Generalized Matching for School Choice

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Abstract

The school choice problem is formulated as a one-sided or a two-sided matching problem. However, neither model adequately captures the features of the market design applications of school choice. In particular, the one-sided matching solution may be politically infeasible; and the two-sided matching solution may involve inefficiencies. We introduce a generalized model that encompasses one-sided and two sided matching models and their hybrid. We propose a natural stability notion; characterize student optimal stable matchings; and provide a student optimal stable matching mechanism that reduces to the Top Trading Cycles algorithm when the problem is a one-sided matching problem and becomes equivalent to the Gale-Shapley student optimal stable matching algorithm when the problem is a two-sided matching problem.

1 Introduction

The theory of matching has played a critical role in the market design of school choice (Abdulkadiroğlu and Sönmez 2003). In particular, it has guided the debate on and the design of student admissions in public school choice programs in several cities, including Boston (Abdulkadiroğlu, Pathak, Roth

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and Sönmez 2006) and New York City (Abdulkadiroğlu, Pathak and Roth 2009).[1]

A typical school choice problem consists of a set of students and schools. Students rank schools in the order of their preferences. Since schools have limited capacities, admissions to popular schools are usually regulated via assignment priorities. For example, a school may give neighborhood priority to students who live within a certain distance; some schools may rank students based on their academic records; and an entrance examination may determine the rankings at some special schools. We will refer to all such rankings by school priorities or simply priorities.

A student's priority is violated at a school if she prefers the school to her assignment yet another student with lower priority is enrolled in that school. In the *one-sided* matching models of school choice, priorities do not bring any restriction to the problem, indeed they can be violated to promote student welfare. In contrast, the *two-sided* matching models assume that priorities cannot be violated at *any* school. Accordingly, an assignment[2] is said to be *stable* if, whenever a student prefers a school to her assignment, that school is enrolled up to its capacity by students who have higher priority at that school. In a two-sided matching model, student welfare is promoted subject to this stability constraint.

However, these two extreme assumptions on priorities are rarely satisfied in real life applications. In reality, districts provide a variety of school options with varying sources and policy implications of priorities at different schools. For example, the Boston Public Schools (BPS) offers exam schools in addition to the regular schools. A student's score in an entrance exam and her GPA determine her ranking at an exam school, whereas her home address and her siblings' schooling status determine her priority at a regular school. Violating student rankings at Boston's exam schools has proved to be politically and legally infeasible even when the violation was the result of a court-ordered desegregation plan. In contrast, during the redesign of the admissions process, BPS and the public considered the option of violating priorities at regular schools to promote student welfare. Likewise, the New York City high school system involves some schools at which respecting priorities emerges as a major policy goal and some other schools where priority

[1] Also see Abdulkadiroglu, Pathak, Roth (2005) and Abdulkadiroglu, Pathak, Roth, Sönmez (2005).

[2] We use assignment and matching interchangably.

violations may not be a cause of concern. We discuss these in more detail in the case studies section below.

Neither a one-sided nor a two-sided matching model captures the different policy implications of priorities at different schools. In particular, a one-sided matching formulation may yield politically infeasible solutions by violating priorities; and a two-sided matching formulation may harm student welfare by imposing additional constraints on the problem. The following simple example demonstrates the last point.

Example 1 *Consider a school choice problem with four students* $\{i_1, i_2, i_3, i_4\}$ *and four schools* $\{s_1, s_2, s_3, s_4\}$ *each with one seat. Student preferences and school priorities are given by*

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$	$\succ_{i_4}$		$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
s_2	s_1	s_1	s_3		i_1	i_2	i_2	i_1
s_1	s_2	s_2	s_4	*and*	i_3	i_1	i_1	i_3
s_3	s_3	s_3	s_1		i_2	i_3	i_3	i_2
s_4	s_4	s_4	s_2		i_4	i_4	i_4	i_4

That is, i_1 *prefers* s_2 *to* s_1*,* s_1 *to* s_3 *and* s_3 *to* s_4; s_1 *gives highest priority to* i_1*, the next highest priority to* i_3*,* i_4 *has the lowest priority at* s_1*.*

