

An adaptive $O(\log n)$ -optimal policy for the online selection of a monotone subsequence from a random sample

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Abstract

Given a sequence of n independent random variables with common continuous distribution, we propose a simple adaptive online policy that selects a monotone increasing subsequence. We show that the expected number of monotone increasing selections made by such a policy is within $O(\log n)$ of optimal. Our construction provides a direct and natural way for proving the $O(\log n)$ -optimality gap. An earlier proof of the same result made crucial use of a key inequality of Bruss and Delbaen [5] and of de-Poissonization.

KEYWORDS

adaptive policy, dynamic programming, Markov decision problem, monotone subsequence, online selection

1 | INTRODUCTION

In the problem of *online* selection of a *monotone increasing* subsequence, a decision maker observes sequentially a sequence of independent non-negative random variables X_1, X_2, \dots with common continuous distribution F and seeks to construct a monotone subsequence

$$X_{\tau_1} \leq X_{\tau_2} \leq \dots \leq X_{\tau_j} \quad (1)$$

where the indices $1 \leq \tau_1 < \tau_2 < \dots < \tau_j$ are stopping times with respect to the σ -fields $\mathcal{F}_i = \sigma\{X_1, X_2, \dots, X_i\}$, $1 \leq i < \infty$, and the trivial σ -field \mathcal{F}_0 . Since the indices are required to be possible values of stopping times, all selection/rejection decisions are terminal. That is, if the decision maker

chooses not to select the value X_i at time i , then that value is lost forever. Similarly, if X_i is selected at time i , then that selection cannot be changed in the future. In general, the stopping times can be chosen to optimize different objective functions, and two main approaches have been considered in the literature. In the first, the decision maker seeks to maximize the expected number of selected elements when n are sequentially revealed [13]. In contrast, in the second approach the decision maker's objective is to minimize the expected time it takes to construct a monotone subsequence with n elements [1]. Here, we confine our attention to the first—more classical—approach.

We then call a sequence of stopping times $1 \leq \tau_1 < \tau_2 < \dots < \tau_j \leq n$ such that (1) holds a *feasible policy*, and we denote the set of all such policies by $\Pi(n)$. For any $\pi \in \Pi(n)$, we then let $L_n(\pi)$ be the random variable that counts the number of selections made by policy π for the sample $\{X_1, X_2, \dots, X_n\}$. That is,

$$L_n(\pi) = \max\{j : X_{\tau_1} \leq X_{\tau_2} \leq \dots \leq X_{\tau_j} \text{ where } 1 \leq \tau_1 < \tau_2 < \dots < \tau_j \leq n\}.$$

Samuels and Steele [13] first studied this selection problem and found that for each $n \geq 1$ there is a unique policy $\pi_n^* \in \Pi(n)$ such that

$$\mathbb{E}[L_n(\pi_n^*)] = \sup_{\pi \in \Pi(n)} \mathbb{E}[L_n(\pi)], \quad (2)$$

and for such optimal policies one has the asymptotic estimate

$$\mathbb{E}[L_n(\pi_n^*)] \sim (2n)^{1/2} \quad \text{as } n \rightarrow \infty. \quad (3)$$

Over the last few decades the understanding of policy π_n^* has substantially evolved. For instance, by formulating (2) as a finite-horizon Markov decision problem one sees that the optimal policy π_n^* is characterized by time and state dependent acceptance intervals. Furthermore, the asymptotic estimate (3) was refined with the much tighter bounds

$$(2n)^{1/2} - O(\log n) \leq \mathbb{E}[L_n(\pi_n^*)] \leq (2n)^{1/2} \quad \text{as } n \rightarrow \infty. \quad (4)$$

The upper bound in (4) was first discovered by Bruss and Robertson [8] while studying the maximal number of elements of a random sample whose sum is less than a specified value. The analysis was instigated by the work of Coffman et al. [9], and it now represents one of the early steps into the domain of resource-dependent branching processes [7]. The result of Bruss and Robertson [8] is actually quite rich; recent extensions and applications are discussed in Steele [14]. The upper bound in (4) also appeared in Gnedin [10] who considered the sequential selection of a monotone increasing subsequence from a random sample with random size.

The $O(\log n)$ lower bound in (4) is much more recent. It first appeared in the work of Bruss and Delbaen [5] who studied the mean-optimal sequential selection of a monotone increasing subsequence when the observations X_1, X_2, \dots are revealed at the arrival epochs of a unit-rate Poisson process on $[0, n]$. While the Bruss and Delbaen [5] result provides compelling evidence that a similar bound should also hold for the discrete-time formulation we consider here—that is the formulation in which the observations are revealed at the times $1, 2, \dots, n$ —the sequential nature of the two selection processes makes the result of Bruss and Delbaen [5] not immediately applicable. The connection between the continuous-time formulation of Bruss and Delbaen [5] and the discrete-time optimization (2) was then argued by Arlotto et al. [2] who used the concavity of the map $n \mapsto \mathbb{E}[L_n(\pi_n^*)]$ and the $O(\log n)$ -bound of Bruss and Delbaen [5] to ultimately confirm the lower bound in (4).

