A New Framework for Enhancing Accuracy for Ghost Methods in Solving FSI Problems

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Outline

1. Ghost Methods for Fluid Structure Interaction
   - General Idea of Ghost Methods
   - GFM for Fluid-Fluid Interactions
   - GFM for Fluid-Structure Interactions
   - Model Equations: 1D and 2D Wave Equations

2. A High Order Framework: Stage 1 - Surrogate Interfaces
   - Surrogate Interfaces on Cartesian Grids
   - Surrogate Interface Values

3. A High Order Framework: Stage 2 - Populating Ghost Values
   - General Idea of Operator Matching in Populating Ghost Values
   - Examples for 1D Operators
   - Examples for 2D Operators

4. Numerical Examples
   - Numerical Examples for 1D Wave Equations
   - Numerical Examples for 2D Wave Equations
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General Idea of Ghost Methods

Ghost Methods

- Ghost methods is usually used for simulating interactions between different materials.
- Computational domain for each medium is extended beyond the physical interface.
- The interface conditions are treated implicitly by applying unified solver and ghost values.
General Idea of Ghost Methods

Features

- Structured meshes can be used, because body-fitted grids are not required.
- No mesh motion throughout the simulation.
- Unified treatment for every single medium, i.e. no special treatments near interfaces.
- Ghost values play a significant role in enforcing correct interface conditions.

Limitations

- Achieving high order accuracy is not nontrivial.
- Conservation is often lost.
- Numerical methods should be explicit in time: domain of dependence is restricted to a few layers.
Ghost values for fluid-fluid interactions

- The variables for both media are the same.
- The interface moves at local flow velocity, indicating contact discontinuity at the interface.
- Continuous variables across the interface can have ghost values beginning copied from the other medium.
- Other variables have the ghost values extrapolated from real domain.

Example from Prof. T. G. Liu

- Designed in account for shocks impacting on interfaces.
- Pressure and velocity are copied.
- Entropy is extrapolated.
Ghost values of fluid for FSI

- The structure has a different set of variables from the fluid.
- Interface (normal) velocity is given by the structure surface, and passed to the fluid side through transmission conditions.
- Generally, no physical variables can be copied from the structure side as ghost values for the fluid side.

Examples

- Real interactions: usually no pressure, energy, etc., on the solid side.
- Forced motion: no status variables on the structure part at all.
GFM for Fluid-Structure Interactions

Compared with fluid-fluid interactions

- Fluid-fluid interaction:
  - Accuracy of numerical scheme may be aided by using physical variables as ghost values.
  - Second order accuracy and third order accuracy are reported in literature.

- Fluid-structure interaction:
  - Velocity at the interface comes from the structure, and it can introduce large velocity gradient for fluid.
  - No natural physical values to be used as ghost values for fluid.
  - Currently no higher than first order accuracy is reported.

A different approach of GFM is required for achieving high order accuracy for FSI.
Model Equations: 1D and 2D Wave Equations

Simplified FSI

- Interface conditions are available for all variables.
- The motion of structure is simply modeled as forced motion.
- Solving linear problem by freezing the coefficients.
- i.e. Wave equations with constant coefficients and known moving interface (boundary) conditions are used as model equations.

1D Wave Equation

The problem:

\[
\begin{align*}
u_t + au_x &= 0 & x \in [0, d(t)] \\
u(d(t), t) &= g(t) & t \in [0, T] \\
u(x, 0) &= u_0(x) & x \in [0, d(0)]
\end{align*}
\]

Well-posedness conditions:

\[
\begin{align*}
g(0) &= u_0(d(0)) \\
d'(t) &> a & t \in [0, T]
\end{align*}
\]
Model Equations: 1D and 2D Wave Equations

2D Wave Equation

- The problem:

\[
\begin{align*}
    u_t + au_x + bu_y &= 0 \\
    u(\tilde{x}(s, t), \tilde{y}(s, t), t) &= g(s, t) \\
    u(x, y, 0) &= u_0(x, y)
\end{align*}
\]

- The smoothly parameterized interfaces: \( \Gamma_t : (\tilde{x}(s, t), \tilde{y}(s, t)) \)

- Well-posedness of the problem is assumed, and there is:

\[ u_0(\tilde{x}(s, 0), \tilde{y}(s, 0)) = g(s, 0) \]

for consistency of data.
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Surrogate Interfaces on Cartesian Grids

Two stages of proposed framework:

- First stage-1: identifying the surrogate interfaces on the fixed Cartesian grid, to take place of the irregular physical interfaces.
- First stage-2: calculating surrogate interface values on the surrogate interfaces.
- Second stage: populating ghost values using surrogate interface locations and values (not the physical interface locations and values).

