Strategic Open Routing in Service Networks

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We study the behavior of strategic customers in an open-routing service network with multiple stations. When a customer enters the network, she is free to choose the sequence of stations that she visits, with the objective of minimizing her expected total system time. We propose a two-station game with all customers present at the start of service and deterministic service times, and we find that strategic customers "herd," i.e., in equilibrium all customers choose the same route. For unobservable systems, we prove that the game is supermodular, and we then identify a broad class of learning rules—which includes both fictitious play and Cournot best-response—that converges to herding in finite time. By combining different theoretical and numerical analyses, we find that the herding behavior is prevalent in many other congested open-routing service networks, including those with arrivals over time, those with stochastic service times, and those with more than two stations. We also find that the system under herding performs very close to the first-best outcome in terms of cumulative system time.

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1. Introduction

In many large entertainment or commercial service environments, various services are provided at different stations, and individuals are often free to strategically choose their route through the system. For example, consider a catered reception with a buffet line in the main dining room and a beverage cart in the adjacent hall. A guest wants to visit both stations, but she may choose to do so in either order. The guest wishes to select the sequence that minimizes the time that she spends waiting in queues, but given that other guests are probably also attempting to do the same, which sequence should she choose?

This example highlights the strategic behavior of rational customers in service networks with *open routing*, i.e., those in which customers can visit the stations in any order. The dynamics of service networks with open routing are more complex than those of systems in which customers merely choose a server, such as a grocery store where customers choose a checkout lane. In that example, once a customer has chosen her preferred lane, she is not impacted by the service process of any lane but her own. However, in a network with open routing, customers who choose different routes can still end up at the same station at the same time, directly impacting each other's waiting times. For instance, a customer who finishes service at station A and joins the queue at station B may encounter another customer there who chose to visit station B first and will visit station A later.

Open routing characterizes a variety of other service environments in addition to the aforementioned catering example. For instance, Baron et al. (2016) study an outpatient medical clinic in which patients receive a battery of tests over several hours. Tests may be administered in any order, and they numerically study the problem of a central planner who seeks to maximize patient satisfaction. Open-routing service systems also include amusement parks (where customers choose the order in which they visit the attractions), shopping centers (where customers choose the order in which they visit the stores), and college orientations (where students must accomplish multiple tasks, such as a campus tour and a residence hall visit, in any order).

Motivated by these applications, we study how strategic customers choose their routes in a stylized open-routing service environment. In order to achieve a broad understanding of the role of open routing, we characterize analytically the behavior of strategic customers in a two-station network under some restrictive assumptions, and employ numerical simulation to relax some of these assumptions and show that the same insights also hold in more realistic settings.

In the base model, we study a two-station network in which customers require services from both stations. Each station serves its customers on a first-come, first-serve (FCFS) basis. To model strategic open routing, we assume that each customer chooses her own route to acquire service from both stations in the network, with the objective of minimizing her expected total time in the system, and that all customers are present prior to the start of service. Intuitively, one might expect that a rational customer would attempt to avoid the popular route in order to receive service as quickly as possible. However, closer inspection reveals that a rational customer's decision is driven instead by the need to get into a favorable position at the more congested station. This leads to a surprising *herding* behavior: in any pure-strategy Nash equilibrium, all customers choose exactly the same route. This behavior is related to the notion of "follow the crowd" as discussed in Hassin and Haviv (2003) and Hassin (2016). After proving that any pure-strategy Nash equilibrium must involve herding, we then demonstrate that any mixed-strategy Nash equilibrium must be unstable. Moreover, we show that a large and intuitive class of adaptive learning dynamics—which includes both fictitious play and Cournot best-response—converges to herding at one of the two stations. The herding equilibria therefore emerge as the focal equilibria of the routing game.

While our base model does not directly solve any of the motivating problems discussed above, it uncovers the herding phenomenon in an open-routing service network, which bears further investigation. To verify the robustness of the herding behavior, we analyze strategic open routing in several different settings and its impact on social welfare. Our base two-station model in which all customers are strategic and present prior to the start of service is relaxed in several directions. Specifically, we relax our modeling assumptions by studying games with non-strategic customers, customers who visit only one of the two stations, and service networks with more than two stations. In all of these settings, we find that strategic customers herd. The sequential version of the original game also displays herding, with the minor twist that the last-moving player alone avoids the crowd. Additionally, we run a simulation study to test the robustness of herding. We find that, in the presence of congestion, herding continues to prevail in systems with stochastic service times and arrivals over time, with herding becoming less prominent as the arrival rate decreases and the systems become less congested. We also find that the social welfare under herding—as measured by the sum of the system times experienced by all customers in the network—is within a constant of the first-best outcome, and such constant is independent of the number of players. For systems with many customers, this result implies that the welfare under herding differs only by a small fraction from the optimal social welfare.

The outline for the rest of the paper is as follows. We review the related literature in Section 2, and we introduce our base model in Section 3. Section 4 discusses the *open routing game*, in which all customers are present in the system when service becomes available and make routing decisions about which station to visit first before learning their relative priority or the decisions of others. In Section 5, we derive the unique subgame perfect equilibrium of the sequential variant of the open routing game; in that setting, customers are aware of their priority and of the routing decisions of those with higher priorities. Section 6 studies the system from the perspective of a central planner attempting to optimize the routing assignments of customers. Section 7 relaxes some assumptions of the open routing game to allow for customers who wish to visit only one of the two stations and systems with more than two stations. Section 8 studies systems which do not experience congestion. Section 9 analyzes the output from a simulation study designed to test the robustness of herding in networks with stochastic service times and arrivals over time. Finally, Section 10 makes concluding remarks. In addition, all proofs for results presented in the paper are provided in Appendices A-D.

2. Literature Review

Our work is related to the literature on queues with strategic customers as well as the literature on congestion games. The first stream was started by Naor (1969), who studied the use of tolls to induce desired behavior among customers in a queue. The second was begun by Braess (1968), in his pioneering work demonstrating the now-famous Braess' Paradox.

Hassin and Haviv (2003) provide a comprehensive survey of the existing strategic queueing literature, and Hassin (2016) provides an extremely thorough coverage of the more recent work in the area. Our model is perhaps most closely related to the work of Parlaktürk and Kumar (2004). Theirs is one of only a few papers incorporating a stochastic network with stations in sequence in which customers choose the order of stations that they visit. The authors demonstrate the existence of unstable Nash equilibria for a two-station network in which every self-interested "job" must have two tasks performed on it, where each station can perform either task on a given job, and the first task takes on average a shorter time. Each station has a queue for Task 1 and another queue for Task 2, and the system planner may choose which queue to serve next at each station. By contrast, in our model each station can perform only one task and each station has only one queue; hence stations cannot dynamically distinguish between different classes of customers.

Other work in the vicinity includes that of Adiri and Yechiali (1974), who relay a model of multiple queues for a single server, arranged and priced by priority. Their results mirror ours in that arriving customers choosing a queue must take into account the possibility of being preempted (cut in front of) by an arriving customer who chooses a higher priority queue. Naturally, an important distinction is that in our model there are two queues for two servers, as opposed to two or more queues for one server in their work. Most recently, Honnappa and Jain (2015) study what they call the "network concert queueing game." For feedforward networks of several structures, they use fluid limits to determine symmetric equilibria and the price of anarchy when nonatomic users are allowed to choose both their arrival times and their routes through the network. Cohen and Kelly (1990) present an interesting analog of Braess' Paradox in a stochastic queueing network. They illustrate that when FCFS nodes are placed in sequence with infinite-server nodes, the mean sojourn time in equilibrium can actually increase if customers are given the ability to switch from one track to the other. The analysis is done in steady state and queue lengths are assumed to be unobservable, simplifying both the queueing and game theoretic portions of the analysis. Other important work in this stream includes Enders et al. (2008) and Glazer and Hassin (1983).

On the empirical side, Pinilla and Prinz (2003) conduct a helpful, mainly simulation-based study of flexible routing schemes for shape deposition manufacturing. They study the standard sequential model and employ simulation to gain insights on a system which allows flexibility. They propose two examples—shape deposition manufacturing and routing in a coffee shop—which can be cast within the open routing framework, and they find from simulation that performance can be significantly improved by dynamically assigning the sequence of tasks to be performed on each item, instead of following a fixed sequence of tasks.

Su and Zenios (2004) model the U.S. kidney allocation system as an $M/M/1$ queueing system in which potential recipients monitor both their position in the queue and the quality of the organ offered to them. They find that the current benchmark of FCFS service results in socially suboptimal allocation of organs because strategic recipients tend to refuse lower quality organs, knowing that they will likely be offered a better organ later. Modifying the queueing discipline to last-come, first-serve leads to the socially optimal outcome, although the authors note that such a discipline will likely be considered too unfair to implement in practice. Schummer (2016) and Leshno (2017) also study allocation of objects to strategic customers on queue-like waiting lists.

Finally, Veeraraghavan and Debo (2009, 2011) investigate competition between two service providers where queues build up and customers have private information regarding the quality of each provider, and they find herding behavior similar to what we discover here. However, the driving force behind their result is quite different from what we observe. In Veeraraghavan and Debo (2009, 2011), customers are motivated by service quality, while in our model the herding behavior occurs because customers require services at both stations and attempt to minimize their expected total time in the system. In our setting, although by starting with the less-crowded station a customer may glean a shorter wait time before beginning her first service, she will afterwards face a severe penalty upon arriving to the congested station and being near the end of the queue there. Additional related work includes Afèche and Mendelson (2004), Debo and Veeraraghavan (2014), and Cui and Veeraraghavan (2016).

Several papers in the congestion literature also warrant discussion. Feldman and Tamir (2012) also consider "jobs" (customers) to be strategic. Their paper focuses mainly on "conflicting congestion effects," both positive and negative, associated with the level of participation or traffic on a network. In their work, customers are allowed to choose from a set of machines working in parallel. They show that best-response dynamics do not always converge to a Nash equilibrium, but that the schedule generated by the longest processing time heuristic is indeed a Nash equilibrium if the number of machines is "right." An important difference from our analysis is that they do not incorporate the ordering of customers on a machine, instead modeling each player's cost function as merely the overall load of the machine chosen by that player. In the standard congestion model

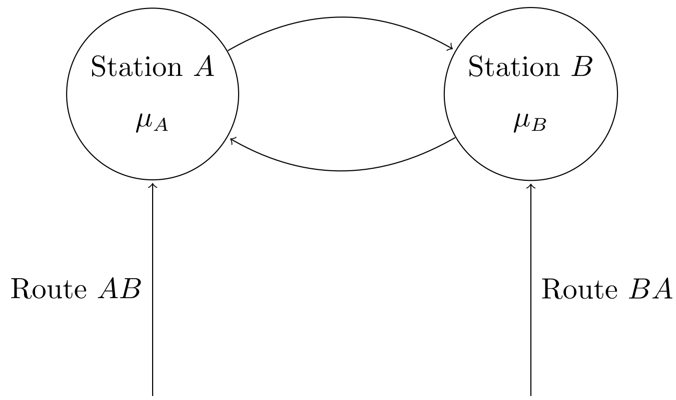


Figure 1 AB **customers visit station** A **first, while** BA **customers visit station** B **first.**

of which Braess' Paradox comprises a special case, Roughgarden and Tardos (2002) show that the price of anarchy is at most 4/3 when the performance measure is the total latency of the system and the latency functions are linear.

3. A Two-Station Service Network

Our setting is that of a service network with two stations, station A and station B, each with a single server, and with service rates μ_A and μ_B, respectively. The network serves N customers (or "players") who are all present in the system when service becomes available. We focus on the case in which the service rates are nonidentical, and thus without loss of generality we assume that the expected service time at station A is greater than that at station B (i.e., we have the service rate relation $\mu_A < \mu_B$). Similar to many service environments, each station operates on a FCFS basis. Figure 1 gives a visual depiction of the network.

Every customer must visit each station exactly once, but the order in which to visit the stations is unrestricted. Customers seek to minimize their expected total time in the system; their action space is the set $\{AB, BA\}$, where the first letter denotes the station visited first. We will at times refer to AB customers to identify those who visit station A first. Similarly, BA customers are those who visit station B first.

We note that the centralized (non-strategic) version of the problem can be viewed as an *open-shop* scheduling problem with jobs that need to be processed by two servers, where either sequence is permissible for each job. A concise summary of the available results on open-shop scheduling can be found in Pinedo (2012, Chapter 8 and Section 13.4). For deterministic systems, there are polynomial-time algorithms available for makespan, but the problem of minimizing total completion time is NP-hard. For stochastic systems, the optimal policy for makespan is the longest expected remaining processing time first rule, and when the metric is expected total completion time, the preemptive shortest expected remaining processing time first policy is optimal among preemptive dynamic policies.

4. The Open Routing Game

In this section, we study the *open routing game*, in which all customers are present in the system when service becomes available and positions in the priority order are drawn uniformly at random. Players are aware that this randomization takes place, but they must make their routing decisions before learning their realized priorities or the routing decisions made by others. The open routing game is therefore equivalent to a symmetric one-shot game in which all players make routing decisions simultaneously. Players seek to minimize their expected total time spent in the system. Service times at stations A and B are first assumed to be deterministic with rates μ_A and μ_B, respectively, such that $\mu_A < \mu_B$. In Section 4.5, we relax the deterministic service assumption to incorporate stochastic service times. These dynamics resemble service environments in which a large number of customers are present before the service starts.

We first make two observations that help us understand the system time experienced by customers choosing each of the two routes through the network.

Property 1. If station B ever becomes idle, then it will never build up a queue again.

To understand Property 1, we note that, because $\mu_A < \mu_B$, the service time at station B is shorter than that at station A. Hence, after the first service begins, the arrivals to station B occur deterministically with an arrival rate that is smaller than station B's service rate. Therefore, once station B becomes idle, arrivals will never occur close enough together to form a queue. So, the system time for a customer who chooses route AB, and who has priority j at station A, depends on whether or not station B becomes idle before the customer departs station A. If station B idles, then the customer's system time is the sum of j service times at station A plus her own service time at station B. If station B does not idle, then the customer's system time is the sum of all of the service times at station B that must be completed up to and including herself. This will include all of the BA players, plus the $j-1$ players in front of her at station A, as well as herself.

Property 2. Station A never idles from the time it begins its first service until it finishes serving all N customers.

Similarly to Property 1, Property 2 follows from the fact that, because $\mu_A < \mu_B$, the service time at station B is shorter than that at station A. As soon as station A begins service, it will complete a service every $1/\mu_A$ units of time, but it will receive an arrival every $1/\mu_B < 1/\mu_A$ units of time until the last BA player departs station B. Station A will then never become idle until it finishes its workload. The one way in which station A can idle is at the beginning, if all N players choose route BA: in that case, station A will idle during the first service time at station B, after which it will work continually until it finishes with all N customers.

One might expect that players minimizing their system times would attempt to avoid each other and seek a less congested route. Instead, when the number of players N is large enough, we find that in equilibrium players "herd"—that is, all players take the same route through the network.

PROPOSITION 1 **(Herding Equilibria in the Open Routing Game)**. *For $N \geq 2\mu_A/\mu_B + 1$, the open routing game has a Nash equilibrium in which all players "herd" at station A, that is, take route AB. Furthermore, if $\mu_B < 2\mu_A$ and $N \geq \max\{\mu_B/\mu_A + 1, (2\mu_A + \mu_B)/(2\mu_A - \mu_B)\}$, then the game also has a Nash equilibrium in which all players "herd" at station B.*

We next give an intuitive explanation of Proposition 1. In the first candidate profile, all players visit station A first, so a player will have on average half of the other customers behind her if she visits station A first. However, if she visits station B first while everyone else visits station A first, then she will be the last customer to receive service at station A. Therefore, if N is large enough—in fact, $N \geq 3$ is sufficient here—then it is in her best interest not to deviate from the candidate profile. The herding equilibrium at station B can be similarly explained; intuitively, if N is large and the rest of the customers are slated to visit station B first, then a customer is better off being in front of an average of half of the other customers at station B. Otherwise, after visiting station A first she will certainly have to wait behind all of the other players at station B.

Proposition 1 establishes the existence of Nash equilibria which exhibit herding behavior. However, we must answer several questions to verify that these herding equilibria are indeed plausible: (i) can the herding equilibria be implemented via simple, decentralized learning dynamics? (ii) are there other, non-herding Nash equilibria? and (iii) what happens when some of the players are not strategic? To address these questions, we establish a key submodularity property for the players' expected system times in the open routing game. This property then allows us to pinpoint the herding profiles as the focal equilibria of the game.