After a careful analysis, Bruss and Delbaen [6] proved that the mean-optimal number of monotone increasing selections with Poisson-many observations is asymptotically normal after centering around $(2n)^{1/2}$ and scaling by $3^{-1/2}(2n)^{1/4}$. Arlotto et al. [2] showed that the same asymptotic limit also holds for the discrete-time problem with n observations so, in summary, we now know that

$$\frac{3^{1/2}\{L_n(\pi_n^*) - (2n)^{1/2}\}}{(2n)^{1/4}} \implies N(0, 1), \quad \text{as } n \rightarrow \infty. \quad (5)$$

However, the analyses of Bruss and Delbaen [5, 6] and Arlotto et al. [2] do not address whether there is a simple *adaptive* online policy—that is, a policy that depends on the value of the last selection and on the number of observations that are yet to be seen—that is $O(\log n)$ optimal. The works of Rhee and Talagrand [12] and Arlotto and Steele [3] tell us that the best non-adaptive policy is $O(n^{1/4})$ optimal, but this optimality gap is too crude. For instance, the expected number of monotone increasing selections made by the best non-adaptive policy cannot even be used to center the random variable $L_n(\pi_n^*)$ around $(2n)^{1/2}$ in the weak law (5).

In this paper, we construct a *simple adaptive online policy* $\hat{\pi}_n$ that is $O(\log n)$ optimal. The policy is characterized by a sequence of functions $\hat{h}_n, \hat{h}_{n-1}, \dots, \hat{h}_1$ such that if the value of the last selection up to and including time i is s and if $k = n - i$ observations remain to be seen then the value X_{i+1} is selected if and only if X_{i+1} falls in the *acceptance interval* $[s, \hat{h}_{n-i}(s)]$. In terms of the stopping times, the policy $\hat{\pi}_n$ corresponds to setting $\hat{\tau}_0 = 0$, $X_{\hat{\tau}_0} = 0$, and then defining the stopping times $\hat{\tau}_1 < \hat{\tau}_2 < \dots < \hat{\tau}_j$ recursively as

$$\hat{\tau}_j = \min\{\hat{\tau}_{j-1} < i \leq n : X_i \in [X_{\hat{\tau}_{j-1}}, \hat{h}_{n-i+1}(X_{\hat{\tau}_{j-1}})]\} \quad \text{for } 1 \leq j \leq n,$$

with the convention that if the set of indices on the right-hand side is empty, then $\hat{\tau}_j = \infty$. The random variable $L_n(\hat{\pi}_n)$ then denotes the number of monotone increasing selections made by policy $\hat{\pi}_n$, and the expected value of $L_n(\hat{\pi}_n)$ satisfies the two bounds given in the next theorem.

Theorem 1 ($O(\log n)$ -Optimal Policy) *For each $n \geq 1$, there is a simple adaptive online policy $\hat{\pi}_n$ such that*

$$(2n)^{1/2} - 2\{\log(n) + 1\} \leq \mathbb{E}[L_n(\hat{\pi}_n)] \leq \mathbb{E}[L_n(\pi_n^*)] \leq (2n)^{1/2}. \quad (6)$$

We discuss the structure of policy $\hat{\pi}_n$ and of the functions $\{\hat{h}_k : 1 \leq k < \infty\}$ in Section 2, and then we turn to the proof of Theorem 1. The upper bound in (6) immediately follows from (4) and the optimality of policy π_n^* , but our analysis requires a generalization of the upper bound (4) which we study in Section 3. The proof of the lower bound (6) then follows in Sections 4 and 5. Finally, in Section 6 we make concluding remarks and underscore some open problems.

2 | POLICY $\hat{\pi}_N$ AND ITS VALUE FUNCTION

For any feasible policy π and any continuous distribution F , we see that the number of selections made by π for the sample $\{X_1, X_2, \dots, X_n\}$ is unchanged if we replace each X_i by its monotone transformation $F^{-1}(X_i)$. Thus, we can assume without loss of generality that the X_i 's are uniformly distributed on $[0, 1]$. Next, we let

$$\hat{h}_k(s) = \min\{s + [2k^{-1}(1-s)]^{1/2}, 1\} \quad \text{for all } s \in [0, 1] \text{ and all } k \geq 1, \quad (7)$$

and we use the sequence of functions $\{\widehat{h}_k : 1 \leq k < \infty\}$ to construct appropriate acceptance intervals. Specifically, if s denotes the value of the last selection when k observations are yet to be seen and x is the k -to-last presented value, then x is selected as element of the subsequence that is under construction if and only if $x \in [s, \widehat{h}_k(s)]$.