Ideas behind surrogate interface:

- From implementation point of view, surrogate interfaces are aligned with Cartesian grid lines, thus they ease the analysis of ghost values in second stage.
- From mathematical point of view, the following two problems are equivalent.
Surrogate Interfaces on Cartesian Grids

A general PDE problem

- The PDE is known to be valid on the domain $\Omega$:
  \[ P(u) = 0 \]
  with known boundary values $u|_\Gamma = g$, $\Gamma \subseteq \partial \Omega$.
- The solution is desired on $\Omega_1 \subset \Omega$, i.e. $u|_{\Omega_1}$ is to be solved. And $g_1 = u|_{\Gamma_1}$ can be somehow obtained, where $\Gamma_1 \subseteq \partial \Omega_1$.

Equivalent approaches

1. Solve $P(u) = 0$ on $\Omega$ with $u|_\Gamma = g$. And restrict the solution to $\Omega_1$.
2. Solve $P(u) = 0$ on $\Omega_1$ with $u|_{\Gamma_1} = g_1$. 

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Several requirements for surrogate interfaces

- Surrogate interfaces should be in the real domain, where the governing equations hold, like $\Gamma_1$.
- In order to utilize Cartesian grids, the surrogate interfaces should be in alignment with grid lines.
- In order to get accurate approximations of $g_1$, the surrogate interfaces $\Gamma_1$ should be close to the real interface $\Gamma$, where $g$ is known.
Motivation Surrogate Interfaces Operator Matching Examples

General Idea S.I.V

Surrogate Interface Values

A few remarks on $g_1 \approx u|_{\Gamma_1}$

- Ideally, $g_1 = u|_{\Gamma_1}$ is favored. But only $g = u|_{\Gamma}$ is known.
- Practically, $g_1 \approx u|_{\Gamma_1}$ to certain order of accuracy can be obtained from the exact $g = u|_{\Gamma}$.

1D wave equation

- The real interface location is $\Gamma = \{d(t)\}$, whereas the surrogate one $\Gamma_1 = \{x_I | x_I \leq d(t) < x_{I+1}\}$.
- A first order accurate $g_1$:
  
  $$g_1(t) = u(d(t), t) = g(t).$$

- A second order accurate $g_1$:
  
  $$g_1(t) = g(t) + (x_I - d(t))u_x(d(t), t) = g(t) + (x_I - d(t)) \cdot \frac{g'(t)}{d'(t) - a}$$

- Third order accurate $g_1$, etc...
Surrogate Interface Values

2D wave equation

- The real interface locations at time \( t \) are \( \Gamma = \{ (\tilde{x}(s,t), \tilde{y}(s,t)) | s \in \mathbb{R} \} \).
- Suppose a particular point on the surrogate interface is \((x_0, y_0)\), where the surrogate interface value \( g_1 \) needs to be populated.
- \((\tilde{x}_0, \tilde{y}_0)\) corresponding to \( s_0 \) is an interface point on \( \Gamma \) close to \((x_0, y_0)\).
- A first order accurate \( g_1 \):
  \[ g_1 = u(\tilde{x}_0, \tilde{y}_0, t) = g(s_0, t) \]
- A second order accurate \( g_1 \):
  \[ g_1 = g(s_0, t) + (x_0 - \tilde{x}_0) \begin{vmatrix} g_t & \tilde{y}_t - b \\ g_s & \tilde{y}_s \end{vmatrix} + (y_0 - \tilde{y}_0) \begin{vmatrix} \tilde{x}_t - a & \tilde{y}_t - b \\ \tilde{x}_s & \tilde{y}_s \end{vmatrix} \]
- Third order accurate \( g_1 \), etc...
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Two approaches in updating data near interfaces

- After building up surrogate interfaces and corresponding surrogate interface values, all the data to be operated are stored on Cartesian nodes.

- Approach 1: using ghost method, enforcing surrogate interface values through ghost values and unified numerical schemes:
  \[ u^{n+1} = \mathcal{L}^o(u^n, u^{n,G}) \]
  for all grid points in real domain.