4.1. Submodularity of Expected System Time

Before deriving the submodularity property of the expected system time, we require some additional notation. Let i be an arbitrary player (or customer) index, s_i the strategy of player i, and $\mathbf{s}_{-i}$ the vector of strategies for all of the other players. We say that $s_i = 1$ if player i chooses route AB, and $s_i = 0$ if she chooses route BA. Similarly, a value of 1 for a given entry in $\mathbf{s}_{-i}$ means that the corresponding player has chosen route AB, while a value of 0 for a given entry in $\mathbf{s}_{-i}$ means that the corresponding player has chosen route BA. Denote by $T(s_i, \mathbf{s}_{-i})$ the expected system time for a player who employs strategy s_i when her opponents play the profile $\mathbf{s}_{-i}$. Note that the uniform priority randomization means that the game is symmetric, and thus we require no player index on T. Following the definition of Topkis (1998) and letting $\leq$ denote the usual partial order, we will show that the function $T(s_i, \mathbf{s}_{-i})$ is *submodular*, i.e., that it has *decreasing differences*. Specifically, we will find that

$$T(s_i, \mathbf{s}_{-i}) - T(\widetilde{s}_i, \mathbf{s}_{-i}) \leq T(s_i, \widetilde{\mathbf{s}}_{-i}) - T(\widetilde{s}_i, \widetilde{\mathbf{s}}_{-i}) \quad \text{for all } \widetilde{s}_i \leq s_i \text{ and } \widetilde{\mathbf{s}}_{-i} \leq \mathbf{s}_{-i}. \tag{1}$$

The decreasing differences condition (1) trivially holds if $\widetilde{s}_i = s_i$, so we can focus on the case in which $\widetilde{s}_i = 0$ and $s_i = 1$. Moreover, because all customers are present when service starts and priorities are drawn uniformly at random, we can replace $\mathbf{s}_{-i}$ with the sum of its entries, m, and $T(s_i, \mathbf{s}_{-i})$ with $T(s_i, m)$. The variable m then simply represents the number of players—excluding player i—who have chosen route AB. With a slight abuse of notation, we can replace the decreasing differences condition (1) with the equivalent condition

$$T(1, m) - T(0, m) \leq T(1, \widetilde{m}) - T(0, \widetilde{m}) \quad \text{for all } 0 \leq \widetilde{m} \leq m \leq N - 1. \tag{2}$$

We now introduce the shorthand

$$d_m := T(1, m) - T(0, m)$$

and rewrite condition (2) as

$$d_m \leq d_{\widetilde{m}} \quad \text{for all } 0 \leq \widetilde{m} \leq m \leq N - 1. \tag{3}$$

The difference d_m represents the relative preference of player i between route AB and route BA, given that m other players chose route AB; a negative value indicates that route AB will result in a shorter expected system time, and a positive value means that route BA will yield a shorter expected system time. Similarly, a relatively smaller value of d_m indicates a relatively greater preference for route AB. Therefore, if equation (3) holds—and we will show that it does—then the greater the number of other customers who have chosen route AB, the greater relative preference each customer will have for route AB.

Proposition 2 **(Submodularity of Expected System Time)**. *If*

$$N > N_{\text{sub}} := \frac{2\mu_B}{\mu_A} + 1,$$

then each player's expected system time in the open routing game is submodular. Moreover, we have

$$d_m < d_{m-1} \quad \text{for all } 1 \leq m < N - \frac{\mu_B}{\mu_A}. \tag{4}$$

Observe that $d_{\widetilde{m}} < 0$ implies that, if a player were aware that exactly $\widetilde{m}$ others were choosing route AB, then her expected system time for route AB would be less than that for route BA, and thus she would prefer route AB. Furthermore, the decreasing differences property gives us that if $d_{\widetilde{m}} < 0$, then $d_m < 0$ also for $\widetilde{m} \leq m$, and therefore for any $\widetilde{m} \leq m$, a player who knew that m others had chosen route AB would also want to choose route AB. Intuitively, a critical mass of players choosing a given route tends to attract the remaining players to the same route; if many players go to station A first, then the others should join them, and similarly for station B. We will revisit this idea in the next subsection when we discuss convergence of adaptive dynamics to the herding equilibria. The strict inequality in equation (4) of Proposition 2 plays a key role in establishing this convergence.

Proposition 2 implies that if, as we henceforth assume, players' utility functions decrease linearly with their expected system times, then their utility functions are supermodular, and the open routing game is a *supermodular game* in the sense of Topkis (1998, Section 4.1). Supermodular games have received much attention in the literature. For example, it is well documented that if a supermodular game has a unique Nash equilibrium, then a wide range of learning rules will converge to it (see Milgrom and Roberts 1990). However, because the game that we study has multiple equilibria, we cannot directly apply the classical convergence result. Nevertheless, in the subsequent subsection, we apply supermodularity and the strict inequalities in equation (4) to prove that in our game, a large class of learning rules converges to one of the herding equilibria. Next, we state a corollary that follows directly from the proof of Proposition 2, which will also be used in the derivation of the subsequent convergence result.

Corollary 1. *If $N > N_{\text{sub}}$, then for any $m = 1, \ldots, N-1$, either $d_m < 0$ or $d_m < d_{m-1}$.*

4.2. Adaptive Dynamics Converge to Herding

As we have seen, when the service rates are close together and N is large, the herding strategy profiles are Nash equilibria. We next show that in addition, a general class of decentralized learning processes will converge to one of these herding equilibria.

We propose a model of learning which allows customers to update their beliefs in each period based on the play observed and also incorporates a "memory" of past actions. First, we assume that in each period players choose their routes to minimize their expected system times based on their current beliefs about other customers' strategies. When a player faces a tie, we assume that the player always chooses route AB. Let $\boldsymbol{\beta}_i^{(t)} = (\beta_{i,0}^{(t)}, \ldots, \beta_{i,N-1}^{(t)})$ be the vector of player i's beliefs, i.e., $\beta_{i,j}^{(t)}$ is player i's probability assessment, at the beginning of period t, that exactly j

players (not including herself) will take route AB in period t, for $j = 0, 1, \ldots, N-1$. We allow the initial beliefs, $\boldsymbol{\beta}_i^{(1)}$, to be an arbitrary vector of probabilities summing to one. Next, let $x^{(t)}$ be the realized total number of players who take route AB in period t, and let $x_i^{(t)}$ be the decision of player i (with $x_i^{(t)} = 1$ for choosing route AB and 0 otherwise). We also let $x_{-i}^{(t)}$ denote the realized number of players who take route AB in period t, excluding player i; that is, $x_{-i}^{(t)} = x^{(t)} - x_i^{(t)}$. Player i is making a decision in period t based on $\boldsymbol{\beta}_i^{(t)}$, her belief at the beginning of period t, which incorporates her experience up to and including period $t-1$.

For $m \in \{0, 1, \ldots, N-1\}$, let $\boldsymbol{e}(m) \in \mathbb{R}^N$ be the vector with a one in the $(m+1)$-st entry, and zeros in all of the remaining entries. Given a sequence of real numbers $\{\alpha_t\}$ with $0 \leq \alpha_t \leq 1$ for all $t = 1, 2, \ldots$, the beliefs in our model satisfy the recursion

$$\boldsymbol{\beta}_i^{(t+1)} = (1-\alpha_t)\boldsymbol{\beta}_i^{(t)} + \alpha_t \boldsymbol{e}(x_{-i}^{(t)}) \quad t = 1, 2, \ldots. \tag{5}$$

We give the name $\{\alpha_t\}$-*learning* to the process in which beliefs evolve according to equation (5). Intuitively, a larger α_t implies that the players are giving more weight to their experience in period t, and less weight to earlier periods. We also note that choosing $\alpha_t = 1/t$ results in the familiar learning rule known as fictitious play introduced in Brown (1951). Within fictitious play learning, players best-respond to the empirical frequency of past moves. Similarly, by letting $\alpha_t = 1$ we recover the Cournot best-response model, in which players best-respond to the path of play realized in the prior period. The interested reader is referred to Fudenberg and Levine (1998) for detailed discussion of the Cournot model (Chapter 1), fictitious play (Chapter 2), and other learning models.

Next we show that under a mild regularity condition on the sequence $\{\alpha_t\}$ and on the initial beliefs $\boldsymbol{\beta}_i^{(1)}$, if all players update their beliefs according to the learning model (5), then play will converge to one of the herding equilibria.

Proposition 3 **(Convergence of $\{\boldsymbol{\alpha_t}\}$-Learning to Herding Equilibria)**. *Define m^* by*

$$m^* = \min\{m \in \{1, \ldots, N\} : d_m \leq 0\},$$

and consider the $\{\alpha_t\}$-learning process in equation (5). *If* (i) $N > N_{\text{sub}}$, (ii) *the sequence $\{\alpha_t\}$ is such that*

$$\lim_{\bar{t} \to \infty} \prod_{t=t'}^{\bar{t}} (1-\alpha_t) = 0 \qquad \text{for all } t' \geq 1, \tag{6}$$

and (iii) *there exists some $\ell \geq 1$ such that $x^{(\ell)} \neq m^*$, then players will converge to one of the herding equilibria in finitely many periods. That is, there exists $t_0 < \infty$ such that either*

$$x^{(\bar{t}+1)} = N \quad \text{or} \quad x^{(\bar{t}+1)} = 0 \qquad \text{for all } \bar{t} \geq t_0.$$

In Proposition 3, condition (6) essentially enforces that players' earlier beliefs and actions must eventually fade so that play does not get stuck on strategy profiles that are anchored to initial beliefs. For example, natural learning rules such as fictitious play and Cournot best-response satisfy this condition. This result reinforces the intuition that many players choosing a given route exerts a pull on others to do the same. The only circumstance which can possibly avoid herding is that in which $x^{(t)} = m^*$ for all $t = 1, 2, \ldots$. For this stagnation to occur, we must have $x^{(1)} = m^*$, an unlikely event if N is large and, say, initial beliefs are drawn as independent random vectors uniformly distributed on the N-dimensional probability simplex. Even if this first-period event is realized, route switching in later periods is inevitable because the beliefs of AB customers will be moving to favor route BA, and those of BA customers will be moving to favor route AB. When this route switching occurs, it must always be perfectly symmetric to maintain $x^{(t)} = m^*$, which is also unlikely.

The proof of Proposition 3 leverages the submodularity of the expected system time established by Proposition 2. Intuitively, if the number of players taking route AB is strictly greater than m^*, then the players currently taking route AB have no incentive to switch, in this period or in any later period, because the beliefs which led them to select route AB in the current period will be further reinforced. Then, given that there will always be more than m^* players on route AB in the future periods, the players who chose route BA will eventually switch to route AB, provided that they sufficiently update their beliefs, as in condition (6). Similarly, if the number of players taking route AB is strictly less than m^* in some period, one can deduce that eventually all players will converge to route BA. This result reiterates the herding phenomenon: many players on a single route attract the others to join them.

Next, we state a corollary which shows that for Cournot best-response, the requirement of $x^{(\ell)} \neq m^*$ is not required when N is large enough.

Corollary 2 **(Convergence of Cournot best-response)**. *Under Cournot best-response, if $N > \max\{N_{\text{sub}}, 2\mu_A/(\mu_B - \mu_A)\}$, then players will converge to one of the herding equilibria in finitely many periods.*

4.3. Equilibrium Refinement

We have established that in the open routing game, players following intuitive learning rules such as Cournot best-response will converge to one of the herding equilibria. We now again invoke the submodularity of the expected system time to show that the herding profiles are the only pure-strategy Nash equilibria for this game.

Corollary 3 **(Equilibrium Refinement)**. *If $N > N_{\text{sub}}$, then the only pure-strategy Nash equilibria of the open routing game are the symmetric herding equilibria of Proposition 1.*

However, Corollary 3 does not completely rule out the existence of Nash equilibria in which players adopt mixed strategies. Our next result rules out any Nash equilibria in which some but not all players use properly mixed strategies, and shows that any mixed-strategy Nash equilibrium is not stable enough to survive even the slightest perturbation in another player's strategy.

Proposition 4 **(Elimination and Instability of Mixed Equilibria)**. *If $N > N_{\text{sub}}$, then the open routing game has no Nash equilibria in which some players mix and others use pure strategies. Moreover, any Nash equilibrium in which all N players use mixed strategies is unstable; that is, a small perturbation in the strategy of any player will cause other players to strictly prefer a pure strategy.*

Therefore, the herding equilibria are the only pure-strategy Nash equilibria of the open routing game, and in any mixed-strategy Nash equilibrium it must be that all players employ properly mixed strategies (i.e, no one plays a pure strategy). Such mixed-strategy equilibria, however, are quite unstable and unlikely to be implemented.

Exploiting the decreasing differences property of the expected system time, we also find that if there is significant service rate disparity—specifically, if service at station B is more than twice as fast as service at station A—then route AB is a strictly dominant strategy for all players.

Corollary 4 **(Dominant Strategy)**. *If $2\mu_A \le \mu_B$ and $N > N_{\text{sub}}$, then route AB is a strictly dominant strategy for all players.*

The dominant strategy result can be understood via the following rough argument. Consider a customer contemplating her move, and suppose that the other $N-1$ customers are playing route BA. If she follows the crowd on route BA, then, on average, she will be behind half of the other customers, first at station B and then again at station A. Since station A is the bottleneck, her expected system time will be on the order of $N/(2\mu_A)$. If instead she chooses route AB, then her system time will be a deterministic N/μ_B. Because $2\mu_A \le \mu_B$, it is better for her to choose route AB over route BA. Then, because $N > N_{\text{sub}}$, if any customer shifts to route AB, then the decreasing differences property implies that her preference for route AB will only increase, and therefore route AB is her optimal strategy regardless of the moves of the other customers. So, if the service rates are close together, then both herding profiles are Nash equilibria, while if they are far apart, then it is a dominant strategy to visit the slower station A first.

With these results in favor of the herding equilibria and the fact that an intuitive class of learning rules converges to them, we see that the herding effect exerts a strong influence on the behavior of rational customers in the open routing game. We note that this herding effect coincides with several examples of open routing. At catered events, anecdotal evidence suggests that guests often flock to the buffet line (the slower station) even when others are doing the same, presumably to avoid waiting in an even longer queue there if they instead got their drink first; also in theme parks, although customers do not all start at the same station, many customers rush to the most popular rides first to experience them before the queue grows long.

4.4. The Open Routing Game with Non-Strategic Customers

In practice some (perhaps many) customers may not attempt to minimize their overall expected system times. For example, at a catered reception a guest may have a preference which dictates a certain route irrespective of its effect on her total system time. Accordingly, we now examine the effect on the system of customers who are not strategic, that is, customers who must visit both stations but who have a pre-determined route which they will follow without contemplating any alternative.

To this end, consider a system as described in Section 3 with N customers, and again assume that priorities are drawn uniformly at random. Suppose now that $N = N_{AB} + N_{BA} + N_S$, where N_{AB} is the number of non-strategic customers who will take route AB no matter what, N_{BA} is the number of non-strategic customers who will take route BA no matter what, and N_S is the number of strategic customers. In this system, a strategic customer could feasibly encounter $N_{AB}, \ldots, N_{AB} + N_S - 1$ other customers choosing route AB. Therefore, we can focus on

$$d_m = T(1,m) - T(0,m), \ \text{ for } m = N_{AB}, \ldots, N_{AB} + N_S - 1.$$

Recall that d_m represents the relative preference between route AB and route BA for a strategic customer, when there are m other customers choosing route AB.

By Proposition 2, we immediately have that d_m is decreasing in m in the range of $N_{AB}, \ldots, N_{AB} + N_S - 1$ as long as $N > N_{\text{sub}}$. Therefore, as in Section 4.1, the expected system times for the customers that are strategic are submodular. The submodularity in turn implies that the herding equilibria (among the strategic customers) prevail as the only pure-strategy Nash equilibria. This is summarized in the next corollary.

Corollary 5 **(Equilibria with Non-Strategic Customers)**. *If $N > N_{\text{sub}}$, then exactly one of the following holds:*

(i) *route AB is a dominant strategy for all strategic customers;*

(ii) *route BA is a dominant strategy for all strategic customers;*

(iii) *both herding profiles are Nash equilibria, and there are no other pure-strategy Nash equilibria.*

We note that Proposition 3 also holds in this setting with non-strategic customers, in the sense that strategic customers will converge to one of the herding equilibria under a general class of learning dynamics. These results suggest that customers who are aware that some others are not rational can still implement an equilibrium profile which involves herding. Moreover, depending on the system parameters, herding at one station can be a dominant strategy.

4.5. The Stochastic Open Routing Game

We now discuss how the herding behavior that we observe in the open routing game continues to emerge when the service times are stochastic. We refer to this new setting as the *stochastic open routing game*. Specifically, for $\mu_A < \mu_B$ we now suppose that the service times at stations A and B are independent and identically distributed random variables with means $1/\mu_A$ and $1/\mu_B$ and variances σ_A^2 and σ_B^2, respectively.

Properties 1 and 2 do not extend to this environment because the uncertainty in service times may induce idle and busy periods that did not arise in the deterministic setting. As a result, we can no longer express customers' expected system times in closed form. Nevertheless, we can bound the expected system times when all customers herd by applying existing results from queueing theory. More specifically, when all customers choose route AB, the waiting times at station B behave as in an underloaded $GI/GI/1$ queue (single-server queue with general independent arrival and service time distributions) that started empty, with arrival rate μ_A and service rate μ_B. Then, a stochastic dominance argument from Müller and Stoyan (2002, Theorem 6.2.1) tells us that we can bound the expected waiting time at station B for any customer in this $GI/GI/1$ queue by the expected steady-state waiting time. Combining this argument with the classical steady-state result of Kingman (1962), we then bound the expected system time for all of the customers who take route AB, and this bound holds for any value of N. Under suitable conditions, we have that our upper bound for the expected system time on route AB is smaller than the expected system time on route BA, and therefore it is a Nash equilibrium for all customers to choose route AB.

Proposition 5 **(Nash Equilibrium with Route AB—Stochastic Service)**. *If*

$$N > 1 + \frac{2\mu_A}{\mu_B} + \frac{\mu_A^2(\sigma_A^2 + \sigma_B^2)}{1 - \mu_A/\mu_B}, \tag{7}$$

then it is a symmetric Nash equilibrium for all customers to choose route AB.