We now define the *critical value* $s_k = \max\{1 - 2k^{-1}, 0\}$ and we note that for $s \in [0, s_k]$ the decision maker is *conservative* and selects the k -to-last value x if and only if it is within $\{2k^{-1}(1 - s)\}^{1/2}$ of the most recent selection s . On the other hand, if $s \in [s_k, 1]$ the decision maker is *greedy* and accepts any k -to-last value x that is larger than the most recent selection s .

If k denotes the number of observations that are yet to be seen and s is the value of the most recent selection, then we let $\widehat{v}_k(s)$ denote the expected number of monotone increasing selections made by the acceptance interval policy characterized by the functions $\widehat{h}_k, \widehat{h}_{k-1}, \dots, \widehat{h}_1$. The functions $\{\widehat{v}_k : 1 \leq k \leq n\}$ are the value functions associated with policy $\widehat{\pi}_n$ and they can be obtained recursively. Specifically, if $\widehat{v}_0(s) = 0$ for all $s \in [0, 1]$, then for $k \geq 1$ we have the recursion

$$\widehat{v}_k(s) = \{1 - \widehat{h}_k(s) + s\}\widehat{v}_{k-1}(s) + \int_s^{\widehat{h}_k(s)} \{1 + \widehat{v}_{k-1}(x)\} dx. \tag{8}$$

To see why this recursion holds, we condition on the k -to-last uniform random value. With probability $1 - \widehat{h}_k(s) + s$ the newly presented value x does not fall in the acceptance interval $[s, \widehat{h}_k(s)]$. In this case, no selection is made and we are left with $k - 1$ remaining observations and with the value of the most recent selection s unchanged. This amounts to an expected number of remaining selections equal to $\widehat{v}_{k-1}(s)$ and it justifies the first summand of our recursion (8). On the other hand, with probability $\widehat{h}_k(s) - s$ the newly presented value x falls in the acceptance interval, and we obtain a reward of one for selecting x plus the expected number of remaining selections over the next $k - 1$ observations when the value of the most recent selection changes to x . Integrating this over all $x \in [s, \widehat{h}_k(s)]$ gives us the second summand of the recursive equation (8). The value functions $\{\widehat{v}_k : 1 \leq k < \infty\}$ are all continuous on $[0, 1]$ and their behavior is well summarized by the following theorem.

Theorem 2 (\widehat{v}_k Bounds) *For all $k \geq 1$ and all $s \in [0, 1]$ one has that*

$$\{2k(1 - s)\}^{1/2} - 2\{\log(k) + 1\} \leq \widehat{v}_k(s) \leq \{2k(1 - s)\}^{1/2}. \tag{9}$$

Since policy $\widehat{\pi}_n$ is characterized by the thresholds $\widehat{h}_n, \widehat{h}_{n-1}, \dots, \widehat{h}_1$ and by the initial state $s = 0$, one then has the equivalence

$$\mathbb{E}[L_n(\widehat{\pi}_n)] = \widehat{v}_n(0) \quad \text{for all } n \geq 1.$$

Hence, Theorem 1 is an immediate corollary of Theorem 2, but the upper bound (9) is a refinement of (4) to an arbitrary initial state. In fact the same upper bound holds for all feasible policies based on acceptance intervals, including the optimal one.

The estimates in Theorem 2 also allow us to provide some intuition for our choice of the threshold functions $\{\widehat{h}_k : 1 \leq k < \infty\}$ in (7). For every $k \geq 1$ and $s \in [0, 1]$ the threshold functions aim to balance the expected reward to-go $\widehat{v}_{k-1}(s)$ that one obtains when skipping the k -to-last observation x , and the reward $1 + \widehat{v}_{k-1}(x)$ that one earns when selecting the k -to-last value x . Since $\widehat{v}_{k-1}(s) \approx \{2(k - 1)(1 - s)\}^{1/2}$, one can solve the equation

$$\{2(k - 1)(1 - s)\}^{1/2} = 1 + \{2(k - 1)(1 - \widehat{x})\}^{1/2} \tag{10}$$

to find the largest value of x that makes selecting the current value worthwhile. Equation (10) then tells us that

$$\widehat{x} = s + [2(k-1)^{-1}(1-s)]^{1/2} - [2(k-1)]^{-1} \approx s + [2k^{-1}(1-s)]^{1/2} - [2k]^{-1},$$

so our choice (7) accounts for the first two terms of the approximation of the solution of equation (10). The truncation in (7) then ensures that all the thresholds $\{\widehat{h}_k : 1 \leq k < \infty\}$ are feasible. At the end of the next section we use an optimization argument to provide further intuition for our choice of the threshold functions.

