C & -L_2F \end{bmatrix} \begin{bmatrix} x \\
\hat{x} \\
\hat{d}
\end{bmatrix} + \begin{bmatrix} BN \\ BN \end{bmatrix} r + \begin{bmatrix} E \\ L_1F \\ L_2F \end{bmatrix} d
\]
\[ y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ \hat{d} \end{bmatrix} + Fd \]

As usual, it is difficult to determine any closed loop properties in these coordinates. Let us instead change coordinates to \[ \begin{bmatrix} \Delta x \\ \Delta d \end{bmatrix} \]. Using the fact that \[ \Delta \dot{d} = -\dot{\hat{d}} \] (because \( d \) is a constant) yields (after some algebra) that:

\[
\begin{bmatrix} \dot{\hat{x}} \\ \dot{\Delta x} \\ \dot{\Delta d} \end{bmatrix} = \begin{bmatrix} A-BK & BK & -BG \\ 0 & A-L_1C & E-L_1F \\ 0 & -L_2C & -L_2F \end{bmatrix} \begin{bmatrix} x \\ \Delta x \\ \Delta d \end{bmatrix} + \begin{bmatrix} BN \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} d
\]

\[ y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta d \end{bmatrix} + Fd \]

It is clear from the above equations that

(a) There is a separation principle: the closed loop eigenvalues are the eigenvalues of the state feedback, \( A-BK \), plus those of the observer for \( \hat{x} \) and \( \hat{d} \). It follows that if \( (A,B) \) is controllable, and \( (\hat{A},\hat{C}) \) is observable, then we may always obtain a stable closed loop system.

(b) If the closed loop system is stable, then the steady state response to a step command \( r(t) = r_0 \mathbf{1}(t) \) and a step disturbance \( d(t) = d_0 \mathbf{1}(t) \) approaches a constant value. Hence, the derivatives of the state variables asymptotically approach zero. Setting the left hand side of the above equation equal to zero thus yields that

\[
\Delta x_{ss} = 0 \\
\Delta d_{ss} = 0
\]

and

\[
y_{ss} := C(-A+BK)^{-1}BNr_0 + \left[ C(-A+BK)^{-1}E + F + C(-A+BK)^{-1}BG \right]d_0
\]
Suppose that \( C(-A + BK)^{-1}B \) is right invertible. (Right invertibility is guaranteed by assumptions (iii) and (iv).) Then setting

\[
N = \left[ C(-A + BK)^{-1}B \right]^\#
\]

and

\[
G = -N \left[ C(-A + BK)^{-1}E + F \right]
\]
yields \( y_{ss} = 1 \).

The only thing that remains to be proven is that our assumptions guarantee that \( (\tilde{A}, \tilde{C}) \) is observable. We do this by showing that

\[
\text{rank} \begin{bmatrix} \lambda I - \tilde{A} \\ \tilde{C} \end{bmatrix} = \text{rank} \begin{bmatrix} \lambda I - A & -E \\ 0 & \lambda I \\ C & F \end{bmatrix} = n + m,
\]

where \( \lambda \) is either equal to an eigenvalue of \( A \), or \( \lambda = 0 \).

**Case 1:** Suppose that \( \lambda \neq 0 \) is an eigenvalue of \( A \). Then

\[
\text{rank} \begin{bmatrix} \lambda I - A & -E \\ 0 & \lambda I \\ C & F \end{bmatrix} = n + m
\]

if and only if this matrix has \( n + m \) linearly independent columns. Because \( \lambda \neq 0 \), it is clear that the last \( m \) columns are linearly independent among themselves and are also linearly independent from the first \( n \) columns. Hence

\[
\text{rank} \begin{bmatrix} \lambda I - A & -E \\ 0 & \lambda I \\ C & F \end{bmatrix} = n + m \iff \text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n
\]

if and only if \( \lambda \) is an observable eigenvalue of \( A \).
Case 2: Consider $\lambda = 0$. The rank test reduces to requiring that

$$\text{rank} \begin{bmatrix} -A & -E \\ C & F \end{bmatrix} = n + m$$

This condition is implied by assumptions (ii) and (v)-(vii).

To summarize:

- Choose $K$ so that $A - BK$ is stable
- Choose $L_1$ and $L_2$ so that $\begin{bmatrix} A & E \\ 0 & 0 \end{bmatrix} - [L_1 \ L_2] [C \ F]$ is stable
- Set $N = \left[ C(-A + BK)^{-1}B \right]^{\#}$ and $G = -N \left[ C(-A + BK)^{-1}E + F \right]$

Then $y(t) \rightarrow r_0$, $\forall d_0, x_0$.

It is interesting to note that this control scheme is also a form of integral control; it is just that the integrators are buried in the observer. Indeed, we have

$$\dot{d} = L_2(y - \hat{y})$$

or, in block diagram form

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\begin{align*}
\dot{y} & \rightarrow L_2 \rightarrow \dot{d} \\
y & \rightarrow \Sigma \rightarrow L_2 \rightarrow \frac{1}{s} \rightarrow \dot{d}
\end{align*}
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