Next, we consider the case in which all customers are choosing route BA. In this case, the waiting times at station A behave as in an overloaded $GI/GI/1$ queue with arrival rate μ_B and service rate μ_A, which does not have a steady-state distribution. To bound the expected waiting time in an overloaded $GI/GI/1$ queue, we apply Lindley's equation to show that the total idling time is represented by the "dual" process of the overloaded system, and that this "dual" process has the same distribution as the waiting time in an underloaded $GI/GI/1$ queue (see Grimmett and Stirzaker 2001, Section 11.5). This allows us to bound the total expected idling time and to show that under suitable conditions it is a Nash equilibrium for all customers to choose route BA. The result is summarized in the next proposition.

PROPOSITION 6 **(Nash Equilibrium with Route *BA*—Stochastic Service)**. *If we have* $\mu_A < \mu_B < 2\mu_A$ *and*

$$N \geq \frac{\mu_B + 2\mu_A}{2\mu_A - \mu_B} + \frac{\mu_A^2 \mu_B^2 (\sigma_A^2 + \sigma_B^2)}{(2\mu_A - \mu_B)(\mu_B - \mu_A)}, \tag{8}$$

then it is a symmetric Nash equilibrium for all customers to choose route BA.

5. The Sequential Open Routing Game

We have thus far assumed that customers had no visibility into the state of the network when making routing decisions, i.e., that they were not aware of their relative priorities or of the decisions made by others. In many applications, however, customers may have some knowledge about their position in the queue and about the routes that others have chosen. For example, in service systems such as amusement parks, at times customers may pre-queue, allowing them at the preset opening time to choose their route conditioned on the actions of those preceding them. To investigate a system where customers have knowledge about their position, we consider the *sequential open routing game*, in which customers make decisions according to their relative position among the N customers and can observe the routes chosen by those who move before them. This setting facilitates a natural representation as an extensive-form game, and we will find its unique subgame perfect equilibrium using backward induction.

We remark that the open routing game of Section 4 represents one extreme, in which customers have no information apart from the system parameters (μ_A, μ_B, and N), while the sequential open routing game that we study here corresponds to the other extreme, in which customers have perfect information regarding their position and the state of the network prior to their decision. Interestingly, in this sequential environment we also observe a phenomenon that is similar to the herding behavior observed in the open routing game: in equilibrium all customers but one visit the slower station A first.

Here we again consider a two-station service network with deterministic service times and N customers present at the beginning of service availability. We note, however, that unlike the open routing game of Section 4, the sequential open routing game is not symmetric, and accordingly the customer index i will no longer be arbitrary. We index customers $i = 1, 2, \ldots, N$, by the order in which they make routing decisions, so customer 1 is the first to move, customer 2 is the second, etc. We also assume that a customer's position in the order corresponds to her priority, so customer 1 will be served first at whichever station she chooses, and customer $i \geq 2$ will wait behind any customers in the set $\{1, \ldots, i-1\}$ who have chosen the same route as her. Next, for $i = 1, 2, \ldots, N$, we define $y_i^A \in \{0, 1, \ldots, i\}$ to be the number of players that have chosen route AB, up to and including customer i. To derive the subgame perfect equilibrium via backward induction, we need to analyze the total time that player N, the last customer to make her routing decision, spends in the system under different strategy profiles. The state observed by player N depends only on y_{N-1}^A, the number of the first $N-1$ customers that chose route AB.

System Time for Customer *N*. Given a value of y_{N-1}^A, we use S_N^A to denote the system time that customer N, the last customer to move, experiences if she takes route AB, and S_N^B to denote the system time that she experiences if she takes route BA.

Because $\mu_A < \mu_B$ and the service times are deterministic, we see that if customer N first visits the faster station B, followed by the slower station A (i.e., she chooses route BA), then she experiences exactly the same system time as if she merely joined a queue of $N-1$ customers all waiting at station A, as long as at least one customer before her chose route AB (i.e., as long as $y_{N-1}^A \geq 1$). If no one else chose route AB, that is, if $y_{N-1}^A = 0$, then the above is still true except that station A sits idle until the first departure from station B, so customer N's system time is increased by $1/\mu_B$. Therefore, we can express S_N^B, the system time for the case in which customer N chooses route BA, as

$$S_N^B = \begin{cases} \dfrac{N}{\mu_A} + \dfrac{1}{\mu_B} & \text{if } y_{N-1}^A = 0, \\ \dfrac{N}{\mu_A} & \text{if } y_{N-1}^A \geq 1. \end{cases} \tag{9}$$

Now, suppose that customer N chooses route AB, visiting the slower station A first, followed by the faster station B. If player N arrives at station B and finds it idle, then S_N^A, the system time that she experiences if she visits station A first, is simply equal to $(y_{N-1}^A + 1)/\mu_A$, the time that she spends at station A, plus the time it takes for her to be served at station B, which is equal to $1/\mu_B$. If player N arrives at station B and finds it busy, then by Property 1, station B must never have been idle since it started service. As player N will be the last customer served at station B, we have then that S_N^A is equal to N/μ_B. To summarize, the system time that customer N experiences from visiting station A first, S_N^A, is given by

$$S_N^A = \begin{cases} \dfrac{y_{N-1}^A + 1}{\mu_A} + \dfrac{1}{\mu_B} & \text{if } \dfrac{y_{N-1}^A + 1}{\mu_A} \geq \dfrac{N-1}{\mu_B}, \\ \dfrac{N}{\mu_B} & \text{if } \dfrac{y_{N-1}^A + 1}{\mu_A} < \dfrac{N-1}{\mu_B}, \end{cases} \tag{10}$$

where $(y_{N-1}^A + 1)/\mu_A \geq (N-1)/\mu_B$ is the condition that ensures that station B is idle upon customer N's departure from station A.

Equilibrium Strategy Profile. With the evaluation of customer N's system times we deduce her strategy in the set of subgame perfect equilibria. Because $\mu_A < \mu_B$, whenever the second condition in equation (10) is met we have that $S_N^A < S_N^B$, so customer N's system time from choosing route AB is the shorter.

If instead the first condition in equation (10) holds, then we have two cases. First, if the number of preceding customers choosing route AB satisfies the inequality $y_{N-1}^A \leq N-2$, then S_N^A is bounded above by $(N-1)/\mu_A + 1/\mu_B$, so we again have that $S_N^A < S_N^B$.

Second, if $y_{N-1}^A = N-1$ and customer N chooses route AB, then she will be served last at station A and finish there after N/μ_A units of time, but she will still require a service time of $1/\mu_B$ at station B. On the other hand, if she takes route BA, then she will immediately be served at station B and leave the system exactly when station A completes its workload, after N/μ_A time units. The extra $1/\mu_B$ from route AB causes S_N^A to be greater than S_N^B for $y_{N-1}^A = N-1$.

Summarizing, we have that $S_N^B < S_N^A$ if and only if $y_{N-1}^A = N-1$, and otherwise $S_N^A < S_N^B$. Therefore, customer N's optimal strategy is to choose route BA if and only if $y_{N-1}^A = N-1$, and route AB otherwise.

With the optimal strategy profile for customer N, we now inductively determine the subgame perfect equilibrium strategy profile for customers $N-n$, for every n in the set $\{1, ..., N-1\}$.

Proposition 7 **(Unique Subgame Perfect Equilibrium)**. *The following strategies form the unique subgame perfect equilibrium of the sequential open routing game:*

1. *Customers* $1, \ldots, N-1$ *visit station* A *first in every subhistory.*
2. *Customer* N *visits station* B *first if and only if she observes* $y_{N-1}^A = N-1$*; otherwise, she visits station* A *first.*

Moreover, along the equilibrium path the first $N-1$ *customers to move visit station* A *first, and the final customer visits station* B *first.*

Intuitively, one might expect that customers should join the queue with the shortest wait time because that minimizes their waiting time before getting served at one of the two stations. However, Proposition 7 states that if customers are rational, then all except the last will first visit station A, which is slower and which each customer except the first one observes to have more customers in its queue than station B. This behavior is best explained by what we call delayed overtaking. If customers later in the order will visit the slower station A first, then it is in the best interest of a customer who moves earlier to immediately join the queue at station A because otherwise, during her time at the faster station B, others will overtake her at station A.

Interestingly, we observe that the optimal actions of the first $N-1$ customers are completely independent of the state that they observe and are essentially driven by the strategies of those later in the order. Note also that the equilibrium actions—i.e., the herding profiles—from the open routing game of Section 4 cannot be supported as equilibria (Nash or subgame perfect) for the sequential game. Namely, if the last customer to move is aware of her position and the decisions of the other customers, then her best response to all of the earlier customers visiting the same station (either A or B) is always to visit the other station, breaking both of the equilibria of Proposition 1. However, in equilibrium only one customer visits station B first here, so the subgame perfect equilibrium of this section is quite similar to the herding equilibrium at station A discussed in Section 4. Therefore, in an environment in which decisions are made sequentially, the pull to the slower station has the strongest impact on behavior, and the delayed overtaking effect dominates.

On the whole, we find that behavior in the sequential open routing game parallels that of the one-shot version. The most important departures are (i) that the delayed overtaking effect always dominates in the sequential environment, supporting a form of herding only at the slower station and (ii) that in equilibrium the last customer, recognizing that she is the last to move, takes the opposite route of all of the other players.

6. Herding vs. Central Optimum

We consider now a central planner who wishes to optimize social welfare in the open-routing service network of Sections 4 and 5. We take as our measure of social welfare the sum of the system times of all customers. The central planner's problem is to minimize this quantity, which we call *cumulative system time*. We compare the cumulative system time under herding with the optimal cumulative system time, first by deriving bounds on the sub-optimality of herding and then by numerically solving for the optimal routing. We end by discussing the implications of these results on the price of anarchy.

Cumulative System Time Under Herding. If $0 \leq x \leq N$ customers are assigned to route AB, then denote by $D(x)$ the corresponding cumulative system time. The functional form of $D(x)$, along with discussion of the intuition behind it, can be found in Appendix B. We note that

$$D(0) = D(N) = \frac{N}{\mu_B} + \frac{N^2 + N}{2\mu_A}, \tag{11}$$

i.e., the cumulative system time is the same under either herding profile. We let D_N^H denote the quantity in equation (11). Let D_N^* denote the minimal cumulative system time, that is,

$$D_N^* = \min_{x \in \{0,\dots,N\}} D(x).$$

Next, we show that the gap, $D_N^* - D_N^H$, is uniformly bounded by a constant.

Proposition 8 **(Optimality Gap of Herding)**. *The optimality gap $D_N^H - D_N^*$ satisfies the bounds*

$$\frac{1}{\mu_A} \leq D_N^H - D_N^* \leq \frac{1}{\mu_A} + \frac{17\mu_B^2 - 6\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B - \mu_A)} \quad \text{for all } N \geq 1 + \frac{2\mu_B}{\mu_A}. \tag{12}$$

The bounds (12) are independent of the population size N, and the ratio of this gap to the optimal cumulative system time rapidly approaches 0 as N grows. As we discuss below, numerical evidence suggests that in most cases assigning exactly one customer to route AB minimizes the cumulative system time. In those cases the optimality gap is equal to one service time at station A.

Exact Central Optimal Solution. With Proposition 8 bounding the optimality gap of herding, we now numerically solve the central planner's problem for a variety of parameter values. The main observation worth noting is that, for all of the parameter combinations that we have studied, the minimal cumulative system time D_N^* is attained by setting the number of AB customers to one. The reason for doing so, instead of assigning all of the customers to herd on route BA (i.e., setting $x = 0$) is to avoid inducing idle time at station A during the first service time at station B. The opposite extreme of assigning all customers to herd on route AB induces a similar idle time problem at station B, resulting in equivalent cumulative system time to the case with $x = 0$. Still, the difference in cumulative system time when setting $x = 0$ instead of $x = 1$ is usually negligible, as we will observe. We compute and study the cumulative system time for a range of parameter values. Fixing $\mu_B = 1$ and taking $N = 50$, we compute the cumulative system time vector (letting $x = 0, 1, \dots, N$) for all values of $\mu_A \in \{1 \times 10^{-5}, 2 \times 10^{-5}, \dots, 99{,}999 \times 10^{-5}\}$. Table 1 shows the maximum percentage difference between the cumulative system time under herding and the minimal cumulative system time, over ranges of μ_A. In all cases the cumulative system time under herding is less than a tenth of a percent above optimal.

Table 1 Maximum Percent Above Optimal Cost Under Herding (i.e., $x = 0$ or $x = N$; $\mu_B = 1$)

μ_A	10^{-5}	$(10^{-5}, .2]$	(.2,.4]	(.4,.6]	(.6,.8]	(.8,1]
Max % Gap	.00004	.078	.078	.077	.077	.078

Price of Anarchy. In Sections 4 and 5, we found that self-interested customers followed the crowd to the same route. The optimal outcome under a social planner is quite similar. Rather than balancing the workload across the two queues, a social planner should send almost all of the customers to the same station first. The intuition has to do with the fact that if the queues are of similar length at both stations, then the first customer to exit the system will not do so until she has waited a substantial amount of time for her second service. The performance of such a load-balancing routing assignment is close to that of the worst possible routing scheme (see also Figure 3 in Appendix B). Conversely, if all customers are assigned to the same route, then the early customers have a short (if any) wait at both stations. Whoever the final customers are to leave the system, they cannot possibly leave until station A processes everyone before them, and that time serves as a lower bound on their system times. But for the customers early in the order, their system times are significantly shorter if customers are concentrated on one route.

These results lead to an interesting observation regarding the price of anarchy. The price of anarchy is defined as the ratio between the social welfare in the worst equilibrium outcome and the best outcome achievable by a social planner (Koutsoupias and Papadimitriou 2009). In the open routing game of Section 4, the price of anarchy is close to 1 because the equilibrium outcome sends everyone to the same station, and the difference in social welfare between this and the optimum is only a single service time at station A. Moreover, as N grows the price of anarchy converges to 1 by Proposition 8. As rational customers will herd in equilibrium, the good overall performance of herding with respect to the cumulative system time is encouraging for managers. In the event that customers are not fully rational, there is a strong argument for managers to incentivize herding among their customers, as it is a simple strategy with near-optimal performance.

7. The Prevalence of Herding

We have seen that the herding behavior predominates in the open routing game. With enough players, the herding equilibria are always the only pure-strategy Nash equilibria of the game. Furthermore, when the service rates are far apart, customers have a strictly dominant strategy which implements the herding equilibrium at station A. The herding equilibria continue to hold even when some customers are not strategic and have their routes fixed in advance, or when we allow for stochastic service times. In the sequential variant of the open routing game, we similarly observe a form of herding, albeit only at the slower station A. In this section, we overview additional settings in which the herding behavior continues to prevail. The technical details associated with this overview are discussed in Appendix C.

Customers Who Visit Only One Station

First, we discuss the impact of "dedicated" customers, who require service only at one of the two stations. In the context of the open routing game of Section 4, we now relax the assumption that all customers must visit both stations. Specifically, we decompose N as $N = N_A + N_B + N_S$, where N_A customers visit only station A, N_B customers visit only station B, and the remaining N_S customers are strategic, i.e., they visit both stations but may choose the order. When there are enough strategic players, both the herding profile at station A and that at station B are Nash equilibria of this game. The intuition behind the proof is similar to that of the open routing game. If all of the strategic customers herd at station A, then a customer who deviates will again be the last customer to be served at station A, and therefore she prefers to follow the crowd. We also observe in several numerical experiments that the expected system time appears to be submodular as long as most of the customers are strategic and there is not too much imbalance in the number of dedicated customers at each station. These numerical experiments are detailed in Appendix C. The herding behavior is robust to the presence of "dedicated" customers who require service only at one of the two stations.

$\mathcal{S}$-Station Open Routing Game

We now discuss systems with more than two stations. Consider a generalized version of the open routing game of Section 4 for a system with $\mathcal{S}$ stations and N players. For stations $\xi = 1, 2, \ldots, \mathcal{S}$, let μ_ξ be the service rate at station ξ, and assume that $\mu_1 < \mu_2 < \cdots < \mu_{\mathcal{S}}$.

If players make routing decisions simultaneously and if priorities are drawn uniformly at random, then players must choose from among the $\mathcal{S}!$ possible routing vectors. In this case, we have a Nash equilibrium which is analogous to that in the open routing game with two stations. Specifically, it is a Nash equilibrium for players to choose a route that visits stations in order of increasing service rate, from the slowest to the fastest. Thus once again we have a herding effect, where players follow the crowd and congregate at a single station.

8. Queueing Networks in Steady State

In all of the systems that we have studied thus far, the open-routing service networks were *congested*, i.e., until all customers had arrived, the service rates were lower than the customer arrival rates.

In all of those systems, the herding phenomenon has prevailed. We now turn to the question of whether herding behavior continues to emerge in a service network which does not face congestion. Specifically, we investigate equilibrium behavior when customers arrive according to a Poisson process with rate λ, rather than all being present in the system when service becomes available. Again every customer must be processed at both station A and station B. Service times at stations A and B are independent and exponentially distributed with rates μ_A and μ_B, respectively. Here we take $\lambda < \mu_A < \mu_B$, and we assume that arriving customers find the queueing network in steady state and are served in order of arrival at each station. Note that $\lambda < \mu_A$ implies that, in contrast to earlier settings, with positive probability station A will have idle time in between services. The state of the system is not observable to the customers, i.e., customers may not condition their actions on the system state which includes the queue lengths and types of customers at both stations.