3 | A REFINED PROPHET UPPER BOUND

In this section, we prove the upper bound (9) by showing that it holds for all policies that are based on acceptance intervals. The adaptive policy $\widehat{\pi}_n$ and the unique optimal policy π_n^* both have this property. The argument we provide draws substantially from the earlier analyses of Gnedin [10] and Bruss and Delbaen [5], but it takes advantage of the flexibility that comes from allowing for an arbitrary initial state. Specifically, here we assume that the first subsequence element can be selected only if it is larger than an arbitrary value $s \in [0, 1]$. In contrast, in the classical formulation one always takes the initial state $s = 0$.

For any $k \geq 1$, an arbitrary acceptance interval policy π_k is given by a sequence h_k, h_{k-1}, \dots, h_1 of functions such that

$$s \leq h_j(s) \leq 1 \quad \text{for all } 1 \leq j \leq k \text{ and all } s \in [0, 1].$$

If s denotes the initial state or the value of the last observation selected prior to being presented with the j -to-last value x , then x is selected if and only if $x \in [s, h_j(s)]$. Next, for any $s \in [0, 1]$ we set $M_0 = s$ and we let M_i denote the maximum between M_0 and the largest of the elements of the subsequence that have been selected up to and including time i . The number of selections from $\{X_1, X_2, \dots, X_k\}$ when $M_0 = s$ is then given by the random variable

$$L_k(\pi_k, s) = \sum_{i=1}^k \mathbb{1}(X_i \in [M_{i-1}, h_{k-i+1}(M_{i-1})]).$$

If we now take expectations on both sides and use the Cauchy-Schwarz inequality to estimate the sum of the products $1 \cdot \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}]$ for $1 \leq i \leq k$, we obtain

$$\begin{aligned} \mathbb{E}[L_k(\pi_k, s)] &= \sum_{i=1}^k 1 \cdot \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}] \\ &\leq k^{1/2} \left\{ \sum_{i=1}^k \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}]^2 \right\}^{1/2}. \end{aligned} \quad (11)$$

The definition of M_i as the maximum between M_0 and the largest of the elements of the subsequence that have been selected up to and including time i tells us that

$$M_i = \begin{cases} M_{i-1} & \text{if } X_i \notin [M_{i-1}, h_{k-i+1}(M_{i-1})] \\ X_i & \text{if } X_i \in [M_{i-1}, h_{k-i+1}(M_{i-1})] \end{cases}$$

and, because X_i is uniformly distributed on the unit interval, we have the identity

$$\mathbb{E}[M_i - M_{i-1} | \mathcal{F}_{i-1}] = \int_{M_{i-1}}^{h_{k-i+1}(M_{i-1})} (x - M_{i-1}) dx = \frac{1}{2} (h_{k-i+1}(M_{i-1}) - M_{i-1})^2.$$

Taking the total expectation then gives

$$\mathbb{E}[(h_{k-i+1}(M_{i-1}) - M_{i-1})^2] = 2\{\mathbb{E}[M_i] - \mathbb{E}[M_{i-1}]\},$$

so a second application of the Cauchy-Schwarz inequality implies the upper bound

$$\mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}]^2 \leq 2\{\mathbb{E}[M_i] - \mathbb{E}[M_{i-1}]\}.$$

If we now sum over $1 \leq i \leq k$ and recall that $\mathbb{E}[M_k] \leq 1$ and $M_0 = s$, we obtain from telescoping that

$$\sum_{i=1}^k \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}]^2 \leq 2\{\mathbb{E}[M_k] - \mathbb{E}[M_0]\} \leq 2(1 - s). \quad (12)$$

This last inequality and the bound in (11) finally give us that

$$\mathbb{E}[L_k(\pi_k, s)] \leq \{2k(1 - s)\}^{1/2} \quad \text{for all } k \geq 1 \text{ and all } s \in [0, 1]$$

and complete the proof of the upper bound (9).

The same upper bound can also be obtained by formulating a simple optimization problem that provides further insight into our choice of the adaptive thresholds $\{\widehat{h}_k : 1 \leq k < \infty\}$. We consider the optimization problem

$$\begin{aligned} w^* &= \max_{d_1, \dots, d_k} \sum_{i=1}^k d_i \\ \text{s.t.} \quad &\sum_{i=1}^k d_i^2 \leq 2(1 - s), \end{aligned} \quad (13)$$

and we obtain from inequality (12) that $d_i = \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}]$ is a feasible solution. This feasible solution has objective

$$\sum_{i=1}^k \mathbb{E}[h_{k-i+1}(M_{i-1}) - M_{i-1}] = \mathbb{E}[L_k(\pi_k, s)],$$

so we have that the optimal objective value of (13) is an upper bound for $\mathbb{E}[L_k(\pi_k, s)]$, and that (13) is a relaxation of the online monotone subsequence problem (2).

