Variance Swaps

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Outline of Presentation

Part I: Background
- Introduction to Variance Swaps
- Pricing Intuition
- The Variance Swap Market

Part II: Replication and Pricing

Part III: Variance Swap Strategies
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Introduction to Variance Swaps

• The Product
  – Offers direct and “simple” exposure to the volatility of an underlying asset
  – Liquid across major equity indices and some large-cap stocks
  – Used for various purposes:
    • Take a volatility view (Long or Short)
    • Diversify returns
    • Trade forward volatility, correlation, dispersion
  – Replication:
    • Exact replication by an infinite continuous portfolio of vanilla options
    • In practice, hedged with a “small” number of options
    • Pricing reflects volatilities across the entire skew surface
    • In practice, VSwaps trade at a slight premium to ATM implied volatilities
Introduction to Variance Swaps

- **The VSwap contract**
  - OTC product: Two parties agree to enter into a swap with maturity T
    - The buyer of the swap receives realized variance, \( \sigma^2 \), over the life of the contract at date T
    - The seller of the swap receives a fixed pre-determined strike \( K^2 \) at date T. The strike reflects market estimates of future volatility (implied volatility) at time t.

\[
\text{payoff}_{\text{long}}(T) = h(T) = N_{\text{Var}}(\sigma^2 - K^2) = \frac{N_{\text{Vega}}}{2K}(\sigma^2 - K^2)
\]

\( N_{\text{Vega}} \) represents the average profit/loss for a 1% (1 vega) change in volatility

- **Measuring realized variance and volatility**
  - Issues
  - Actual method used: “RMS” (Root-Mean-Squared) = ignore mean
    - Simplifies calculation (a little), error made not too big, mean is typically around zero

\[
\sigma^2 = \frac{252}{N} \sum_{i=1}^{N} \left[ \ln\left( \frac{S_i}{S_{i-1}} \right) \right]^2
\]

where \( S_i \) is the price of the underlying at closing and \( N \) is the number of trading days during the length of the contract.
VSwap Mark-to-Market

- Variance is additive, which simplifies the MTM, we need:
  - Realized variance since the start of the swap
  - Implied variance (new Strike) from $t$ until expiry $T$
  - Additivity equation:

\[
(T - t_0) \text{var}_{\lambda \rightarrow T}(S) = (t - t_0) \text{var}_{\lambda \rightarrow t}(S) + (T - t) \text{var}_{t \rightarrow T}(S) \Rightarrow \begin{cases}
  t_0 = 0 \\
  \text{var}_{\lambda \rightarrow T}(S) = \sigma_{0,T}^2 \\
  \text{var}_{\lambda \rightarrow t}(S) = \sigma_{0,t}^2 \\
  \text{var}_{t \rightarrow T}(S) = \sigma_{t,T}^2
  \end{cases}
\]

\[
(0, T) \Rightarrow \text{Payoff } (0, T) = N[\sigma_{0,T}^2 - K_0^2] \Rightarrow \text{PV}(t, T) = Ne^{-r(T-t)} \{[\sigma_t^2 - K_0^2] \lambda + [K_t^2 - K_0^2](1 - \lambda)\}
\]

- $t_0$ set by the market at time $t$ to make Swap Value $= 0$
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Pricing Intuition

- The strike $K$, reflects the market’s best guess of future volatility **but** with a premium (actually 2 premiums)
  - Convexity premium:
    - Variance swaps are convex in volatility: The gain from an increase in volatility is greater in absolute terms than the loss from the corresponding decrease
    - To take this into account, traders charge a premium to the ATM implied volatilities
  
- Volatility risk premium (replication premium):
  - Theoretical price calculated from prices of replicating options, so the strike $K$ can be thought of a weighted average of vanilla option implied vols. In the presence of skew and skew convexity, avg. vols will usually be above ATM vol, making the VSwap more expensive.