The distribution of the queue length encountered by a customer at the first station in her route is the steady-state distribution of the queue length at that station. In addition, classical results from Kelly (1979) give us that the distribution for the queue length observed by a customer at the second station in her route also matches the steady-state distribution for the queue length at that station. Next, we state a lemma which is a version of Corollary 3.5 from Kelly (1979) modified to our queueing network, which has customer types AB and BA, and queues A and B.

Lemma 1 **(Kelly 1979, Corollary 3.5)**. *When a customer of type $\psi \in \{AB, BA\}$ reaches station $\xi \in \{A, B\}$, the probability that she finds κ customers at station ξ is equal to the steady-state probability that there are κ customers at station ξ.*

We use "type" to refer to a realized route through the network (AB or BA) and "group" to refer to a subset of the population that shares an ex-ante routing probability. Specifically, let $\mathbf{p} = (p_1, p_2, \ldots)$ be a (possibly infinite-dimensional) vector of routing probabilities corresponding to the strategies chosen by the corresponding groups, e.g., p_1 is the probability that a customer in group 1 chooses route AB. Let $\mathbf{z} = (z_1, z_2, \ldots)$ be a vector of strictly positive numbers summing to 1 representing the percentage of the population in each group. The vectors $\mathbf{p}$ and $\mathbf{z}$ together specify a strategy profile for the entire population. Then, we have that external arrivals of each type occur via independent Poisson streams with rate λp_A to route AB and λp_B to route BA, where p_A and p_B are given by

$$p_A = \langle \mathbf{p}, \mathbf{z} \rangle \text{ and } p_B = 1 - p_A.$$

The following result implies that the expected system time is the same for both routes under any strategy profile.

Corollary 6 **(Expected System Time in Steady-State)**. *For any strategy profile, i.e., for any pair of* $\mathbf{p}$ *and* $\mathbf{z}$*, the expected system time is the same for customers taking route* AB *and* BA*.*

In this setting, we define our notion of equilibrium to be that customers of every group must be best-responding to the strategy profile defined by $\mathbf{p}$ and $\mathbf{z}$. As the expected system time is the same for either route, customers of every group are playing a best-response to all other groups, and there is no incentive for members of any group to change their strategy. Therefore, any $\mathbf{p}$ and $\mathbf{z}$ form an equilibrium strategy profile.

Observe that when the system is allowed to reach steady state, the herding profiles in which all customers choose the same route are indeed equilibria. However, as any other feasible routing profiles also form an equilibrium, there is no reason to give special preference to the herding profiles in this setting. Consequently, these results suggest that the herding behavior which prevails in congested service networks no longer predominates in systems which are not very busy.

9. Simulation Study for Systems with Stochastic Arrivals

We have shown that herding occurs in congested open-routing service networks if all customers are present at the start of service. Moreover, our results in Section 8 suggest that herding probably does not occur in a system which is not congested, i.e., with a service rate faster than the arrival rate. These analyses lead naturally to the following hypothesis.

Hypothesis 1. *Herding occurs when a service system is congested, that is, the arrival rate is higher than the service rates of either station until the arrival of the last customer.*

We are interested in Hypothesis 1 because the arrival rate to a service system is often not constant, and a service system may experience a high customer arrival rate especially at the start of its service availability. Note that if we assume that customers are fully rational in the open routing game with stochastic arrival times, then each customer would be required to perform prohibitively difficult analysis to solve for a Bayesian Nash Equilibrium. Even highly intelligent customers are unlikely to implement such outcomes. Accordingly, we next perform a simulation study designed to test Hypothesis 1. We assume that the simulated customers learn about the system through repeated rounds of play, and we simulate systems in which customers arrive stochastically over time and the service times at both stations are stochastic.

9.1. Simulation Setup

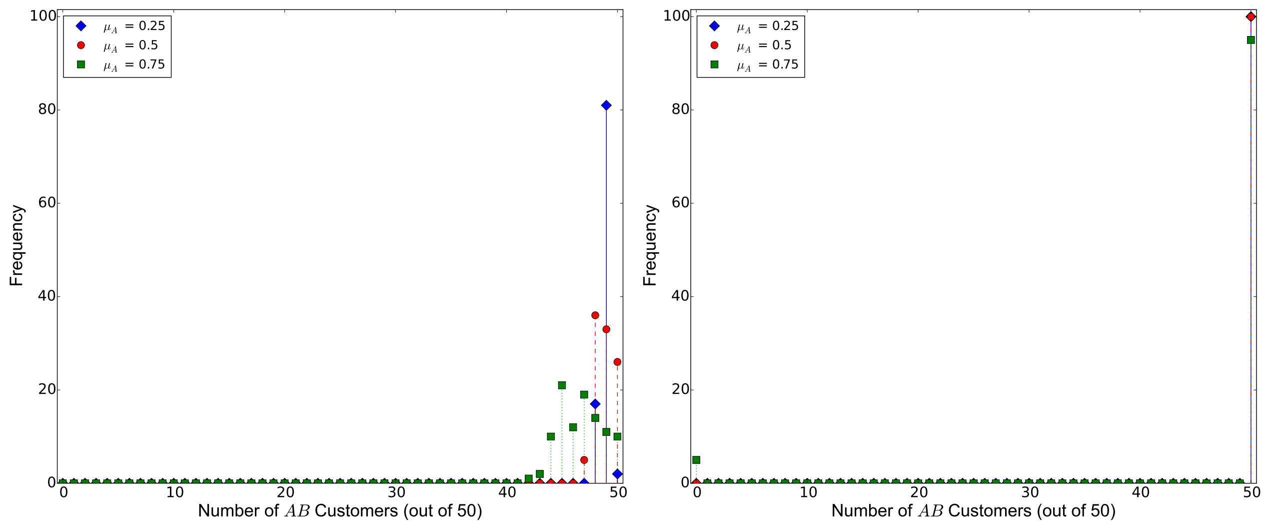
The setting for the simulation is as follows. Play proceeds for multiple rounds, and customers update their beliefs after each round of play based on the wait time that they experience. In each round, customers also observe what their wait time would have been had they chosen the opposite route, fixing the moves of the other customers. Thus, in each round all customers get samples for both routes. Each customer's assessment of her expected system time on a particular route is equal to the empirical average of her own samples for that route across all rounds. Moves are randomly generated in the first round, and thereafter each customer chooses the route with the smaller expected system time according to her beliefs in that round.

We define the parameters γ and ϕ to control the mean and variance of the arrival times. Specifically, the arrival time of customer $i \in \{1, \ldots, N\}$ is assumed to be uniformly distributed on the interval $[i\gamma - \phi, i\gamma + \phi]$. Thus the mean arrival time for customer i is equal to $i\gamma$, and the width of the uniform distribution is 2ϕ. Arrival times of different customers are mutually independent. If ϕ is large enough relative to γ, then successive intervals may overlap and customers may not always arrive in the same order. For example, if $\gamma = .1$ and $\phi = .25$, then the arrival time of customer 1 will be uniformly distributed on $[-.15, .35]$, the arrival time of customer 2 will be uniformly distributed on $[-.05, .45]$, etc. Service times at each station are taken to be exponentially distributed, where the service rate μ_B at station B is fixed at 1 and the rate μ_A at station A varies in each experiment.

We simulated all combinations of $\gamma \in \{.001, .1, .25, .5, .75, 1\}$, $\phi \in \{0, .25, .5, .75, 1\}$, and $\mu_A \in \{.1, .25, .5, .75\}$, for a total of 120 experiments. For each parameter combination we ran 100 independent trials with 250 rounds of play in each trial. In all cases we consider $N = 50$ customers. We analyze the results of the simulation study in the next subsection.

9.2. Results of Simulation

In Figure 2, we depict the empirical frequencies for the number of simulated customers who choose route AB in the final round (round 250) for three different values of the service rate μ_A. The left panel shows the results when the arrival times are deterministic and the arrival order is fixed, i.e., when $\gamma = .001$ and $\phi = 0$. This example closely resembles the sequential open routing game of Section 5, where customers are all present at time zero and make their moves sequentially. In the simulation, unlike in Section 5, customers are not able to observe the moves of those before them in the order when making their routing decisions. Similarly, customers do not explicitly incorporate the strategies of later arrivals. Instead, each customer decides her route based on her experience from the historical rounds.



(a) $\gamma = .001$; $\phi = 0$ (b) $\gamma = .001$; $\phi = .75$

Figure 2 Frequency charts for number of AB customers in round 250 (100 trials).

Recall from Proposition 7 that in the subgame perfect equilibrium of the sequential game, the first $N-1$ customers choose route AB, and the final customer chooses route BA. Referring to the plot in the left of Figure 2, we see that the outcome of the simulation is similar to the equilibrium; the density is mostly concentrated close to 50. As noted, in our simulated system, customers do not condition their strategies on the moves of earlier arrivals, and we therefore do not always see exactly $N-1$ customers choosing route AB. Still, experience from the previous rounds has taught them that if they take route BA, then they may be overtaken at station A by later arrivals, resulting in more total time in the system. The same observation persists for all values of μ_A.

In the right panel of Figure 2, we see results for the case in which again customers all arrive very close to the start of service, but now there is significantly more variance in the arrival order. We thus have a setting which is reminiscent of the one-shot open routing game of Section 4. The arrival times have the same tightly-spaced means—$\gamma = .001$—as in part (a). Now, however, we have $\phi = .75$, which is much greater than the successive difference in mean arrival times γ. Far from a fixed order, each player now could arrive in any position among the 50 customers. As she forms her beliefs over several rounds, the empirical averages will reflect this randomness in her priority.

Continuing to consult the right plot of Figure 2, when the service rate at station A is 75% of that at station B, we see that the outcomes in the final round are split between the two herding profiles, with more than 90% of trials ending up with herding at route AB and the rest ending with herding at route BA. When the service rate at station A is less than half of that at station B (25% and 50%), in all 100 trials we see play converging to herding at route AB. These outcomes match our analytical results from Section 4; that is, the herding equilibria are the only pure-strategy Nash

Table 2 **Summary statistics with $\mu_A = .75$ for number of AB customers (out of 50) in the final round.**

(a) Sample Mean

	ϕ = 0	.25	.5	.75	1
γ = .1	46.22	46.61	45.42	47.45	49.2
.25	46.55	45.83	46.22	46.07	45.72
.5	45.2	45.27	45	44.64	44.32
.75	41.68	42.7	41.59	41.73	42.46
1	35.9	37.38	36.47	36.23	38.17

(b) Sample First Quartile

	ϕ = 0	.25	.5	.75	1
γ = .1	44	45	45	47	49
.25	45	44	44.75	44	44
.5	43	43	42	42	42
.75	39.75	41	38.75	40	40.75
1	33.75	34.75	34	34	35

equilibria of the open routing game; that a large class of learning rules will converge to one of the herding profiles; and that when the service rates differ by a factor greater than 2, route AB is a strictly dominant strategy for all customers.

We have thus far discussed examples with arrivals that are very close together ($\gamma = .001$), which matches closely with our theoretical models. We now discuss settings for which arrivals are more spaced out in time, i.e. $\gamma \geq .1$. We begin by fixing μ_A at .75, varying γ within $[.1, 1]$ and ϕ within $[0, 1]$. We note that in our simulations, herding on route BA never occurs when $\gamma \geq .1$. Because herding at station B does not arise, the number of AB customers in the final round—more specifically, how close this number is to 50—is a reasonable measure of the strength of herding. The mean and first quartile for this quantity can be found in Table 2, over ranges of γ and ϕ. We observe that all three summary statistics for the number of AB customers tend to decrease as γ increases. That is, as the arrivals occur less frequently, herding begins to dissipate. For $\gamma \leq .5$ we see a marked tendency toward herding. For example, when $\gamma = .25$, the mean number of AB customers in the final round for any ϕ is above 45, which is more than 90% of the total number of customers. However, once γ nears one, the herding effect is less pronounced.

Results of the simulation runs with smaller values of μ_A are qualitatively similar. First, recall that our theoretical results for service networks with all customers present at the start of service tell us that route AB is a strictly dominant strategy if $2\mu_A \leq \mu_B$. Our numerical analysis suggests that this extends to cases with stochastic arrivals as well. In our simulations, we find that across all trials and for all parameter combinations such that $2\mu_A \leq \mu_B$, the *minimum* number of AB customers in the final round is 44. We now discuss the particular case of $\mu_A = .5$, the summary statistics for which are reported in Table 3. We note that every entry in Table 3 is at least 47, and that the values in Table 3 are greater than the corresponding values in Table 2 (with $\mu_A = .75$) in all cases. This pattern of successively more prominent herding continues as μ_A decreases further. Such a pattern indicates, as also suggested by the proofs establishing the herding equilibria, that an important driver of the herding behavior is the severity of the penalty of being behind one additional customer.

Table 3 **Summary statistics with $\mu_A = .5$ for number of AB customers (out of 50) in the final round.**

(a) Sample Mean

γ \ ϕ	0	.25	.5	.75	1
.1	48.47	48.53	49.98	50	50
.25	48.34	48.39	47.81	48.61	49.18
.5	48.05	48.07	48.3	48.1	47.82
.75	47.88	47.9	47.81	47.8	47.76
1	47.48	47.42	47.47	47.52	47.33

(b) Sample First Quartile

γ \ ϕ	0	.25	.5	.75	1
.1	48	48	50	50	50
.25	48	48	47	48	49
.5	47	47	48	47.75	47
.75	47	47	47	47	47
1	47	47	47	47	47

The outcomes of our simulation study provide strong evidence for Hypothesis 1, i.e., that herding emerges when the open routing service network is congested. Moreover, our study shows that as arrivals begin to be spaced further apart in time, herding correspondingly diminishes in prevalence, verifying our analysis in Section 8. Additionally, when the service rates are far apart (i.e., when $2\mu_A < \mu_B$), the pull to the slower station A is strong enough that herding is more robust to slower arrivals. These numerical results mirror qualitatively the theoretical results from previous sections, reinforcing the plausibility of herding in congested service systems in more realistic settings.

10. Conclusion

We model customer behavior for service networks in which self-interested customers require service at each station and are permitted to determine their routes through the network. In our base two-station model, customers are present in the system when service becomes available and make decisions about which station to visit first. We find that the expected system time for each customer is a submodular function, and we exploit this property throughout.

In equilibrium, customers herd at one station; that is, all of the customers take the same route through the network. This behavior is motivated by the need to avoid arriving late to the congested station. If all of the other customers are visiting the same station first, then a customer who visits the other station guarantees herself to be served last at the busy one, and she is thus better off following the crowd. However, if the service rates are far apart, then it is a dominant strategy for all players to visit the slower station first. We see also that the herding behavior is stable enough that a large class of learning rules converges to one of the herding equilibria, even if players play both strategies early on. In addition, we find from an investigation of the central optimum that herding is good for social welfare. The cumulative system time—the sum of the system times for all customers—is at its lowest when all but one customer takes route BA. Interestingly, in the presence of herding the welfare loss is usual only a single service time.

Further theoretical and numerical analyses suggest that the herding behavior highlighted by our base model is prevalent whenever the system is congested. Alternatively, we also find that when the arrival rate is less than the bottleneck service rate of the system, herding no longer occurs. Our findings thus suggest that in a service network with open routing, strategic customers will herd if the network is congested.

Our paper provides the following two main takeaways to practitioners. First, in a congested open-routing service network, experienced customers are likely to herd. This insight is important because if system planners assume that strategic customers choose routes at random or join the shortest queue, then they would incorrectly evaluate the system. Second, our analysis shows that herding achieves a near-optimal cumulative system time. Therefore, if cumulative system time is the main objective, instead of requiring customers to take a certain route, system planners can attempt to incentivize herding, or simply help the customers to understand that herding is their best strategy.

Service networks with open routing are an exciting area of study, and this paper generates multiple avenues for future work. For example, in some cases, customers might have complex utility functions; if customers were risk averse or if they preferred not to wait too long at any one station, then the equilibrium analysis would be substantially different from ours. Another system worth studying would be one with more than two stations wherein each customer visits only a subset of the stations, extending our results on two-station systems with dedicated customers. Intriguing also would be a system with more than two stations where customers dynamically choose their routes after each successive service completion. Finally, it would be of interest to learn about customer routing behavior in a system when jockeying between queues is allowed.

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Appendix A: Proofs of Results in Sections 4 and 5

Proof of Proposition 1. The fact that priorities are drawn uniformly at random implies that the game is symmetric, and we thus consider player (or customer) i, where i is an arbitrary player index. Suppose that all customers take route AB and that $N \geq 2\mu_A/\mu_B + 1$. The system time experienced by player i depends on which priority she is assigned. Because all customers are taking route AB, by Property 1, customer i will always find station B idle when she finishes service at station A. When assigned priority j, customer i will wait for $j-1$ players to be served at station A, be served herself, and then immediately be served at station B. Thus, as a function of her priority j, customer i's total system time $S^A(j)$ is given by

$$S^A(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}, \quad j = 1, \ldots, N.$$

Let $T(1,m)$ denote customer i's expected system time if she chooses route AB and m other customers also choose route AB, and let $T(0,m)$ denote customer i's expected system time if she chooses route BA and m other customers choose route AB. As priorities are drawn uniformly at random, customer i's expected system time when following the candidate equilibrium strategy is given by

$$T(1, N-1) = \sum_{j=1}^{N} \frac{1}{N} S^A(j) = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.$$

If she takes route BA, then player i will be behind all $N-1$ other players when she gets to station A. So, her total system time $T(0, N-1)$ is deterministic and is given by

$$T(0, N-1) = \frac{1}{\mu_B} + \left(\frac{N}{\mu_A} - \frac{1}{\mu_B}\right) = \frac{N}{\mu_A}.$$

Now, our assumption that $N \geq 2\mu_A/\mu_B + 1$ implies that

$$\frac{1}{\mu_B} + \frac{1}{2\mu_A} \leq \frac{N}{2\mu_A} \implies \quad \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_A}.$$

Therefore, customer i has no incentive to deviate as $T(1, N-1) \leq T(0, N-1)$, and we have a Nash equilibrium.