The optimal value of (13) can be easily estimated. The problem has a linear objective function and convex constraints; by the Karush–Kuhn–Tucker (KKT) conditions, its optimal solution $(d_1^*, d_2^*, \dots, d_k^*)$ is given by

$$d_i^* = [2k^{-1}(1 - s)]^{1/2} \quad \text{for all } 1 \leq i \leq k,$$

and its optimal value is

$$w^* = \{2k(1-s)\}^{1/2} \quad \text{for all } k \geq 1 \text{ and } s \in [0, 1].$$

It then follows that the adaptive thresholds $\{\widehat{h}_k : 1 \leq k < \infty\}$ defined by (7) are *reoptimized* thresholds that use the optimal solution of the relaxation (13) for any given $k \geq 1$ and $s \in [0, 1]$. Of course, a small nuisance arises, and one needs to make sure that the reoptimized thresholds are indeed feasible. Since there are values $s \in [0, 1]$ such that $s + [2k^{-1}(1-s)]^{1/2} > 1$, one needs the truncation introduced in (7) and separately consider the conservative and greedy regimes.

4 | RESIDUAL FUNCTIONS AND RESIDUAL DIFFERENCES

The upper bound (9) tells us that $\widehat{v}_k(s) \leq \{2k(1-s)\}^{1/2}$ for all $s \in [0, 1]$ and all $1 \leq k < \infty$, so if we consider the *residual function*

$$r_k(s) = \{2k(1-s)\}^{1/2} - \widehat{v}_k(s) \quad \text{for } s \in [0, 1] \text{ and } 1 \leq k < \infty, \quad (14)$$

we can obtain a recursion for \widehat{v}_k that is equivalent to (8). Specifically, when we substitute the value function $\widehat{v}_{k-1}(y)$ with the difference $[2(k-1)(1-y)]^{1/2} - r_{k-1}(y)$ for all $y \in [0, 1]$ on the right-hand side of the recursive equation (8) and rearrange, we obtain that

$$\begin{aligned} \widehat{v}_k(s) &= [2(k-1)(1-s)]^{1/2} \\ &- \int_s^{\widehat{h}_k(s)} \{[2(k-1)(1-s)]^{1/2} - [2(k-1)(1-x)]^{1/2} - 1\} dx \\ &- \{1 - \widehat{h}_k(s) + s\} r_{k-1}(s) - \int_s^{\widehat{h}_k(s)} r_{k-1}(x) dx. \end{aligned} \quad (15)$$

Next, we define the function $\delta_k : [0, 1] \rightarrow \mathbb{R}$ as

$$\begin{aligned} \delta_k(s) &= [2k(1-s)]^{1/2} - [2(k-1)(1-s)]^{1/2} \\ &+ \int_s^{\widehat{h}_k(s)} \{[2(k-1)(1-s)]^{1/2} - [2(k-1)(1-x)]^{1/2} - 1\} dx, \end{aligned} \quad (16)$$

so when we plug-in the recursion (15) on the right-hand side of (14) and use the definition of $\delta_k(s)$, we see that the residual function (14) can be written as

$$r_k(s) = \delta_k(s) + \{1 - \widehat{h}_k(s) + s\} r_{k-1}(s) + \int_s^{\widehat{h}_k(s)} r_{k-1}(x) dx. \quad (17)$$

For each $k \geq 1$, the residual function $r_k(s)$ is continuous and defined on a compact interval, so if we maximize with respect to s we find the *maximal residual*

$$r_k = \max_{0 \leq s \leq 1} r_k(s) \quad \text{for } 1 \leq k < \infty. \quad (18)$$

If we now substitute the residual functions on the right-hand side of (17) with their maximal value, we find the inequality

$$r_k(s) \leq \delta_k(s) + r_{k-1} \quad \text{for all } s \in [0, 1] \text{ and all } 1 \leq k < \infty. \quad (19)$$

This inequality implies that the residual difference $r_k(s) - r_{k-1}$ is bounded above by $\delta_k(s)$, and we will shortly see that the behavior of the function δ_k changes in the conservative and greedy regimes. Nevertheless, the function δ_k is appropriately bounded for each $k \geq 1$.

Proposition 3 (Residual Differences Bound) *For each $k \geq 1$ one has that*

$$r_k(s) - r_{k-1} \leq \delta_k(s) \leq \frac{2}{k} \quad \text{for all } s \in [0, 1]. \quad (20)$$

The first inequality of (20) is just a rearrangement of (19), so we focus on the second inequality. Its proof is given in the next two sections. Section 4.1 studies the conservative regime, while Section 4.2 deals with the greedy regime.

4.1 | The function δ_k in the conservative regime

We begin our analysis of the function δ_k in the conservative regime by stating the following monotonicity lemma.