- Approach 2: using standard methods for BVP, enforcing surrogate interface values directly, which results in different numerical operators near the interfaces.
  \[ u^{n+1} = \mathcal{L}^o(u^n) \]
  for all grid points in real domain and away from surrogate interfaces.

  \[ u^{n+1} = \mathcal{L}^b(u^n) \]
  for all grid points in real domain and close to surrogate interfaces.
General Idea of Operator Matching

Operator matching

- Approach 2 achieves desired order of accuracy of $\mathcal{L}^o$ with proper $\mathcal{L}^b$.
- Approach 1 should also achieve the desired order of accuracy for $\mathcal{L}^o$, if the ghost values $u^n,G$ are defined in a way such that for all real grid points close to the surrogate interfaces:

\[
\mathcal{L}^b(u^n) = \mathcal{L}^o(u^n, u^{n,G})
\]

Remarks

- For FDM, operator matching is straightforward.
- For FVM, interface values are applied at cell faces rather than cell centers where data are stored. And operator matching becomes a flux-matching problem, such as:

\[
\text{Find } u^{n,G}_r, \text{ such that } F(u^n_l, u^{n,G}_r) = f_{\text{sur.interf}}
\]
Examples for 1D Operators

### Notations

- $\lambda$: CFL number, defined by $a \Delta t / \Delta x$.
- $I$: Surrogate interface (which is a grid point) index.
- $u_{I+j}^{n,G}$: Ghost value for node $x_{I+j}$ in ghost domain.

### 1D Lax-Wendroff

- $L^o$:
  
  $$ u_{i}^{n+1} = u_{i}^{n} - \frac{1}{2} \lambda (u_{i+1}^{n} - u_{i-1}^{n}) + \frac{1}{2} \lambda^2 (u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}) $$

- $L^b$:
  
  $$ u_{I}^{n+1} = u_{I}^{n} - \lambda (u_{I}^{n} - u_{I-1}^{n}) $$

- After operator matching:
  
  $$ u_{I+1}^{n,G} = 2u_{I}^{n} - u_{I-1}^{n} $$
Examples for 1D Operators

1D 3rd order upwind-biased method in space and RK3 in time

- TVD RK3, suppose the generic operator for a forward Euler step is $L$:

  \[
  u^{(1)} = L(u^n) \\
  u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}L(u^{(1)}) \\
  u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}L(u^{(2)})
  \]

- $L^o$ for a single stage of RK3:

  \[
  u^{n+1}_i = u^n_i - \frac{1}{6}\lambda(-u^n_{i+2} + 6u^n_{i+1} - 3u^n_i - 2u^n_{i-1})
  \]

- $L^b$ for a single stage of RK3:

  \[
  u^{n+1}_I = u^n_I - \frac{1}{2}\lambda(3u^n_I - 4u^n_{I-1} + u^n_{I-2}) \\
  u^{n+1}_{I-1} = u^n_{I-1} - \frac{1}{2}\lambda(u^n_I - u^n_{I-1})
  \]

- After operator matching:

  \[
  u^{n,G}_{I+j} = \frac{1}{2}(j+1)(j+2)u^n_I - j(j+2)u^n_{I-1} + \frac{1}{2}j(j+1)u^n_{I-2} \quad 1 \leq j \leq 6
  \]
Examples for 2D Operators

2D Lax-Wendroff

- Let $D_0$, $D_+$, $D_-$ be the central difference operator, forward difference operator and backward difference operator respectively, then $\mathcal{L}^O$:

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{1}{2} a D_0,x u_{i,j}^n - \frac{1}{2} b D_0,y u_{i,j}^n + \frac{1}{2} a^2 D_+,x D_-,x u_{i,j}^n + \frac{1}{2} b^2 D_+,y D_-,y u_{i,j}^n + \frac{1}{4} ab D_0,x D_0,y u_{i,j}^n$$

- $\mathcal{L}^b$: complicated, and depends on local geometry of real grid points.