With the herding equilibrium at station A established, assume now that $\mu_B < 2\mu_A$ and $N \geq \max\{\mu_B/\mu_A + 1, 2\mu_A + \mu_B/(2\mu_A - \mu_B)\}$, and suppose that all customers take route BA. We will evaluate whether any customer has incentive to deviate. The condition $N \geq (\mu_B/\mu_A) + 1$ ensures that, if player i deviates and takes route AB, then station B will not finish serving all $N-1$ other players before player i finishes at station A. Applying the same notation as the previous case, customer i's total system time $T(1,0)$ from taking route AB is deterministic and is given by

$$T(1,0) = \frac{N}{\mu_B}.$$

If customer i takes route BA, then her priority at station B is drawn uniformly at random, and with probability $1/N$ she will be in position j, for $j = 1, \ldots, N$. Suppose that she draws priority j; she will wait for $j-1$ customers to be served at station B, be served herself there, and then wait in a queue at station A. Because all customers take route BA, station A will idle for the first $1/\mu_B$ units of time, and then will work continuously until it has processed all N customers. Player i's system time corresponds to the time when station A finishes with the j-th job. Thus, we have that customer i's system time $S^B(j)$ is

$$S^B(j) = \frac{j}{\mu_A} + \frac{1}{\mu_B}.$$

We can now calculate her expected total system time $T(0,0)$ as

$$T(0,0) = \sum_{j=1}^{N} \frac{1}{N} S^B(j) = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.$$

By our assumption that $\mu_B < 2\mu_A$ and $N \geq (2\mu_A + \mu_B)/(2\mu_A - \mu_B)$, we have

$$2\mu_A + \mu_B \leq (2\mu_A - \mu_B)N \implies \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \leq \frac{N}{\mu_B}.$$

Therefore we have $T(0,0) \leq T(1,0)$, implying that no customer has incentive to deviate, and we have a Nash equilibrium. $\square$

Proof of Proposition 2. As discussed, submodularity is equivalent to the decreasing differences condition (3). For our proof, we divide $T(1,m)$, the expected system time for a fixed player choosing route AB when m other players are also choosing the same route, into two components. We first define $\widetilde{T}(1,m)$ to be the expected system time for the player choosing route AB, not counting any waiting time that she may experience in the queue at station B. Because priorities are drawn uniformly at random, we have

$$\widetilde{T}(1,m) := \sum_{j=1}^{m+1} \frac{1}{m+1}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{m+2}{2\mu_A}. \tag{13}$$

Next, we define δ_m as the difference between the total expected system time and the expression in equation (13), i.e., $\delta_m := T(1,m) - \widetilde{T}(1,m)$. By definition, then, δ_m is equal to the expected time spent waiting in the queue at station B.

Station A never idles until it finishes as long as at least one player chooses route AB. Therefore, for $m \geq 1$, the expected system time for a player choosing route BA when m other players choose route AB is given by

$$T(0,m) = \sum_{\ell=1}^{N-m} \frac{1}{N-m}\left(\frac{m+\ell}{\mu_A}\right) = \frac{N+m+1}{2\mu_A}. \tag{14}$$

By equations (13) and (14), we have that for all $m \geq 1$,

$$d_m := T(1,m) - T(0,m) = \delta_m + \widetilde{T}(1,m) - T(0,m) = \delta_m + \frac{1}{\mu_B} - \frac{N-1}{2\mu_A}, \tag{15}$$

which implies that for all $1 \leq \widetilde{m} \leq m \leq N-1$,

$$d_m \leq d_{\widetilde{m}} \text{ if and only if } \delta_m \leq \delta_{\widetilde{m}}. \tag{16}$$

For $m \geq 1$, define $\delta_m^{(j)}$ to be the wait time in queue at station B experienced by the fixed player when m other players choose route AB, given that she chooses route AB and receives priority j at station A. Then we have the representations

$$\delta_m^{(1)} = \left(\frac{N-m-1}{\mu_B} - \frac{1}{\mu_A}\right)^+, \text{ and } \delta_m^{(j+1)} = \left(\delta_m^{(j)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ \text{ for all } j = 1, \ldots m.$$

Because $1/\mu_B - 1/\mu_A < 0$, we have that

$$\delta_m^{(j+1)} \le \delta_m^{(j)} \text{ for all } j = 1, \dots, m. \tag{17}$$

Moreover, because the function $f(x) := (x - 1/\mu_A)^+$ is non-decreasing, we also have

$$\delta_{m+1}^{(1)} \le \delta_m^{(1)},$$

and therefore, for $j = 2, \dots, m+1$,

$$\delta_{m+1}^{(j)} = \left(\delta_{m+1}^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ \le \left(\delta_m^{(j-1)} + \frac{1}{\mu_B} - \frac{1}{\mu_A}\right)^+ = \delta_m^{(j)}. \tag{18}$$

We have thus established that $\delta_m^{(j)}$ is monotonically decreasing both in j for any given m and in m for any given j. Next, because priorities are drawn uniformly at random, we have that

$$\delta_m = \frac{1}{m+1} \sum_{j=1}^{m+1} \delta_m^{(j)} \quad \text{for all } m \ge 1. \tag{19}$$

Equation (19) expresses δ_m as the average of the terms $\delta_m^{(1)}, \dots, \delta_m^{(m+1)}$. By equation (17), $\delta_m^{(j)}$ is non-increasing in j, and therefore the average of $\delta_m^{(1)}, \dots, \delta_m^{(m+1)}$ is bounded above by the average of $\delta_m^{(1)}, \dots, \delta_m^{(m)}$. We then have

$$\delta_m = \frac{1}{m+1} \sum_{j=1}^{m+1} \delta_m^{(j)} \le \frac{1}{m} \sum_{j=1}^{m} \delta_m^{(j)} \le \frac{1}{m} \sum_{j=1}^{m} \delta_{m-1}^{(j)} = \delta_{m-1} \text{ for all } 2 \le m \le N-1, \tag{20}$$

where the second inequality comes from the monotonicity in m in equation (18). Equations (16) and (20) then imply that

$$d_m \le d_{m-1} \quad \text{ for all } 2 \le m \le N-1, \tag{21}$$

which satisfies the decreasing differences condition. Moreover, when $2 \le m < N - \mu_B/\mu_A$, we have that

$$\delta_m^{(1)} = \left(\frac{N-m-1}{\mu_B} - \frac{1}{\mu_A}\right)^+ < \left(\frac{N-m}{\mu_B} - \frac{1}{\mu_A}\right)^+ = \delta_{m-1}^{(1)}, \tag{22}$$

so we conclude that the inequalities in equations (20) and (21) hold strictly in this range.

Lastly, we directly evaluate and compare d_1 and d_0. Observe that $N > N_{\text{sub}}$ implies that $2/\mu_A < (N-1)/\mu_B$, so when $m = 0$ or $m = 1$, if the fixed player chooses route AB, then she always faces a queue at station B. Therefore,

$$T(1,1) = \frac{1}{2}\left(\frac{N-1}{\mu_B} + \frac{N}{\mu_B}\right) = \frac{2N-1}{2\mu_B} \quad \text{and} \quad T(1,0) = \frac{N}{\mu_B}.$$

When the player chooses route BA, we note that station A will not idle if $m = 1$, but it idles for $1/\mu_B$ units of time if $m = 0$. Therefore,

$$T(0,1) = \frac{1}{N-1} \sum_{j=1}^{N-1} \frac{j+1}{\mu_A} = \frac{N+2}{2\mu_A} \quad \text{and} \quad T(0,0) = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right) = \frac{1}{\mu_B} + \frac{N+1}{2\mu_A}.$$

Applying the equations above, we have the relation

$$d_1 = \frac{N-1}{\mu_B} - \frac{N+1}{2\mu_A} + \left(\frac{1}{2\mu_B} - \frac{1}{2\mu_A}\right) < \frac{N-1}{\mu_B} - \frac{N+1}{2\mu_A} = T(1,0) - T(0,0) = d_0. \tag{23}$$

Equations (21) and (23) imply that the expected system time has decreasing differences and is therefore submodular, and equations (22) and (23) imply equation (4). $\square$

Proof of Corollary 1. If $m < N - \mu_B/\mu_A$, then the statement is immediately verified by Proposition 2. If $m \geq N - \mu_B/\mu_A$, we then must have

$$\frac{N-m}{\mu_B} \leq \frac{1}{\mu_A},$$

which implies that the queue at station B clears before the first AB customer arrives. Recall that $\delta_m = T(1,m) - \widetilde{T}(1,m)$ is equal to the expected time spent waiting in the queue at station B. Because the queue at station B clears before the first AB customer arrives, we have $\delta_m = 0$, and equation (15) implies that

$$d_m = \frac{1}{\mu_B} - \frac{N-1}{2\mu_A} < \frac{1}{\mu_B} - \frac{1}{\mu_A} < 0,$$

where the first inequality follows because $N > N_{\text{sub}} > 3$. $\square$

Proof of Proposition 3.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the standard vector product, and let $\boldsymbol{d}$ be the vector of differences $d_m = T(1,m) - T(0,m)$ of waiting times for choosing routes AB and BA, i.e., $\boldsymbol{d} := (d_0, d_1, \ldots, d_{N-1})$. Define $\pi_i^{(t)}$ as the expected difference in system times for player i, given her belief at time t, which is given by

$$\pi_i^{(t)} := \langle \boldsymbol{\beta}_i^{(t)}, \boldsymbol{d} \rangle.$$

We proceed by cases.

Case 1: suppose that $x^{(\ell)} \geq m^* + 1$. Then for any player i who chose route AB in period ℓ, we must have $x_{-i}^{(\ell)} \geq m^*$, and also $\pi_i^{(\ell)} \leq 0$ because otherwise player i would have chosen route BA in period ℓ. This implies that

$$\pi_i^{(\ell+1)} = (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell \langle \boldsymbol{e}(x_{-i}^{(\ell)}), \boldsymbol{d} \rangle \leq (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell d_{m^*} \leq 0, \tag{24}$$

where the first inequality follows from the submodularity of the expected system time, and the second inequality follows by the definition of m^*. Equation (24) implies that all of the customers who chose route AB in period ℓ will do so again in periods $\ell+1$, $\ell+2$, ... Therefore, for any $t \geq \ell$, we have that $x_i^{(t)} = 1$ for each player i that chose route AB in period ℓ, and this implies that

$$m^* + 1 \leq x^{(\ell)} \leq x^{(t)} \qquad \text{for all } t \geq \ell.$$

For $\bar{t} \geq \ell$ and for any customer i who chose route BA in period ℓ, we have that

$$\begin{aligned}
\pi_i^{(\bar{t}+1)} &= \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \sum_{t=\ell}^{\bar{t}} \alpha_t \langle \boldsymbol{e}(x_{-i}^{(t)}), \boldsymbol{d} \rangle \prod_{s=t+1}^{\bar{t}} (1-\alpha_s) \\
&\leq \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \sum_{t=\ell}^{\bar{t}} \alpha_t d_{m^*+1} \prod_{s=t+1}^{\bar{t}} (1-\alpha_s) \\
&= \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\right) d_{m^*+1},
\end{aligned}$$

where the inequality follows by the submodularity of the expected system time. Note that we always have $d_{m^*+1} < 0$, because if $d_{m^*} < 0$, then $d_{m^*+1} \leq d_{m^*} < 0$; and if $d_{m^*} = 0$, then by Corollary 1, we have $d_{m^*+1} < 0$. Combining this with the assumption in equation (6), there must exist t_0 such that for all $\bar{t} \geq t_0$,

$$\pi_i^{(\bar{t}+1)} \leq \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\right) d_{m^*+1} < 0.$$

This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 1$ for every customer i that chose route BA in period ℓ.

Case 2: suppose that $x^{(\ell)} \leq m^* - 1$. Then for any customer i who chose route BA in period ℓ, we must have $\pi_i^{(\ell)} > 0$ and $x_{-i}^{(\ell)} \leq m^* - 1$. This implies that

$$\pi_i^{(\ell+1)} = (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell \langle \boldsymbol{e}(x_{-i}^{(\ell)}), \boldsymbol{d} \rangle \geq (1-\alpha_\ell)\pi_i^{(\ell)} + \alpha_\ell d_{m^*-1} > 0,$$

where the first inequality follows from the submodularity of the expected system time, and the second inequality follows from the definition of m^*. Therefore, for any $t \geq \ell$, we have that $x_i^{(t)} = 0$ for every customer i that chose route BA in period ℓ. Finally, for $\bar{t} \geq \ell$ and for any customer i that chose route AB in period ℓ, we have

$$\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\right) d_{m^*-2}.$$

The assumption in equation (6) and the fact that $d_{m^*-2} > 0$ together imply that there must exist t_0 such that for all $\bar{t} \geq t_0$,

$$\pi_i^{(\bar{t}+1)} \geq \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\pi_i^{(\ell)} + \left(1 - \prod_{t=\ell}^{\bar{t}} (1-\alpha_t)\right) d_{m^*-2} > 0.$$

This implies that for all $\bar{t} \geq t_0$, we have $x_i^{(\bar{t}+1)} = 0$ for each customer i that chose route AB in period ℓ. $\square$

Proof of Corollary 2. By Proposition 3, we know that if there exists some $\ell \geq 0$ such that $x^{(\ell)} \neq m^*$, then Cournot best-response will converge to herding because it is a special case of $\{\alpha_t\}$-learning. We proceed to show that with Cournot best-response, there always exists some such ℓ. Consider an arbitrary path of play in which customers play Cournot best-response, that is, $\{\alpha_t\}$-learning with $\alpha_t = 1$ for all $t \geq 1$. In period t, $x^{(t)}$ players choose route AB. If $x^{(t)} \neq m^*$, then Proposition 3 implies that play will converge to herding in finitely many periods. If instead $x^{(t)} = m^*$, then we have $d_{x^{(t)}} \leq 0$ and $d_{x^{(t)}-1} > 0$ by the definition of m^*. Clearly, if N is odd, then $m^* \neq N/2$, and next, we show that $m^* \neq N/2$ when N is even. Define $Q_{\frac{N}{2}}$ by

$$Q_{\frac{N}{2}} := \min\{\frac{N}{2}, \lfloor \frac{\mu_A(\frac{N}{2}-1)}{\mu_B - \mu_A} \rfloor\}.$$

The quantity $Q_{\frac{N}{2}}$ represents the number of AB customers who face a queue at station B when they depart station A, given that a total of $N/2$ customers chose route AB. We now have

$$T(1, \frac{N}{2}-1) = \sum_{k=1}^{Q_{\frac{N}{2}}} \frac{1}{(\frac{N}{2})} \left(\frac{k + \frac{N}{2}}{\mu_B} \right) + \sum_{k=Q_{\frac{N}{2}}+1}^{\frac{N}{2}} \frac{1}{(\frac{N}{2})} \left(\frac{k}{\mu_A} + \frac{1}{\mu_B} \right)$$

and

$$\begin{aligned} T(0, \frac{N}{2}-1) &= \sum_{j=1}^{\frac{N}{2}+1} \left(\frac{1}{\frac{N}{2}+1} \right) \left(\frac{j + \frac{N}{2} - 1}{\mu_A} \right) \\ &\geq \sum_{k=1}^{\frac{N}{2}} \frac{1}{(\frac{N}{2})} \left(\frac{j + \frac{N}{2} - 1}{\mu_A} \right), \end{aligned} \tag{25}$$

where the inequality in equation (25) comes from the fact that the average of a set of $(N/2)+1$ real numbers is larger than the average of the smallest $N/2$ numbers in the set. We can then write

$$\begin{aligned}d_{\frac{N}{2}-1} &= T(1,\frac{N}{2}-1) - T(0,\frac{N}{2}-1)\\ &\le \sum_{k=1}^{Q_{\frac{N}{2}}} \frac{1}{(\frac{N}{2})}\left(\frac{k+\frac{N}{2}}{\mu_B} - \frac{k+\frac{N}{2}-1}{\mu_A}\right) + \sum_{k=Q_{\frac{N}{2}}+1}^{\frac{N}{2}} \frac{1}{(\frac{N}{2})}\left(\frac{k}{\mu_A} + \frac{1}{\mu_B} - \frac{k+\frac{N}{2}-1}{\mu_A}\right).\end{aligned}$$