Lemma 4 (Monotonicity of the δ -Functions in the Conservative Regime) *For each $k \geq 3$ and $s_k = 1 - 2k^{-1}$, the function $\delta_k : [0, 1] \rightarrow \mathbb{R}$ defined by (16) is non-decreasing on $[0, s_k]$. Moreover, we have that*

$$\delta_k(s) \leq \delta_k(s_k) \leq \frac{4}{3k} \quad \text{for all } s \in [0, s_k]. \quad (21)$$

We prove the monotonicity of δ_k on $[0, s_k]$ by showing that the derivative δ'_k is non-negative on $(0, s_k)$. For $k \geq 3$ and $s \in [0, s_k]$, we see from (7) that the function $\widehat{h}_k(s) = s + [2k^{-1}(1-s)]^{1/2}$, and we obtain from (16) that δ_k is continuous and differentiable on $(0, s_k)$. Furthermore, we also have that $0 \leq [k(1-s)]^{-1} \leq 2^{-1}$, so when we differentiate both sides of (16) and rearrange, we obtain that

$$\delta'_k(s) = \left[\frac{(k-1)(1-s)}{2} \right]^{1/2} \gamma_k \left(\frac{1}{k(1-s)} \right), \quad (22)$$

where γ_k is defined for all $y \in [0, 2^{-1}]$ by

$$\gamma_k(y) = 2 + yk \left[1 - (1 - k^{-1})^{1/2} \right] - 2[1 - (2y)^{1/2}]^{1/2} - (2y)^{1/2} \{ 2 - [1 - (2y)^{1/2}]^{1/2} \}.$$

Of course, here we have that the first factor of (22) is non-negative because $k \geq 1$ and $s \in [0, 1]$, and we prove the non-negativity of the second factor by showing that γ_k is non-negative on the whole interval $[0, 2^{-1}]$. As first step we recall the inequality

$$\frac{1}{2} = k \left[1 - \left(1 - \frac{1}{2k} \right) \right] \leq k \left[1 - (1 - k^{-1})^{1/2} \right],$$

and we replace the second summand of γ_k with this last lower bound. This then implies that for all $y \in [0, 2^{-1}]$, one has that

$$2 + \frac{1}{2}y - 2[1 - (2y)^{1/2}]^{1/2} - (2y)^{1/2} \{2 - [1 - (2y)^{1/2}]^{1/2}\} \leq \gamma_k(y).$$

The next lemma confirms that the left-hand side of this last inequality is non-negative, completing the proof of the monotonicity property in Lemma 4.

Lemma 5 *If $y \in [0, 2^{-1}]$, then one has the inequality*

$$(2y)^{1/2} \{2 - [1 - (2y)^{1/2}]^{1/2}\} \leq 2 + \frac{1}{2}y - 2[1 - (2y)^{1/2}]^{1/2}. \quad (23)$$

The proof of Lemma 5 uses an infinite series representation of the two quantities that appear on each side of inequality (23). The next lemma represents an intermediate step, and it isolates an important property of the coefficients that appear in such series.

Lemma 6 (Coefficient Difference Inequality) *If*

$$a_j = \frac{(2j)!}{(2j-1)(j!)^2 2^{3j/2}},$$

then one has the inequality

$$2^{1/2}a_j - a_{j-1} \geq 0 \quad \text{for all } j \geq 3.$$

Proof By definition of a_j we have $a_j \geq 0$ for all $j \geq 1$. Moreover, we also have the recursion

$$a_j = \left(\frac{2j-3}{2^{1/2}j} \right) a_{j-1}.$$

It then follows that the difference

$$2^{1/2}a_j - a_{j-1} = \left(\frac{2j-3}{j} \right) a_{j-1} - a_{j-1} = \left(\frac{j-3}{j} \right) a_{j-1} \geq 0 \quad \text{for all } j \geq 3,$$

just as needed. ■

With the inequality of Lemma 6 at our disposal, we now carry out the proof of Lemma 5.

Proof of Lemma 5 For $j \geq 1$, we let

$$a_j = \frac{(2j)!}{(2j-1)(j!)^2 2^{3j/2}},$$

and we note that for $y \in [0, 2^{-1}]$ we have the infinite series representation

$$-[1 - (2y)^{1/2}]^{1/2} = -1 + \sum_{j=1}^{\infty} a_j y^{j/2}. \quad (24)$$

The infinite series representation (24) and the fact that $a_1 = 2^{-1/2}$ implies that the left-hand side of (23), $\ell = (2y)^{1/2} \left\{ 2 - [1 - (2y)^{1/2}]^{1/2} \right\}$, can be written as

$$\ell = (2y)^{1/2} + y + \sum_{j=2}^{\infty} 2^{1/2} a_j y^{(j+1)/2}.$$

Similarly, the infinite series representation (24) and the estimates $a_1 = 2^{-1/2}$ and $a_2 = 4^{-1}$ tell us that the right-hand side of (23), $r = 2 + 2^{-1}y - 2[1 - (2y)^{1/2}]^{1/2}$, can be expressed as

$$r = (2y)^{1/2} + y + \sum_{j=3}^{\infty} 2a_j y^{j/2}.$$

In turn, one obtains the difference

$$r - \ell = \sum_{j=3}^{\infty} 2a_j y^{j/2} - \sum_{j=2}^{\infty} 2^{1/2} a_j y^{(j+1)/2} = 2^{1/2} \sum_{j=3}^{\infty} \{2^{1/2} a_j - a_{j-1}\} y^{j/2},$$

and one has that $r - \ell \geq 0$ because of the non-negativity of the differences $\{2^{1/2} a_j - a_{j-1} : j \geq 3\}$ established in Lemma 6. ■