- Operator matching for a particular problem:

  - Type A
    $$u_{i,j}^{n,G} = 2u_{i-1,j}^n - u_{i-2,j}^n$$
  - Type B
    $$u_{i,j}^{n,G} = 2u_{i,j-1}^n - u_{i,j-2}^n$$
  - Type C
    $$u_{i,j}^{n,G} = \frac{1}{2} \left[ (2u_{i-1,j}^n - u_{i-2,j}^n) + (2u_{i,j-1}^n - u_{i,j-2}^n) \right]$$
  - Type D
    $$u_{i,j}^{n,G} = \frac{1}{2} \left[ (2u_{i-1,j}^{n,G} - u_{i-2,j}^n) + (2u_{i,j-1}^{n,G} - u_{i,j-2}^n) \right]$$
Examples for 2D Operators

2D 3rd order upwind-biased method in space and RK3 in time

- \( L^0 \) for a single stage of RK3:

\[
    u_{i,j}^{n+1} = u_i^n - \frac{1}{6} \lambda_x (-u_{i+2,j}^n + 6u_{i+1,j}^n - 3u_{i,j}^n - 2u_{i-1,j}^n) + \frac{1}{6} \lambda_y (-u_{i,j+2}^n + 6u_{i,j+1}^n - 3u_{i,j}^n - 2u_{i,j-1}^n)
\]

- Operator matching for a particular problem:

\[
    u_{i,j}^{n,G} = \frac{1}{2} \left[ (3u_{i-1,j}^n - 3u_{i-2,j}^n + u_{i-3,j}^n) + (3u_{i,j-1}^n - 3u_{i,j-2}^n + u_{i,j-3}^n) \right]
\]

\[
    u_{i,j}^{n,G} = \frac{1}{2} \left[ (3u_{i-1,j}^{n,G} - 3u_{i-2,j}^{n,G} + u_{i-3,j}^{n,G}) + (3u_{i,j-1}^{n,G} - 3u_{i,j-2}^{n,G} + u_{i,j-3}^{n,G}) \right]
\]
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Numerical Examples for 1D Wave Equations

Sample 1D wave equation with moving interface values

- Problem:
  \[u_t - 2u_x = 0\]
  \[d(t) = 1.6 - 0.1 \sin(4t)\]
  \[u(x, 0) \equiv 1.2\]

- The analytical solution:
  \[u(x, t) = \begin{cases} 
  -1.2 & x + 2t \leq 1.6 \\
  g(t') & d(t') + 2t' = x + 2t > 1.6, \ x \leq d(t) 
\end{cases}\]

Numerical schemes

- Lax-Wendroff.
- Second order upwind method in space with TVD RK2 in time.
- Third order upwind-biased method in space with TVD RK3 in time.
Numerical Examples for 1D Wave Equations

Lax-Wendroff

![Graph showing Lax-Wendroff convergence](image)
Numerical Examples for 1D Wave Equations

2nd order upwind method in space and TVD RK2 in time
Numerical Examples for 1D Wave Equations

3rd order upwind-biased method in space and TVD RK3 in time
Numerical Examples for 1D Wave Equations

Remarks

- The best performance is given by the most accurate surrogate interface values together with ghost values defined using operator matching.
- The order of accuracy is limited by the order of accuracy of surrogate interface values. i.e. if second order accuracy is desired, at least second order accurate surrogate interface values must be used.
- Desired order of accuracy is achieved for both second order schemes with errors measured by their $L_1$ norm.
- All other cases the orders of accuracy are slightly below than desired.
- The reason is simple: the analytical solution does not have continuous second order derivatives.
Sample 2D wave equation with moving interface values

- Problem:

\[ u_t - 2u_x - u_y = 0 \]
\[ u(\tilde{x}(\theta, t), \tilde{y}(\theta, t), t) = g(\theta, t) \]
\[ \tilde{x}(\theta, t) = 5 - (2 + 0.5 \cos(t)) \cos(\theta) \]
\[ \tilde{y}(\theta, t) = 5 - (2 + 0.5 \cos(t)) \sin(\theta) \]
\[ g(\theta, t) = \sin(\tilde{x}(\theta, t) + 2t) + \sin(\tilde{y}(\theta, t) + t) \]
\[ u(x, y, 0) = \sin(x) + \sin(y) \]

- The analytical solution:

\[ u(x, y, t) = \sin(x + 2t) + \sin(y + t) \]

Numerica schemes

- Lax-Wendroff.
- Third order upwind-biased method in space with TVD RK3 in time.
Numerical Examples for 2D Wave Equations

Lax-Wendroff

![Graphs showing L∞ and L1 convergence for Lax-Wendroff method](image)

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Numerical Examples for 2D Wave Equations

3rd order upwind-biased method in space and TVD RK3 in time
Remark

- The analytical solution has smooth derivatives of any order. And in both tests, the desired orders of accuracy for both measures of errors are achieved, when sufficient accurate surrogate interface values and operator matching are used.