From the assumption that $N > 2\mu_A/(\mu_B - \mu_A)$, we get

$$\frac{k+\frac{N}{2}}{\mu_B} - \frac{k+\frac{N}{2}-1}{\mu_A} = -\frac{N(\mu_B-\mu_A)}{2\mu_A\mu_B} + \frac{k}{\mu_B} - \frac{k-1}{\mu_A} < \frac{k-1}{\mu_B} - \frac{k-1}{\mu_A} \le 0;$$

which implies that every term in the first summation above is strictly negative. Also, by the assumption that $N \ge N_{\text{sub}} + 1 = 2\mu_B/\mu_A + 2$, we have

$$\frac{1}{\mu_B} - \frac{N}{2\mu_A} + \frac{1}{\mu_A} \le \frac{1}{\mu_B} - \frac{\mu_B}{\mu_A^2} < \frac{1}{\mu_B} - \frac{1}{\mu_A} < 0;$$

which implies that every term in the second summation is strictly negative. By this reasoning and the decreasing differences condition, we conclude that

$$d_{\frac{N}{2}} \le d_{\frac{N}{2}-1} < 0,$$

contradicting the definition of m^*, which requires that $d_{m^*-1} > 0$. Therefore $m^* \neq N/2$. If $x^{(t)} = m^*$, then in period $t+1$ the customers who played route AB in period t will switch to route BA, and the customers who played route BA will switch to route AB. The fact that $m^* \neq N/2$ implies that $x^{(t+1)} \neq m^*$, and therefore play will converge to herding by Proposition 3. □

Proof of Corollary 3. Assume by contradiction that there exists a pure-strategy Nash equilibrium in which $0 < N_{AB} < N$ players choose route AB, and $N_{BA} = N - N_{AB}$ players choose route BA. These assumptions imply that $d_{N_{AB}} \ge 0$ and $d_{N_{AB}-1} \le 0$. But by Proposition 2, we must also have $d_{N_{AB}} \le d_{N_{AB}-1}$. This implies that $d_{N_{AB}} = d_{N_{AB}-1} = 0$, which contradicts Corollary 1. Therefore, there exist no pure-strategy Nash equilibria besides the herding equilibria of Proposition 1. □

Proof of Corollary 4. Proposition 2 tells us that if $N > N_{\text{sub}}$, then the game has decreasing differences, i.e., $d_{N-1} \le d_{N-2} \le \cdots \le d_1 \le d_0$. The relation $2\mu_A \le \mu_B$ then implies that

$$\frac{\mu_B}{2\mu_A} > \frac{N-1}{N+1} \implies \frac{N-1}{\mu_B} < \frac{N+1}{2\mu_A} \implies d_0 = T(1,0) - T(0,0) = \frac{N-1}{\mu_B} - \frac{N+1}{2\mu_A} < 0.$$

Decreasing differences and the fact that $d_0 < 0$ gives us that $d_m = T(1,m) - T(0,m) < 0$ for all $m \in \{0,1,\ldots,N-1\}$. Therefore a player's expected system time is always smaller for route AB than route BA, no matter how many other players choose route AB. □

Proof of Proposition 4. For players $i = 1, 2, \ldots, N$, let $s_i \in [0,1]$ denote player i's strategy—specifically, the probability that player i chooses route AB. Assume by way of contradiction that there exists a Nash equilibrium with some players adopting mixed strategies and other players adopting pure strategies. Let N_{AB} be the number of customers playing the pure strategy of choosing route AB, N_{BA} be the number of players playing the pure strategy of choosing route BA, and $N_M := N - N_{AB} - N_{BA}$ be the number of customers playing "properly" mixed strategies, i.e., placing strictly positive probability on both routes. Let the index i be defined such that $s_i = 1$ for $i = 1, \ldots, N_{AB}$; $s_i = 0$ for $i = N_{AB}+1, \ldots, N_{AB}+N_{BA}$; and $0 < s_i < 1$ for $i = N_{AB}+N_{BA}+1, \ldots, N$. By our assumption, $N_{AB}+N_{BA} < N$ and either $N_{AB} \geq 1$ or $N_{BA} \geq 1$.

Let Γ_i denote the difference between player i's expected system time from choosing route AB and the expected system time from choosing route BA. Let η_k be the probability, given their strategies, that exactly k players choose route AB among players in the set $\{N_{AB}+N_{BA}+2, \ldots, N\}$. Player $N_{AB}+N_{BA}+1$'s difference between her expected system times from taking routes AB and BA can be expressed as

$$0 = \Gamma_{N_{AB}+N_{BA}+1} = \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k}, \tag{26}$$

where the fact that $\Gamma_{N_{AB}+N_{BA}+1} = 0$ is true by assumption; if player $N_{AB}+N_{BA}+1$ is employing a mixed strategy in this Nash equilibrium, then she must be indifferent between route AB and route BA.

Assume first that $N_{AB} \geq 1$. Then player 1 is choosing route AB. In this case, the difference between her expected system times from taking routes AB and BA can be expressed as

$$\begin{aligned}
\Gamma_1 &= s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} \\
&= (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1},
\end{aligned}$$

where the last equality holds by equation (26). Moreover, because player 1 is choosing route AB, that route must be weakly better for her, implying that

$$\sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} \leq 0. \tag{27}$$

Now, combining equations (26), (27), and the decreasing differences property, we get

$$\sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k-1} = \sum_{k=0}^{N_M-1} \eta_k d_{N_{AB}+k} = 0 \tag{28}$$

If the decreasing differences condition holds strictly (that is, if $d_{N_{AB}+k} < d_{N_{AB}+k-1}$) for some $k \in \{0, 1, \ldots N_M - 1\}$, then equation (28) cannot hold. Therefore we must have

$$d_{N_{AB}-1} = d_{N_{AB}} = \ldots = d_{N_{AB}+N_M-1} = 0. \tag{29}$$

Equation (29) contradicts Corollary 1, and thus N_{AB} must be equal to zero and therefore $N_{BA} \geq 1$. In that case, player 1 is choosing route BA, and the difference between her expected times from choosing routes AB and BA can be expressed as

$$\Gamma_1 = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{1+k} + (1 - s_{N_{AB}+N_{BA}+1}) \sum_{k=0}^{N_M-1} \eta_k d_k = s_{N_{AB}+N_{BA}+1} \sum_{k=0}^{N_M-1} \eta_k d_{1+k},$$

where the last equality holds by equation (26). Moreover, because player 1 is choosing route BA, that route must be weakly better for her, implying that

$$\sum_{k=0}^{N_M-1} \eta_k d_{1+k} \geq 0. \tag{30}$$

Now, applying a similar argument as in the case with $N_{AB} \geq 1$, combining equations (26), (30), and the decreasing differences property gives us

$$d_0 = d_1 = \ldots = d_{N_M} = 0. \tag{31}$$

Equation (31) also contradicts Corollary 1. Therefore, there are no Nash equilibria of the open routing game in which some players adopt mixed strategies and other players adopt pure strategies.

Consider now an equilibrium in which all N players mix, that is, $N_M = N$ and $N_{AB} = N_{BA} = 0$. Let ν_k be the probability, given their strategies, that exactly k players choose route AB among players in the set $\{3, \ldots, N\}$. Player 1's expected savings from taking route BA instead of route AB is given by

$$\Gamma_1 = s_2 \sum_{k=0}^{N-2} \nu_k d_{1+k} + (1 - s_2) \sum_{k=0}^{N-2} \nu_k d_k = 0,$$

by assumption since player 1 must be indifferent if she plays a mixed strategy in equilibrium. Denote by $\widetilde{\Gamma}_1^\epsilon$ the perturbed expected savings for player 1 if player 2 increases her probability of route AB to $s_2 + \epsilon$. We have

$$\begin{aligned}
\widetilde{\Gamma}_1^\epsilon &= (s_2 + \epsilon) \sum_{k=0}^{N-2} d_{1+k}\nu_k + (1 - s_2 - \epsilon) \sum_{k=0}^{N-2} d_k \nu_k \\
&= \epsilon \left(\sum_{k=0}^{N-2} \nu_k (d_{1+k} - d_k) \right) \\
&< 0,
\end{aligned}$$

where the strict inequality follows from decreasing differences and the fact that $d_1 < d_0$, from the proof of Proposition 2. Therefore, for any $\epsilon > 0$, if player 2 perturbs her strategy by increasing her probability of route AB to $s_2 + \epsilon$, then s_1 is no longer a best response for player 1, and thus this mixed-strategy Nash equilibrium is not "ϵ-stable." The same argument applies to the perturbation of any one player's strategy. It should be observed that this notion of ϵ-stability is related to but much stronger than the notion of an "evolutionarily stable strategy" as defined in Hassin and Haviv (2003). $\square$

Proof of Corollary 5. Similar to the argument of Corollary 3, suppose that there is a pure-strategy Nash equilibrium besides the herding equilibria. Then there must exist some $N_{AB} < m < N_{AB} + N_S - 1$ such that $d_m \geq 0$ and $d_{m-1} \leq 0$. Combining this with Proposition 2, we must have $d_m = d_{m-1} = 0$, but this contradicts Corollary 1. Therefore, there cannot be non-herding pure-strategy Nash equilibria. Next, we identify three regimes corresponding to the three conclusions in the statement of the proposition.

Case 1: $d_{N_{AB}} < 0$.

If $d_{N_{AB}} < 0$, then even if only non-strategic customers take route AB, it is in the interest of each strategic customer to deviate to route AB. Decreasing differences implies that it is a dominant strategy for all of the strategic players to choose route AB.

Case 2: $d_{N_{AB}+N_S-1} > 0$.

If $d_{N_{AB}+N_S-1} > 0$, then even if all of the other strategic customers are taking route AB, a strategic customer would rather take route BA. Decreasing differences gives that it is a dominant strategy for all of the strategic players to choose route BA.

Case 3: $d_{N_{AB}} \geq 0$ and $d_{N_{AB}+N_S-1} \leq 0$.

In this case, both herding profiles are Nash equilibria. □

Proof of Proposition 5. Assume that all N players visit station A first, so station B is initially empty until the first departure from station A. Observe that if all N customers visit station A first, then station B behaves like a $GI/GI/1$ queueing system with arrival rate μ_A and service rate μ_B (recall that $\mu_A < \mu_B$, so such a system would be stable). Let $F_{W_0^B}$ be the distribution function for a random variable which is independent of the arrival and service processes and which may modify the initial state of the queueing system. Let W_k^B, $k \geq 1$, be the waiting time that the k-th departure from station A experiences at station B, and let $F_{W_k^B}$ be the distribution function of W_k^B. Note that with probability 1 we have $W_0^B = 0$ and $W_1^B = 0$ because station B is initially empty. Therefore, $F_{W_1^B}$ stochastically dominates $F_{W_0^B}$, which we denote as

$$F_{W_0^B} \leq_{st} F_{W_1^B}.$$

Define $F_{W_\infty^B}$ as the stationary waiting-time distribution function for a $GI/GI/1$ queueing system with arrival rate μ_A and service rate μ_B. By Theorem 6.2.1 in Müller and Stoyan (2002), we then have that

$$F_{W_k^B} \leq_{st} F_{W_\infty^B} \quad \text{for all } k = 1, 2, \ldots. \tag{32}$$

Armed with the stochastic dominance relation (32), we consider the candidate equilibrium profile in which all customers visit station A first and evaluate the prospect of deviating to route BA. If the customer follows her current strategy, she will receive priority k at station A, for $k = 1, \ldots N$, with probability $1/N$. Conditional on her priority k, her expected total system time $\mathbb{E}[S^A|k]$ is given by

$$\mathbb{E}[S^A|k] = \frac{k}{\mu_A} + \mathbb{E}[W_k^B] + \frac{1}{\mu_B}.$$

Equation (32) implies that $\mathbb{E}[W_k^B] \leq \mathbb{E}[W_\infty^B]$, where $\mathbb{E}[W_\infty^B]$ is the steady-state expected waiting time from the distribution function $F_{W_\infty^B}$. We then have the bound

$$\mathbb{E}[S^A|k] \leq \frac{k}{\mu_A} + \mathbb{E}[W_\infty^B] + \frac{1}{\mu_B} \leq \frac{k}{\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B}, \tag{33}$$

where the last inequality follows from bounds for the steady-state expected waiting time in queue for a $GI/GI/1$ queue found in Kingman (1962). Taking expectation over the priority in equation (33), we have

$$\begin{aligned}\mathbb{E}[S^A] &\leq \frac{1}{N}\sum_{k=1}^{N}\left(\frac{k}{\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B}\right) \\ &= \frac{N+1}{2\mu_A} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B}.\end{aligned}$$

If a customer deviates and visits station B first, then she will be the last to be served at station A, so her expected system time $\mathbb{E}[S^B]$ satisfies $\mathbb{E}[S^B] \geq N/\mu_A$. Finally, by equation (7), we have

$$\begin{aligned}
&\frac{2\mu_A}{\mu_B} + \frac{\mu_A^2(\sigma_A^2+\sigma_B^2)}{1-\mu_A/\mu_B} < N-1 \\
&\Rightarrow \frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B} < \frac{N-1}{2\mu_A} \\
\Longrightarrow \mathbb{E}[S^A] \leq \frac{N+1}{2\mu_A} + &\frac{\mu_A(\sigma_A^2+\sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B} < \frac{N}{\mu_A} \leq \mathbb{E}[S^B].
\end{aligned}$$

As a customer's expected total system time is shorter if she follows the candidate profile and visits station A first, she has no incentive to deviate. We conclude that it is a Nash equilibrium for all customers to visit station A first. $\square$

Proof of Proposition 6. Let X_k^A and X_k^B be the service time experienced by the k-th customer to be served at stations A and B, respectively. Suppose that all players visit station B first, and let W_k^A be the waiting time at station A experienced by the k-th departure from station B. Define U_k by

$$U_k := X_k^A - X_{k+1}^B.$$

By Lindley's equation, then, we have

$$\begin{aligned}
W_1^A &= 0 \\
W_2^A &= \max\{0, W_1^A + U_1\} = \max\{0, U_1\} \\
&\cdots \\
W_{k+1}^A &= \max\{0, U_k, U_k + U_{k-1}, \ldots, U_k + U_{k-1} + \cdots + U_1\}.
\end{aligned}$$

Now, define I_k^A to be the cumulative idle time experienced by station A before the arrival of the k-th customer, when all customers visit station A first. We can relate the waiting time W_k^A of the k-th customer to the idle time and the excess workload by

$$\begin{aligned}
I_k^A &= W_k^A - \left(\sum_{i=1}^{k-1} X_i^A - \sum_{j=1}^{k} X_j^B\right) \\
&= W_k^A - \sum_{i=1}^{k-1} U_i + X_1^B \\
&= X_1^B + \max\left\{-\sum_{i=1}^{k-1} U_i, -\sum_{i=1}^{k-2} U_i, \ldots, -U_1, 0\right\}.
\end{aligned}$$

Note then that $-U_i$ has the same distribution as $X_{k-i}^B - X_{k-i+1}^A$, and therefore, $I_k^A - X_1^B$ has the same distribution as

$$W_k^B := \max\left\{\sum_{i=1}^{k-1}(X_i^B - X_{i+1}^A), \sum_{i=2}^{k-1}(X_i^B - X_{i+1}^A), \ldots, (X_{k-1}^B - X_k^A), 0\right\}, \tag{34}$$

which is the "dual process" of W_k^A. Moreover, by Lindley's equation, we can view the equation (34) as the wait time of a single server queue with interarrival time distribution corresponding to station A's service time distribution, and service time distribution corresponding to station A's interarrival time distribution. From equation (32) in the proof of Proposition 5 and Kingman (1962), we have that

$$\mathbb{E}[W_k^B] \le \mathbb{E}[W_\infty^B] \le \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)}. \tag{35}$$

Let S^A be the system time associated with choosing route AB, and S^B be the system time associated with choosing route BA. Now, given a priority k at station B, we have

$$\begin{aligned}\mathbb{E}[S^B|k] &= \frac{k}{\mu_A} + \mathbb{E}[I_k^A] \\ &= \frac{k}{\mu_A} + \frac{1}{\mu_B} + \mathbb{E}[W_k^B] \\ &\le \frac{k}{\mu_A} + \frac{1}{\mu_B} + \mathbb{E}[W_\infty^B],\end{aligned}$$

where the equality follows from the fact that $I_k^A - X_1^B$ has the same distribution as W_k^A, while the inequality follows from equation (35). Applying this inequality, we get that

$$\begin{aligned}\mathbb{E}[S^B] &\le \mathbb{E}[W_\infty^B] + \frac{1}{\mu_B} + \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{\mu_A}\right) \\ &= \mathbb{E}[W_\infty^B] + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A} \\ &\le \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)} + \frac{1}{\mu_B} + \frac{N+1}{2\mu_A},\end{aligned}$$

with the last inequality follows from equation (35). Finally, because a player deviating to route AB will be the last customer served at station B, we clearly have

$$\mathbb{E}[S^A] \ge \frac{N}{\mu_B}.$$

Equation (8) then gives

$$\begin{aligned}&\frac{1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A\mu_B(\sigma_A^2 + \sigma_B^2)}{2(\mu_B - \mu_A)} \le N\left(\frac{2\mu_A - \mu_B}{2\mu_A\mu_B}\right) \\ \Longrightarrow\ \mathbb{E}[S^B] \le &\frac{N+1}{2\mu_A} + \frac{1}{\mu_B} + \frac{\mu_A(\sigma_A^2 + \sigma_B^2)}{2(1-\mu_A/\mu_B)} \le \frac{N}{\mu_B} \le \mathbb{E}[S^A].\end{aligned}$$

As it is, the expected system time from choosing route BA is less than that from choosing route AB. Therefore, no customer has incentive to deviate, and we have that it is a Nash equilibrium for every player to choose route BA. □

Proof of Proposition 7. We establish the subgame perfect equilibrium using backward induction. First, equations (9) and (10) tell us that the unique optimal strategy for customer N is to take route BA if $y_{N-1}^A = N-1$ and route AB otherwise. Note that this strategy ensures that in equilibrium always at least one customer will take route AB. Consequently, being the slower station, in equilibrium station A will never idle until it has processed all N customers. Next, we use induction to identify the optimal strategy for all other players. For the induction hypothesis, assume for some integer $2 \le n \le N-1$ that the strategy of subsequent customers $N - n'$, where $1 \le n' < n$, is to always take route AB, and that the final customer to move, customer N, follows the optimal strategy derived above.