The upper bound (21) then follows after one notices that

$$\delta_k(s_k) = 2 \left[1 - \left(1 - \frac{1}{k} \right)^{1/2} - \frac{1}{k} \right] + \frac{4}{3k} \left(\frac{k-1}{k} \right)^{1/2}.$$

For any $y \in [0, 1]$ we know that $1 - (1-y)^{1/2} \leq y$ so we see that the first summand on the right-hand side is non-positive while the second summand is simply bounded by $4(3k)^{-1}$. This then completes the proof of the upper bound in Lemma 4.

4.2 | The function δ_k in the greedy regime

If $s \in [s_k, 1]$ for each $k \geq 1$ then the decision maker selects any feasible observation and the function $\widehat{h}_k(s) = 1$. Thus, we see from carrying the integration in (16) that δ_k can be written as

$$\delta_k(s) = (1-s)^{1/2} \{ (2k)^{1/2} - [2(k-1)]^{1/2} \} + (1-s) \left\{ \frac{1}{3} [2(k-1)(1-s)]^{1/2} - 1 \right\}. \quad (25)$$

We now recall that $s_k = \max\{0, 1 - 2k^{-1}\}$ and that for $s \in [s_k, 1]$ one has that

$$0 \leq 1 - s \leq 2k^{-1}.$$

If we now replace the two factors $(1-s)^{1/2}$ in (25) with their upper bound $(2k^{-1})^{1/2}$ and rearrange, we obtain the inequality

$$\delta_k(s) \leq 2 \left[1 - \left(1 - \frac{1}{k} \right)^{1/2} \right] + (1-s) \left[\frac{2}{3} \left(\frac{k-1}{k} \right)^{1/2} - 1 \right].$$

Again, for any $y \in [0, 1]$ one has that $1 - (1 - y)^{1/2} \leq y$ so the first summand on the right-hand side is bounded by $2k^{-1}$. Moreover, we also see that the second summand is non-positive and we finally obtain that

$$\delta_k(s) \leq \frac{2}{k} \quad \text{for all } s \in [s_k, 1].$$

This last bound and inequality (21) together complete the proof of (20).

5 | COMPLETION OF THE PROOF OF THE LOWER BOUND

Inequality (20) in Proposition 3 plays a crucial role in the proof of the lower bound (9). In fact, it suffices to recall from the definition of the maximal residual (18) that

$$r_k = \max_{0 \leq s \leq 1} \{[2k(1 - s)]^{1/2} - \widehat{v}_k(s)\}$$

and to write r_k as the telescoping sum

$$r_k = \sum_{j=1}^k \{r_j - r_{j-1}\}.$$

The bound (20) tells us that each summand on the right-hand side is bounded by $2j^{-1}$ so we obtain that

$$r_k \leq 2\{\log(k) + 1\} \quad \text{for all } k \geq 1,$$

completing the proof of Theorem 2.

6 | CONNECTIONS AND OBSERVATIONS

Policy $\widehat{\pi}_n$ is a simple adaptive online policy that selects a monotone increasing subsequence. It is then easy to see how it could be generalized to the sequential selection of a unimodal or d -modal subsequence considered by Arlotto and Steele [3]. For instance, in the unimodal case with n observations, one could set the turning time $\bar{n} = \lfloor n/2 \rfloor$ and run policy $\widehat{\pi}_{\bar{n}}$ to construct an increasing segment with the first \bar{n} observations, and a decreasing version of $\widehat{\pi}_{n-\bar{n}}$ to obtain a decreasing segment over the next $n - \bar{n}$ observations. Theorem 1 then would immediately apply to this construction, proving a $O(\log n)$ -optimality gap for the sequential selection of unimodal and d -modal subsequences.

It is also reasonable to expect that policy $\widehat{\pi}_n$ could generalize to the sequential selection of coordinatewise increasing subsequences from a uniform random sample on the m -dimensional hypercube. Thus far, the best optimality-gap estimate comes from the work of Baryshnikov and Gnedin [4] who use a non-adaptive policy to derive a $O(n^{1/(2m+2)})$ bound. The adaptive character of policy $\widehat{\pi}_n$ could be fruitful one more time and help establish a $O(\log n)$ -optimality gap for this multidimensional problem.