- **VSwaps usually trade 1-2 vegas above ATM volatility**
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The Variance Swap Market

• First mentioned in the 1990’s – took off early 2000’s
  – Mostly on index underlyings, EURO STOXX 50, S&P 500
  – Also on large-cap constituents allowing for dispersion trades

• Steady growth over past few years
  – Marketed as an alternative to options without path dependence issues and transaction costs resulting from delta-hedging

• Significant increases in liquidity
  – Variance swaps moved from exotics desks into flow-trading
  – Bid/offer spreads on indices at around 0.4 vegas, 1-2 on single-names
  – Liquid maturities ranging from 3 months to 2 years
  – VIX represents theoretical prices of VSwaps on S&P
  – Around 30% of the vega traded in the market is done so via Vswaps

• Less liquidity in other assets (bonds, fx, commodities)
  – In theory, this should change
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Delta Hedging in Black-Scholes

- P/L over small price movement, in terms of gamma

\[
[C(S + dS) − C_S(S)(S + dS)] − [C(S) − C_S(S)S] = C_S dS + \frac{1}{2} C_{SS} dS^2 − C_S dS + O(dS^3)
\]

\[\approx \frac{1}{2} C_{SS} dS^2 = \frac{1}{2} \Gamma dS^2\]

- Dollar gamma
  - Dollar gamma measure P/L in terms of return
    \[\$\Gamma = \Gamma S^2 / 100\]
  - P/L in terms of dollar gamma: connection to return
    \[\frac{P}{L} = \frac{1}{2} \Gamma dS^2 = \Gamma S^2 \left(\frac{dS}{S}\right)^2 = 50\$\Gamma R^2\]
**Delta Hedging in Black-Scholes**

- **There ain’t no such thing as a free lunch**
  - At the first glance, delta hedge always give positive P/L
  - We have only considered asset price’s movement, omitting the theta
    \[ \theta = C_t = -\frac{1}{2} \Gamma S^2 \sigma^2 \] (We assume zero risk free rate)
  - P/L with theta
    \[ \frac{P}{L} = \frac{1}{2} \Gamma dS^2 + \theta dt = \frac{1}{2} \Gamma S^2 [R^2 - \sigma^2 dt] = 50\Gamma [R^2 - \sigma^2 dt] \]

- **Realised volatility**
  - What is return: \( 1+R=(S+dS)/S=S_{dt}/S_0 \)
  - What is realised volatility: (assume \( dt \) is 1 day)
    \[ \sigma_{real}^2 = 252[\ln(S_{dt}/S_0)]^2 = 252[\ln(1+R)]^2 \approx 252R^2 \]
    \( R^2 = \sigma_{real}^2 dt \implies \frac{P}{L} = 50\Gamma dt(\sigma_{real}^2 - \sigma^2) \)
Delta Hedging v.s. Variance Swap

• Delta hedging and variance swap are similar
  – If realised volatility is higher than implied volatility, delta hedging gains.
    Otherwise, delta hedging loses money.
  – If realised volatility is higher than the strike, variance swap gains.
    Otherwise, variance swap loses money.

• Delta hedging and variance swap are different
  – Delta hedging
    • For high dollar gamma (asset price close to strike), the option has high exposure to spread
      between implied and realised volatilities.
    • For low dollar gamma (asset price far away from the strike), the option has little exposure
      to volatilities.
    • Such exposure to volatility is path dependent.
  – Variance swap
    • Whatever the price the underlying asset has, variance swap has constant exposure to the
      spread between realised and strike volatilities.
    • Such exposure to volatility is path independent
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Replicating a Variance Swap

- General idea of replicating variance swap
  - P/L of variance swap: const \times (\sigma_{real}^2 - \sigma_{Strike}^2)
  - P/L of delta hedging: variable \times (\sigma_{real}^2 - \sigma_{impl}^2)
  - Replicating: choose proper weights of options, to achieve constant gamma.

- A mathematical point of view
  - What price function has constant dollar gamma
  - Suppose the price is C(S):
    - Gamma is: \Gamma = C_{SS}
    - Dollar gamma is: $\Gamma = \Gamma S^2/100 = C_{SS} S^2/100 = \text{constant}$

- Payoff of the constant dollar gamma portfolio at T:
  - Dollar gamma is: $\Gamma = \Gamma S^2/100 = C_{SS} S^2/100 = a: a > 0$
Replicating a Variance Swap

- Constant weighted case v.s. $1/K$-weighted case

- $1/K^2$-weighted case & Aggregate dollar gamma
Replicating a Variance Swap

• Use payoffs of vanilla options to construct such payoff:

\[
\int_0^{S_0} \frac{(K - S_T)^+}{K^2} dK + \int_{S_0}^{\infty} \frac{(S_T - K)^+}{K^2} dK = \int_{S_0}^{S_T} \frac{S_T - K}{K^2} dK \\
= -\ln(S_T) - \frac{1}{S_0} S_T + \ln(S_0) - 1
\]

• Conclusion:
  – We can construct a portfolio of calls and puts initially, weighted as 1/strike-squared, and such portfolio has constant dollar gamma
  – This is a static portfolio, no dynamic trading of options is required.