To serve as the base case, we first verify the induction hypothesis for $n=2$ by deriving the equilibrium strategy for customer $N-1$. Player N, the last to move, is the only customer that follows player $N-1$, and in equilibrium customer N will follow the strategy of visiting station B first if $y^A_{N-1}=N-1$, and station A otherwise. So, if customer $N-1$ visits station B first, then customer N will visit station A first, and customer $N-1$ will be the last person served at station A. She will then experience system time given by

$$S^B_{N-1}=\frac{N}{\mu_A} \tag{36}$$

because, as noted, station A never idles given player N's equilibrium strategy. If customer $N-1$ takes route AB, then her system time will depend on how many players before her made the same choice. If a small enough number of them chose route AB that when player $N-1$ departs from station A she will find station B busy, then her system time S^A_{N-1} is given by

$$S^A_{N-1}=\frac{N-1}{\mu_B}<\frac{N-1}{\mu_A}<\frac{N-1}{\mu_A}+\frac{1}{\mu_A}=S^B_{N-1}. \tag{37}$$

On the other hand, if enough customers before customer $N-1$ chose route AB that she will find station B idle when she departs from station A, then her system time S^A_{N-1} is given by

$$S^A_{N-1}=\frac{y^A_{N-2}+1}{\mu_A}+\frac{1}{\mu_B}<\frac{N-1}{\mu_A}+\frac{1}{\mu_A}=S^B_{N-1} \tag{38}$$

Thus, the equilibrium strategy for customer $N-1$ is to visit station A first in every subhistory. Combined with the equilibrium strategy for customer N, equations (36)-(38) verify the induction hypothesis for $n=2$.

Now, assume that the induction hypothesis holds for some integer $2\le n\le N-1$. Let S^B_{N-n} be the system time that customer $N-n$ experiences if she chooses to join station B first. Similarly, let S^A_{N-n} be the system time that customer $N-n$ experiences if she chooses to join station A first. If customer $N-n$ chooses route AB, then everyone after her will join station A first, and customer $N-n$ will be the last to be served at station A. Because station A never idles, we then must have $S^B_{N-n}=N/\mu_A$. To study S^A_{N-n}, we need to consider the following two cases.

Case 1: The number of AB customers before customer $N-n$, denoted by y^A_{N-n-1}, is small enough that customer $N-n$ finds station B busy when she finishes at station A. Then by Property 2, station B has never idled since starting service and we must have $S^A_{N-n}=(N-n)/\mu_B$. Moreover,

$$S^A_{N-n}=\frac{N-n}{\mu_B}<\frac{N-n}{\mu_A}<\frac{N-1}{\mu_A}+\frac{1}{\mu_A}=S^B_{N-n}. \tag{39}$$

Case 2: The number of AB customers before customer $N-n$, denoted by y^A_{N-n-1}, is big enough that customer $N-n$ finds station B idle when she finishes at station A. In this case, we have $S^A_{N-n}=(y^A_{N-n-1}+1)/\mu_A+(1/\mu_B)$. Therefore, we have that

$$S^A_{N-n}=\frac{y^A_{N-n-1}+1}{\mu_A}+\frac{1}{\mu_B}<\frac{N-1}{\mu_A}+\frac{1}{\mu_A}=S^B_{N-n}.$$

In either case, choosing route AB results in a strictly shorter system time for customer $N-n$. Thus, if the induction hypothesis holds for some $n\ge 2$, we now have that for any $n\le\widetilde{n}\le N-1$, the unique optimal strategy for customer $N-\widetilde{n}$ is to take route AB regardless of the subhistory, given that all subsequent customers act optimally. Having verified the induction hypothesis for $n=2$, we obtain the unique subgame perfect equilibrium of our game, comprised of the strategies stated in the proposition. Furthermore, inspection of the strategies reveals that the resulting equilibrium path entails the first $N-1$ players taking route AB, and the final player N taking route BA. $\square$

Appendix B: Additional Discussion and Proofs for Section 6

We now study the *cumulative system time*, that is the sum of the system times of all customers, under different routing assignments. Suppose that x customers are assigned to route AB and the remaining $N-x$ customers are assigned to route BA. Let Q be the number of AB customers who, after their service at station A, find station B busy. We use $D_A^i(x), i \in \{1,\dots,x\}$, to denote the total time spent in the system by the i-th AB customer; similarly, we use $D_B^j(x), j \in \{1,\dots,N-x\}$, to denote the total time spent in the system by the j-th BA customer. Finally, define $D(x)$ to be the cumulative system time, i.e.,

$$D(x) := \sum_{i=1}^{x} D_A^i(x) + \sum_{j=1}^{N-x} D_B^j(x). \tag{40}$$

If the i-th AB customer finds station B busy upon completing her service at station A, then her total system time is given by

$$D_A^i(x) = \frac{N-x}{\mu_B} + \frac{i}{\mu_B}, \quad i = 1,\dots,Q. \tag{41}$$

We may understand this by recalling from Property 1 that if the i-th customer finds station B busy, then station B must necessarily have never been idle since the start of service availability. Thus, the i-th AB customer will enter service at station B exactly when station B has finished processing all $N-x$ of the BA customers plus the $i-1$ customers who visited station A first and are in front of the i-th AB customer. Adding her own service time, she will experience the total system time related in equation (41).

On the other hand, if the i-th AB customer finishes service at station A and finds station B idle, then her total system time is given by

$$D_A^i(x) = \frac{i}{\mu_A} + \frac{1}{\mu_B}, \qquad \text{for } i = Q+1,\dots,x. \tag{42}$$

Finding station B idle is equivalent to not facing a wait at station B, so a customer's total system time is the sum of the time that she spends at station A and her service time at station B. She must wait at station A for the service of the $i-1$ customers in front of her there, and then she must be served. She then moves to station B, where she immediately enters service, resulting in the system time given by equation (42).

Finally, we have that the j-th BA customer has total system time given by

$$D_B^j(x) = \frac{x+j}{\mu_A}, \qquad \text{for } j = 1,\dots,N-x, \tag{43}$$

if $x > 0$. To build some intuition around this last quantity, we note that the j-th BA customer will enter service at station A exactly when station A finishes processing all x of the AB customers plus all $j-1$ of the BA customers in front of her at station B. We recall from Property 2 that station A never idles as long as $x > 0$, so the time until she enters service at station A is given by the sum of the service times of $x+j-1$ customers at station A. Adding her own service time at station A, we see that her system time is as expressed in equation (43). Note that if $x = 0$, then we have $D_B^j(x) = j/\mu_A + 1/\mu_B$, i.e., the quantity in equation (43) plus $1/\mu_B$, where $1/\mu_B$ is the time that station A is idle before its first customer departs from station B.

Therefore, the cumulative system time, $D(x)$, is simply the sum of all of the terms in equations (41)-(43), and we present this quantity in the following lemma as a function of the number x of customers that visit station A first.

Lemma 2 (**Cumulative System Time**). *Take $x \in \{0, \dots, N\}$, and define $\tilde{Q}$ as*

$$\tilde{Q} := \max\left\{k \in \mathbb{N} : \frac{k}{\mu_A} \le \frac{N-x+k-1}{\mu_B}\right\} = \left\lfloor \frac{\mu_A(N-x-1)}{\mu_B-\mu_A} \right\rfloor.$$

The number Q of AB customers who find station B busy upon service completion at station A is given by

$$Q = \min\{x, \tilde{Q}\},$$

and the cumulative system time $D(x)$ is given by

$$D(x) = \begin{cases} \dfrac{N}{\mu_B} + \dfrac{N^2+N}{2\mu_A} & x = 0, \\ \dfrac{x(2N-x+1)}{2\mu_B} + \dfrac{(N-x)(N+x+1)}{2\mu_A} & 1 \le x \le \dfrac{\mu_A(N-1)}{\mu_B}, \\ \dfrac{\tilde{Q}(2N-2x+\tilde{Q}-1)+2x}{2\mu_B} + \dfrac{N+N^2-\tilde{Q}-\tilde{Q}^2}{2\mu_A} & \dfrac{\mu_A(N-1)}{\mu_B} < x \le N - \dfrac{\mu_B}{\mu_A}, \\ \dfrac{x}{\mu_B} + \dfrac{N^2+N}{2\mu_A} & N - \dfrac{\mu_B}{\mu_A} < x \le N. \end{cases} \tag{44}$$

Substituting $x = N$ into the function $D(x)$, we observe that the cumulative system time is the same under either herding profile, i.e., $D(0) = D(N)$.

Proof. First, observe that

$$\begin{aligned} x &> \frac{\mu_A(N-1)}{\mu_B} \\ \mu_B x - \mu_A x &> \mu_A(N-1) - \mu_A x \\ x &> \frac{\mu_A(N-1-x)}{\mu_B-\mu_A} \\ x &> \left\lfloor \frac{\mu_A(N-1-x)}{\mu_B-\mu_A} \right\rfloor = \tilde{Q}, \end{aligned}$$

where the last equivalence condition holds because x is an integer. Therefore we have

$$Q = \tilde{Q} < x \text{ if } x > \frac{\mu_A(N-1)}{\mu_B} \quad \text{and} \quad Q = x \text{ if } x \le \frac{\mu_A(N-1)}{\mu_B}. \tag{45}$$

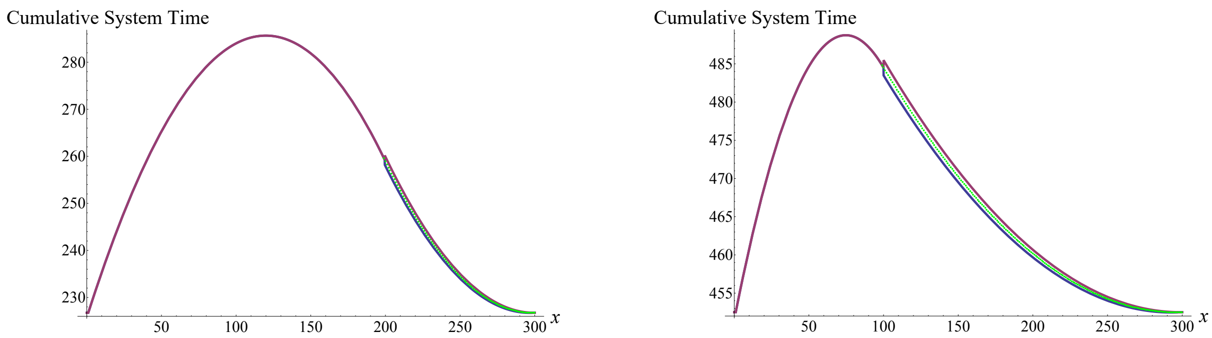
Next, we will study $D(x)$ under the four cases listed in equation (44).

Case 1: $x = 0$. In this case, station A will be idle for the first $1/\mu_B$ units of time. Since all customers visit station B first, each customer must wait for the customers in front of her to finish at station A before she is served there. Hence the customer with priority j at station B will finish at station A exactly when station A completes its j-th service, which because of the idling will occur after $(1/\mu_B) + (j/\mu_A)$ units of time. Thus, the cumulative system time is equal to

$$D(x) = \sum_{j=1}^{N} D_B^j(0) = \sum_{j=1}^{N} \left(\frac{1}{\mu_B} + \frac{j}{\mu_A}\right) = \frac{N}{\mu_B} + \frac{N^2+N}{2\mu_A}.$$

Case 2: $1 \le x \le \mu_A(N-1)/\mu_B$. By equation (45), here we have $Q = x$. Equations (40) and (41) give us that

$$D(x) = \sum_{i=1}^{x} D_A^i(x) + \sum_{j=1}^{N-x} D_B^j(x) = \frac{x(2N-x+1)}{2\mu_B} + \frac{(N-x)(N+x+1)}{2\mu_A}.$$



(a) $N = 300$, $\mu_A = 2/3$, $\mu_B = 1$. (b) $N = 300$, $\mu_A = 1/3$, $\mu_B = 1$.

Figure 3 Cumulative system time as a function of the number of AB customers x.

Case 3: $\mu_A(N-1)/\mu_B < x \le N - \mu_B/\mu_A$. By equation (45), in this case $Q = \tilde{Q}$. By equations (40), (41), and (42), we get

$$D(x) = \sum_{i=1}^{\tilde{Q}} D_A^i(x) + \sum_{i=\tilde{Q}+1}^{x} D_A^i(x) + \sum_{j=1}^{N-x} D_B^j(x) = \frac{\tilde{Q}(2N - 2x + \tilde{Q} - 1) + 2x}{2\mu_B} + \frac{N + N^2 - \tilde{Q} - \tilde{Q}^2}{2\mu_A}.$$

Case 4: $N - \mu_B/\mu_A < x \le N$. Recall that if station B idles then it will never build up a queue again. In this case, even the first AB customer does not face a queue at station B, and therefore the cumulative system time is equal to

$$D(x) = \sum_{i=1}^{x} D_A^i(x) + \sum_{j=1}^{N-x} D_B^j(x) = \frac{x}{\mu_B} + \frac{N^2 + N}{2\mu_A}. \quad \square$$

The four ranges for x that appear in equation (44) correspond to four regimes for the number of AB customers: if $x = 0$, then there are no AB customers and station A will be idle during the first $1/\mu_B$ time periods; if $1 \le x \le \mu_A(N-1)/\mu_B$, then all of the AB customers will face a queue at station B; if $\mu_A(N-1)/\mu_B < x \le N - \mu_B/\mu_A$, then some of the earlier AB customers will face a queue at station B, while the later AB customers will be served at station B immediately; and finally, if $N - \mu_B/\mu_A < x \le N$, then no AB customers will face a queue at station B.

Lemma 2 provides an explicit formula for the cumulative system time $D(x)$, when x customers visit station A first. Because $\tilde{Q}$ is a discrete function of x, we now develop tight upper and lower bounds for $D(x)$ when $x \in \{\lceil \mu_A(N-1)/\mu_B \rceil, \ldots, \lfloor N - (\mu_B/\mu_A) \rfloor\}$. Both the upper and lower bounds are outputs of continuous functions, which are much more amenable to analysis, and the lower bound will be used in the proof of Proposition 8.

Lemma 3 **(Bounds on Cumulative System Time)**. *Defining the functions $S_1(\cdot)$ and $S_2(\cdot)$ and the quantity S_3 by*

$$S_1(t) = \frac{t}{2}\left(\frac{N - x + 1}{\mu_B} + \frac{N - x + t}{\mu_B}\right),$$

$$S_2(t) = \frac{x - t}{2}\left(\frac{t+1}{\mu_A} + \frac{1}{\mu_B} + \frac{x}{\mu_A} + \frac{1}{\mu_B}\right),$$

$$\text{and } S_3 = \frac{N - x}{2}\left(\frac{x+1}{\mu_A} + \frac{N}{\mu_A}\right),$$

we have that if $\mu_A(N-1)/\mu_B < x \le N-(\mu_B/\mu_A)$*, then* $D(x)$ *is bounded by*

$$D(x) \ge S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}-1\right)+S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right)+S_3, \qquad (46)$$
$$D(x) \le S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right)+S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}-1\right)+S_3.$$

Proof. Recall from the proof of Lemma 2 that if $(\mu_A(N-1)/\mu_B) < x \le N-(\mu_B/\mu_A)$, then

$$D(x) = \sum_{i=1}^{\tilde{Q}} \frac{N-x+i}{\mu_B} + \sum_{i=\tilde{Q}+1}^{x}\left(\frac{i}{\mu_A}+\frac{1}{\mu_B}\right)+\sum_{j=1}^{N-x}\frac{x+j}{\mu_A} = S_1(\tilde{Q})+S_2(\tilde{Q})+S_3.$$

The function $S_1(t)$ is non-decreasing on $\mathbb{R}_+$, and the function $S_2(t)$ is non-increasing on $\mathbb{R}_+$. Here we have that $x \le N-\mu_B/\mu_A$, so

$$1 \le \tilde{Q} \le \frac{\mu_A(N-x-1)}{\mu_B-\mu_A},$$

and the monotonicity properties of $S_1(\cdot)$ and $S_2(\cdot)$ give us that

$$D(x) \ge S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}-1\right)+S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right)+S_3$$
$$\text{and } D(x) \le S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right)+S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}-1\right)+S_3. \quad \square$$

The lemma follows from the upper and lower bounds on $\tilde{Q}$ given by

$$\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}-1 < \tilde{Q} = \left\lfloor \frac{\mu_A(N-x-1)}{\mu_B-\mu_A} \right\rfloor \le \frac{\mu_A(N-x-1)}{\mu_B-\mu_A}.$$

In turn, the bounds that we obtain are extremely tight when N is large, as they are essentially the bounds for the discretization error. Figure 3 plots the function $D(x)$, along with these upper and lower bounds, for two problem instances.

Proof of Proposition 8. We will establish that $D(1)$ is a lower bound on the value of $D(x)$ everywhere except the region where $\tilde{Q}$ appears in the expression for $D(x)$, and we will bound the difference between $D(1)$ and the minimum possible value of $D(x)$ in that region.