Many interesting questions remain open, however. Theorem 2 tells us that

$$\{2k(1 - s)\}^{1/2} - \widehat{v}_k(s) \leq 2\{\log(k) + 1\} \quad \text{for all } s \in [0, 1] \text{ and all } k \geq 1,$$

and it would be worthwhile to understand how the right-hand side changes with the initial state value s . When $s = 0$ we have from Theorem 1 that

$$\mathbb{E}[L_n(\pi_n^*)] \leq \mathbb{E}[L_n(\widehat{\pi}_n)] + 2\{\log(n) + 1\} \quad \text{for all } n \geq 1,$$

but the actual expected performance of policy $\widehat{\pi}_n$ seems to be much tighter. Based on an extensive numerical analysis,¹ we conjecture that there is a constant $0 < c < \infty$ such that

$$\mathbb{E}[L_n(\pi_n^*)] \leq \mathbb{E}[L_n(\widehat{\pi}_n)] + c \quad \text{for all } n \geq 1.$$

Of course, this would be a substantial improvement and it would also imply that the functions $\widehat{h}_n, \widehat{h}_{n-1}, \dots, \widehat{h}_1$ are remarkably close to their analog that arise when implementing the optimal dynamic programming algorithm.

Furthermore, it would also be interesting to study in greater detail the difference

$$g(n) = (2n)^{1/2} - \mathbb{E}[L_n(\pi_n^*)].$$

In the closely related problem in which the sequence X_1, X_2, \dots, X_n is given by a uniform random permutation of the integers $\{1, 2, \dots, n\}$, Peng and Steele [11] showed that the corresponding difference is $O(\log n)$ as $n \rightarrow \infty$. In our context with independent and identically distributed observations, the simplest open question is whether $g(n)$ diverges as n grows to infinity, but its answer is unlikely to be easy.

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REFERENCES

- [1] A. Arlotto, E. Mossel, and J. M. Steele, *Quickest online selection of an increasing subsequence of specified size*, *Random Struct. Algorithms* **49** (2016), no. 2, 235–252.
- [2] A. Arlotto, V. V. Nguyen, and J. M. Steele, *Optimal online selection of a monotone subsequence: a central limit theorem*, *Stochastic Process. Appl.* **125** (2015), no. 9, 3596–3622.
- [3] A. Arlotto and J. M. Steele, *Optimal sequential selection of a unimodal subsequence of a random sequence*, *Combin. Probab. Comput.* **20** (2011), no. 6, 799–814.
- [4] Y. M. Baryshnikov and A. V. Gnedin, *Sequential selection of an increasing sequence from a multidimensional random sample*, *Ann. Appl. Probab.* **10** (2000), no. 1, 258–267.
- [5] F. T. Bruss and F. Delbaen, *Optimal rules for the sequential selection of monotone subsequences of maximum expected length*, *Stochastic Process. Appl.* **96** (2001), no. 2, 313–342.
- [6] F. T. Bruss and F. Delbaen, *A central limit theorem for the optimal selection process for monotone subsequences of maximum expected length*, *Stochastic Process. Appl.* **114** (2004), no. 2, 287–311.
- [7] F. T. Bruss and M. Duerinckx, *Resource dependent branching processes and the envelope of societies*, *Ann. Appl. Probab.* **25** (2015), no. 1, 324–372.
- [8] F. T. Bruss and J. B. Robertson, “Wald’s lemma” for sums of order statistics of i.i.d. random variables, *Adv. Appl. Probab.* **23** (1991), no. 3, 612–623.
- [9] E. G. Coffman, Jr., L. Flatto, and R. R. Weber, *Optimal selection of stochastic intervals under a sum constraint*, *Adv. Appl. Probab.* **19** (1987), no. 2, 454–473.
- [10] A. V. Gnedin, *Sequential selection of an increasing subsequence from a sample of random size*, *J. Appl. Probab.* **36** (1999), no. 4, 1074–1085.

¹We estimated numerically the value functions that solve the recursion (8) and its optimal counterpart on a discretized state space with a grid size of 10^{-5} and with k ranging from 1 to 10^4

- [11] P. Peng and J. M. Steele, *Sequential selection of a monotone subsequence from a random permutation*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4973–4982.
- [12] W. Rhee and M. Talagrand, *A note on the selection of random variables under a sum constraint*, J. Appl. Probab. **28** (1991), no. 4, 919–923.
- [13] S. M. Samuels and J. M. Steele, *Optimal sequential selection of a monotone sequence from a random sample*, Ann. Probab. **9** (1981), no. 6, 937–947.
- [14] J. M. Steele, *The Bruss-Robertson inequality: elaborations, extensions, and applications*, Math. Appl. **44** (2016), no. 1, 3–16.

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