• It is not commonly used in practice. Why?
Replicating a Variance Swap

• Other possibilities to construct constant dollar gamma portfolio
  – Use a single vanilla option, buy or sell over time, to keep constant dollar gamma.
    • Dynamic trading is needed
    • The position could end up with enormous amounts of the option.
  – Start with an ATM option, and on each re-hedging step, either sell or hold the previous option, and buy an amount of new ATM to achieve constant dollar gamma
    • Still, dynamic trading of options is needed
  – However, such portfolios are actually used in practice

• Drawbacks of theoretical portfolio
  – Traded strikes are not continuous
  – Lack of liquidity in OTM strikes, especially for puts

• We just use the theoretical model for a fair price
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Derive the Strike

- **Ito’s formula**
  
  For arbitrary smooth $f(S)$
  
  $$f(S_T) - f(S_0) = \int_0^T f'(S_t) dS_t + \int_0^T \frac{1}{2} S_t^2 f''(S_t) \sigma_t^2 dt$$
  
  Assume
  
  $$f(S_t) = \frac{2}{T} \left[ \ln\left(\frac{S_0}{S_t}\right) + \frac{S_t}{S_0} - 1 \right] \Rightarrow f''(S_t) = \frac{2}{T S_t^2}$$
  
  Average realised variance:
  
  $$\frac{1}{T} \int_0^T \sigma_t^2 dt = \frac{2}{T} \left[ \ln\left(\frac{S_0}{S_T}\right) + \frac{S_T}{S_0} - 1 \right] - \frac{2}{T} \int_0^T \left[ \frac{1}{S_0} - \frac{1}{S_t} \right] dS_T$$
  
  $$= \frac{2}{T} \int_0^S_0 \frac{(K - S_T)^+}{K^2} dK + \frac{2}{T} \int_0^\infty \frac{(S_T - K)^+}{K^2} dK - \frac{2}{T} \int_0^T \left[ \frac{1}{S_0} - \frac{1}{S_t} \right] dS_T$$
Derive the Strike

- Fair strike:

\[
K_{VAR}^2 = \frac{2e^{rT}}{T} \left[ \int_0^{S_0} \frac{P_0(K)}{K^2} dK + \int_{S_0}^{\infty} \frac{C_0(K)}{K^2} dK \right]
\]

- The last term in realised volatilities
  - It is equivalent as holding \(1/S_t - 1/S_0\) units of underlying asset at time \(t\).
  - Initially, no underlying asset is needed.

- Strike calculation in practice
  - Use discrete strike values

\[
K_{VAR}^2 = \frac{2e^{rT}}{T} \left[ \sum_{K_i \leq S_0} \frac{\Delta K_i}{K_i^2} P_0(K_i) + \sum_{K_i > S_0} \frac{\Delta K_i}{K_i^2} C_0(K_i) \right] - \frac{1}{T} \left( \frac{S_0}{K_0} - 1 \right)^2
\]
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A Real Life Application

- Underlying asset price: $2,935.02
- Chosen strikes: equally apart, every 5% steps
- Calculated strike for variance swap: 16.625%
- Variance notional: 10,000

Floating leg: €2,701,397.53

Fixed leg: €2,701,355.88

Present value: \( \approx 0 \)
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• Uses of Variance Swaps
• Rolling Short Variance
Uses of Variance Swaps

• Express local or macro volatility view
  – Buy Swap on Index ahead of recession, sell vol into a spike
  – Better than straddles because not affected by trending underlying, no path dependence, no active management of delta-hedging needed

• Hedging purposes
  – Volatility tends to go up in a slumping market

• Rolling short variance
  – Strategy consists of selling short-dated index variance
  – This is to try and take advantage of the volatility risk premium described earlier

• Diversification
• Index Variance spreads (Vol on S&P 500 vs. EuroStoxx)
• Single-Stock volatility pairs trading
• Correlation and dispersion trading (long index, short basket)
• And many many other….
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Rolling Short Variance

- Idea: Take advantage of premiums embedded in Vswap
  - Volatility Risk Premium and Convexity Premium
- Short N=$1 Variance Swap at start of each month
- Implied strike taken from VIX in graph below => Actual P&L will be higher
Rolling Short Variance

- Note: this example is a conservative estimate
  - Using VIX bid-side instead of actual VS Strikes
  - Not reinvesting gains / accounting losses for tax purposes
End of Presentation

• Questions?