Case 1: $x=0$. Substituting 1 for x in equation (44), we get

$$\begin{aligned} D(1) &= \frac{N}{\mu_B}+\frac{(N-1)(N+2)}{2\mu_A} \\ &= \frac{N}{\mu_B}+\frac{N^2+N}{2\mu_A}-\frac{1}{\mu_A} \\ &= D(0)-\frac{1}{\mu_A}, \end{aligned}$$

and therefore $1/\mu_A \le D_N^H - D_N^*$.

Case 2: $1 \le x \le \mu_A(N-1)/\mu_B$. In this interval, the continuous extension of $D(x)$ is a concave quadratic function. Thus, the minimum of $D(x)$ in this region must be at one of the endpoints. We evaluate $D(\mu_A(N-1)/\mu_B)$ and get

$$D\left(\frac{\mu_A(N-1)}{\mu_B}\right) = \frac{N^2+N}{2\mu_A}+\frac{N^2(\mu_A^2\mu_B-\mu_A^3)}{2\mu_A\mu_B^3}+\frac{N(2\mu_A^3+\mu_A^2\mu_B-\mu_A\mu_B^2)}{2\mu_A\mu_B^3}+\frac{\mu_B^2-2\mu_A\mu_B-\mu_A^2}{2\mu_B^3}.$$

The difference $D(\mu_A(N-1)/\mu_B)-D(1)$ is then given by

$$D\left(\frac{\mu_A(N-1)}{\mu_B}\right)-D(1) = \frac{N^2\mu_A(\mu_B-\mu_A)}{2\mu_B^3}-\frac{N(\mu_B-\mu_A)(2\mu_A+3\mu_B)}{2\mu_B^3}+\frac{(\mu_B-\mu_A)(\mu_A^2+3\mu_A\mu_B+2\mu_B^2)}{2\mu_A\mu_B^3}$$

The larger root of this convex quadratic function of N occurs at $N = 1 + 2\mu_B/\mu_A$, and thus

$$N \geq 1 + \frac{2\mu_B}{\mu_A} \implies D\left(\frac{\mu_A(N-1)}{\mu_B}\right) - D(1) \geq 0.$$

We deduce therefore that $D(1)$ is the minimum value of $D(x)$ for $1 \leq x \leq \mu_A(N-1)/\mu_B$.

Case 3: $\mu_A(N-1)/\mu_B < x \leq N - \mu_B/\mu_A$. In this region we will work relative to the lower bound in equation (46). Evaluating this bound, we have

$$\begin{aligned} D(x) &\geq S_1\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A} - 1\right) + S_2\left(\frac{\mu_A(N-x-1)}{\mu_B-\mu_A}\right) + S_3 \\ &= x^2\left(\frac{\mu_A}{2\mu_B(\mu_B-\mu_A)}\right) + x\left(\frac{5}{2(\mu_B-\mu_A)} - \frac{\mu_A(2N+1)}{2\mu_B(\mu_B-\mu_A)}\right) \\ &\qquad + N^2\left(\frac{\mu_B-\mu_A}{2\mu_A\mu_B} + \frac{1}{2(\mu_B-\mu_A)}\right) + N\left(\frac{\mu_B+\mu_A}{2\mu_A\mu_B} - \frac{2}{\mu_B-\mu_A}\right) + \frac{2\mu_A+\mu_B}{2\mu_B(\mu_B-\mu_A)} \\ &=: \underline{D}(x). \end{aligned}$$

We note that $\underline{D}(x)$ is a convex quadratic function in x. Differentiating with respect to x, we get

$$\frac{\partial \underline{D}}{\partial x} = \frac{5\mu_B - \mu_A(1+2N-2x)}{2\mu_B(\mu_B-\mu_A)}.$$

Because the first-order condition is sufficient for a minimum, we set the derivative $\partial\underline{D}/\partial x$ equal to zero and solve for the root x^*, obtaining

$$\begin{aligned} 5\mu_B - \mu_A(1+2N-2x^*) &= 0 \\ 2\mu_A x^* &= \mu_A(2N+1) - 5\mu_B \\ x^* &= N + \frac{1}{2} - \frac{5\mu_B}{2\mu_A}. \end{aligned}$$

We then have that $\underline{D}(x^*)$ is a lower bound on the value of $D(x)$ for any $\mu_A(N-1)/\mu_B < x \leq N - \mu_B/\mu_A$. The bound holds whether or not x^* falls within the relevant interval, as $\underline{D}(x^*)$ is the global minimum of the quadratic function. Evaluating $\underline{D}(x^*)$, we get

$$\begin{aligned} \underline{D}(x^*) &= \frac{N^2+N}{2\mu_A} + \frac{N}{\mu_B} - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)} \\ &= D(0) - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)} \\ &= D(1) + \frac{1}{\mu_A} - \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}. \end{aligned}$$

Because $\mu_A < \mu_B$, we have

$$\begin{aligned} 0 &< 17\mu_B^2 - 6\mu_A\mu_B - 7\mu_A^2 \\ 8\mu_B^2 - 8\mu_A\mu_B &< 25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2 \\ \frac{1}{\mu_A} &< \frac{25\mu_B^2 - 14\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B-\mu_A)}, \end{aligned}$$

which implies that $\underline{D}(x^*) < D(1)$.

Case 4: $N - \mu_B/\mu_A < x \le N$. The cumulative system time $D(x)$ is an increasing linear function on the half-open interval $(N - \mu_B/\mu_A, N]$. Therefore, the infimum of $D(x)$ on this interval is given by the output of the linear function evaluated at $N - \mu_B/\mu_A$, i.e.,

$$\begin{aligned}\inf_{x \in (N-\mu_B/\mu_A, N]} D(x) &= \frac{N - \mu_B/\mu_A}{\mu_B} + \frac{N^2 + N}{2\mu_A} \\ &= \frac{N}{\mu_B} + \frac{N^2+N}{2\mu_A} - \frac{1}{\mu_A} \\ &= D(1).\end{aligned}$$

Collecting the results from the four cases, we conclude that

$$\frac{1}{\mu_A} \le D_N^H - D_N^* \le \frac{1}{\mu_A} + \frac{17\mu_B^2 - 6\mu_A\mu_B - 7\mu_A^2}{8\mu_A\mu_B(\mu_B - \mu_A)}. \quad \square$$

Appendix C: Results from Section 7

Customers Who Visit Only One Station

Proposition 9 **(Nash Equilibrium with Route *AB*—Dedicated Customers)**. *If*

$$N_S \ge (N_B + 1)(\frac{2\mu_A}{\mu_B}) + 1 - N_A, \tag{47}$$

then it is a Nash equilibrium for all N_S strategic players to visit station A first.

Proof. Suppose that all of the strategic players choose route AB. If so, then a player who deviates to take route BA will be served at station A after all N_A of the "A-only" players and after the other $N_S - 1$ strategic players. We therefore have the following lower bound on the expected system time $\mathbb{E}[S^B]$ from deviation given by

$$\mathbb{E}[S^B] \ge \frac{N_A + N_S}{\mu_A}. \tag{48}$$

Next we calculate an upper bound on the expected system time from following the profile and taking route AB. Here we note that Property 1 still applies to this system. Namely, if station B ever idles, then it will never build up a queue again. First, consider the case with no B-only players (that is, $N_B = 0$). In this case, there will never be a queue at station B, and strategic customers departing from station A will immediately enter service at station B. Therefore, with no B-only customers we have an exact expression for $\mathbb{E}[S^A]$ given by

$$\mathbb{E}[S^A] = \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B}\right).$$

Now, for a given priority j, if N_B is greater than 0, then the maximum increase in the system time S^A from the case with no B-only players is equal to N_B/μ_B, the increased initial workload at station B. Thus, for any value of N_B we have the bound

$$\mathbb{E}[S^A] \le \sum_{j=1}^{N_A+N_S} \frac{1}{N_A+N_S}\left(\frac{j}{\mu_A} + \frac{1}{\mu_B} + \frac{N_B}{\mu_B}\right) = \frac{N_A+N_S+1}{2\mu_A} + \frac{N_B+1}{\mu_B}. \tag{49}$$

Equations (47), (48), and (49) give

$$
\begin{aligned}
&(N_B+1)(\frac{2\mu_A}{\mu_B})+1 \le N_A+N_S \\
\implies \mathbb{E}[S^A] \le \frac{N_B+1}{\mu_B}+\frac{N_A+N_S+1}{2\mu_A} &\le \frac{N_A+N_S}{\mu_A} \le \mathbb{E}[S^B].
\end{aligned}
$$

We conclude that strategic players have no incentive to deviate, and therefore it is a symmetric Nash equilibrium for all strategic players to visit station A first. □

Proposition 10 **(Nash Equilibrium with Route BA—Dedicated Customers)**. *If $\mu_B < 2\mu_A$ and*

$$
N_S \ge \frac{\mu_B(2N_A+1)-\mu_A(N_B-2)}{2\mu_A-\mu_B}, \tag{50}
$$

then it is a Nash equilibrium for all N_S strategic players to visit station B first.

Proof. Suppose that all of the strategic customers are following the profile of visiting station B first, and consider a player who contemplates deviating and visiting station A first. If she deviates, then she will certainly not enter service at station B until after station B processes all N_B B-only players as well as the other N_S-1 strategic customers. Therefore, we can bound her expected system time $\mathbb{E}[S^A]$ from deviating by

$$
\mathbb{E}[S^A] \ge \frac{N_B+N_S}{\mu_B}. \tag{51}
$$

Next, suppose that she follows the profile and visits station B first, and further suppose that she receives priority k at station B. Let Z_k be the random variable representing the number of strategic customers among the first $k-1$ players at station B. Given that there are N_B+N_S-1 other players at station B, N_S-1 of which are strategic, the random variable Z_k has a hypergeometric distribution with $k-1$ trials from a population of size N_B+N_S-1 containing N_S-1 successes. Therefore, its mean is

$$
\mathbb{E}[Z_k] = (k-1)\frac{N_S-1}{N_B+N_S-1}. \tag{52}
$$

With a large number of B-only players it is possible that station A will become idle while some strategic customers have not yet been served at station B. Consider a strategic customer who is assigned priority k at station B. The greatest amount of idle time that could possibly be introduced at station A is the sum of the service times at station B of the B-only customers with a higher priority. There are exactly $k-1-Z_k$ such players. Define I_k as the amount of time that station A spends idle before the customer with priority k at station B finishes her service at station B. If $N_A=0$, then station A would also idle for $1/\mu_B$ before it receives its first customer. An upper bound on the mean of $\mathbb{E}[I_k]$ is then

$$
\mathbb{E}[I_k] \le \frac{1}{\mu_B}+\mathbb{E}\left[\frac{k-1-Z_k}{\mu_B}\right] = \frac{1}{\mu_B}+\frac{k-1-\mathbb{E}[Z_k]}{\mu_B}. \tag{53}
$$

We can now express the expected system time from choosing route BA by

$$
\begin{aligned}
\mathbb{E}[S^B] &= \sum_{k=1}^{N_B+N_S}\frac{1}{N_B+N_S}\left(\frac{N_A+1+\mathbb{E}[Z_k]}{\mu_A}+\mathbb{E}[I_k]\right) \\
&\le \sum_{k=1}^{N_B+N_S}\frac{1}{N_B+N_S}\left(\frac{N_A+1+\mathbb{E}[Z_k]}{\mu_A}+\frac{k-1-\mathbb{E}[Z_k]}{\mu_B}+\frac{1}{\mu_B}\right) \\
&= \frac{1}{\mu_B}+\frac{N_A+1}{\mu_A}+\frac{N_S-1}{2\mu_A}+\frac{N_B}{2\mu_B},
\end{aligned} \tag{54}
$$

Table 4 Percent Deviation from Submodularity with Dedicated Customers ($N_S = 50, \mu_B = 1$).

	μ_A				
N_A, N_B	1.1	1.3	1.5	1.7	1.9
0,5	2	0	0	0	0
0,10	0	2	0	0	0
5,0	0	2	2	4	4
5,10	0	0	0	0	0
10,0	2	0	4	4	6
10,5	0	0	0	2	4
10,10	0	0	0	0	0

where the inequality follows from equation (53) and the last equality follows from substituting the expression in equation (52) for $\mathbb{E}[Z_k]$ and then evaluating the summation. Because $\mu_B < 2\mu_A$, equations (50), (51), and (54) then give

$$\begin{aligned} &\frac{\mu_B(2N_A+1)-\mu_A(N_B-2)}{2\mu_A-\mu_B} \le N_S \\ &\implies \frac{2N_A+1}{2\mu_A} - \frac{N_B-2}{2\mu_B} \le N_S\left(\frac{1}{\mu_B} - \frac{1}{2\mu_A}\right) \\ \implies \mathbb{E}[S^B] \le \frac{1}{\mu_B} + \frac{N_A+1}{\mu_A} + \frac{N_S-1}{2\mu_A} + \frac{N_B}{2\mu_B} &\le \frac{N_B+N_S}{\mu_B} \le \mathbb{E}[S^A]. \end{aligned}$$

Because $\mathbb{E}[S^B] \le \mathbb{E}[S^A]$, we conclude that no player has incentive to deviate, and thus it is a Nash equilibrium for all N_S strategic players to visit station B first. □

We also observe in a series of numerical experiments that, if there is not too much imbalance in the number of dedicated customers at each station (i.e., only station A vs. only station B), then the expected system time for the strategic customers is often submodular (or close to submodular) even in the presence of dedicated customers. The results of the numerical study are summarized in Table 4. We fix $N_S = 50$ and $\mu_B = 1$. The number of customers who are dedicated to each station varies; there are always 50 strategic customers, and any non-strategic customers increase the total population size. For each routing profile of the strategic customers (i.e., for $x = 0, 1, \ldots, 50$ customers on route AB, with the remainder on route BA), we simulate the system 1,000 times. Service times are deterministic, so the only randomness in the system is the arrangement of customers of different types in the queues. Using the sample average system time for AB and BA customers for each routing profile, we compute the sample differences vector. Table 4 depicts the percentage of difference comparisons (i.e., d_0 vs. d_1, d_1 vs. d_2, etc.) which do *not* satisfy the decreasing differences condition. We note that all instances satisfy the condition within a .5% tolerance. These numerical experiments provide evidence that the expected system time continues to be submodular or almost submodular in the presence of dedicated customers, if most of the customers are strategic and there is not too much imbalance in the number of customers dedicated to each station.

$\mathcal{S}$-Station Open Routing Game

Proposition 11 **(Nash Equilibrium for Unobservable $\mathcal{S}$-Station System)**. *If we have*

$$N \ge 1 + 2\mu_1 \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi}, \tag{55}$$

then the unobservable $\mathcal{S}$-station open routing game has a Nash equilibrium in which all players choose the routing vector $(1,2,\ldots,\mathcal{S})$.

Proof. Suppose that all players are following the routing vector $(1,2,\ldots,\mathcal{S})$, and consider a player who contemplates deviation. On path (that is, if everyone follows the prescribed profile), the only time any customer will face a queue is at station 1. All of the other stations start empty and then receive arrivals only when the station immediately before them completes a service. Because the routing vector is in order of increasing service rate, a queue will never build up at any station other than station 1. Thus, if the customer follows the prescribed profile, then her expected system time $\mathbb{E}[S^{\mathrm{EQ}}]$ is given by

$$\mathbb{E}[S^{\mathrm{EQ}}] = \sum_{j=1}^{N} \frac{1}{N}\left(\frac{j}{\mu_1}\right) + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} = \frac{N+1}{2\mu_1} + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi}.$$

Because the player will not face a queue at any station besides station 1, she cannot possibly improve her system time by changing the order of stations that she visits after station 1. Thus the only deviations that we must consider are those which involve a vector that starts at a station other than station 1. If the player starts at station $\xi \geq 2$, then she will necessarily be the last customer to be served at station 1. Thus, we easily have a lower bound on her expected system time $\mathbb{E}[S^{\mathrm{D}}]$ from deviating given by

$$\mathbb{E}[S^{\mathrm{D}}] \geq \frac{N}{\mu_1}.$$

Equation (55) then gives

$$\begin{aligned} N &\geq 1 + 2\mu_1 \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} \\ \implies \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} &\leq \frac{N-1}{2\mu_1} \\ \implies \mathbb{E}[S^{\mathrm{EQ}}] = \frac{N+1}{2\mu_1} + \sum_{\xi=2}^{\mathcal{S}} \frac{1}{\mu_\xi} &\leq \frac{N}{\mu_1} \leq \mathbb{E}[S^{\mathrm{D}}]. \end{aligned}$$

The player's expected system time is less if she follows the prescribed profile, and we therefore have a symmetric Nash equilibrium where all customers herd at station 1 and follow the routing vector $(1,2,\ldots,\mathcal{S})$. □

Appendix D: Proof of Result from Section 8

Proof of Corollary 6. Let $\mathbb{E}[S^A]$ be the expected system time for a customer who takes route AB, and similarly let $\mathbb{E}[S^B]$ be the expected system time for a customer who takes route BA. Also, take $L_A(\mathbf{p},\mathbf{z})$ and $L_B(\mathbf{p},\mathbf{z})$ to be the steady-state expected number of customers at stations A and B, respectively, when the strategy profile specified by $\mathbf{p}$ and $\mathbf{z}$ is played. By Lemma 1, we have

$$\mathbb{E}[S^A] = \frac{L_A(\mathbf{p},\mathbf{z})}{\mu_A} + \frac{L_B(\mathbf{p},\mathbf{z})}{\mu_B} = \mathbb{E}[S^B]. \quad \square$$