There is a unique stable assignment $\mu = (i_1 - s_1, i_2 - s_2, i_3 - s_3, i_4 - s_4)$*, which assigns* i_1 *and* i_2 *their second choices. Also consider the matching* $\upsilon = (i_1 - s_2, i_2 - s_1, i_3 - s_3, i_4 - s_4)$*.*

υ *is not stable because* i_3*'s priority at* s_1 *is violated;* i_3 *prefers* s_1 *to his assignment and has higher priority at* s_1 *than* i_2*, who is assigned* s_1*. If violating priorities at* s_1 *is acceptable, then* υ *is better than* μ *because* υ *improves both* i_1 *and* i_2*'s assignments without harming the other assignments. Yet* υ *cannot be achieved in a two-sided formulation of the problem.*

This inefficiency problem is a consequence of the standard stability notion, which requires respecting priorities at *all* schools. This source or inefficiency has been overlooked by the literature. To overcome the problem, we introduce a generalized matching model by relaxing the stability requirement. In our model, a school is either stability-constrained or unconstrained and both types of schools may coexist. However, priorities have to be respected only at stability-constrained schools. Consequently, our model encompasses both one-sided and two-sided matching models. It reduces to a one-sided matching model if all schools are unconstrained, to a two-sided matching model if all schools are stability-constrained and it is a hybrid of the two otherwise.

A matching is **pseudo-stable** if it does not violate priorities at stability-constrained schools. A pseudo-stable matching is **stable** if there is no student-school pair (i, s) such that student i prefers s to her assignment, she is eligible for s and s has an empty seat. A pseudo-stable matching is **student optimal stable** if there is no pseudo-stable matching that every student weakly prefers to it and some strictly prefer.

Given a pseudo-stable matching μ, a **stable transfer cycle** consists of an ordered list of schools and students $c = (s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$ such that for all $k = 1, ..., K$,

(i) either student i_k is assigned school s_k at μ, or if i_k is not assigned s_k then there is an empty seat at s_k and i_k is eligible for enrollment at s_k; and

(ii) i_k prefers s_{k+1} to her assignment at μ; and

(iii) if s_{k+1} is a stability-constrained school, i_k has the highest priority among all students who prefer s_{k+1} to their assignment; otherwise i_k is eligible for s_{k+1}.

Given a stable transfer cycle $c = (s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$, updating μ by transferring i_k to s_{k+1}, $k = 1, ..., K$, produces a new matching υ which improves student welfare upon μ. More importantly, if s_{k+1} is a stability-constrained school, then i_k has the highest priority among all students who prefer s_{k+1} to their assignment, therefore υ continues to be pseudo-stable.

Our main result states that if a pseudo-stable matching is not student optimal, then there exists a stable transfer cycle. In other words, a pseudo-stable matching is student optimal stable if and only if it does not admit any stable transfer cycle.

In a stable transfer cycle, a student can be transferred to any unconstrained school that she is eligible for. However, in order to preserve pseudo-stability, a student can be transferred to a stability-constrained school only if she has the highest priority among all students who prefer that school to their assignment. This idea was originally introduced by Erdil and Ergin (2008) to remove inefficiencies from a stable assignment that are due to ties in school priorities in a two-sided matching environment. In contrast, we identify a substantially different source of inefficiency, namely respecting priorities at schools where priorities can be violated. Since Erdil and Ergin's model is a special case of ours, their stable improvement cycle also becomes a special case of stable transfer cycle. More importantly, stable transfer cycles process assignments to both stability-constrained and unconstrained schools.

Our finding yields a simple student optimal stable matching algorithm: Start with the null matching, which is trivially pseudo-stable, but not stable. Given a pseudo-stable matching, if there exists a stable transfer cycle, update the matching by implementing the cycle. Repeat this until no more cycle exists.

In general, there may exist multiple cycles and there is no unique way of selecting a cycle. We offer a particular selection method, which we will refer as *top stable transfer cycles*. That selection method yields a new algorithm, *Stable Transfer Cycles* (STC), which reduces to the Top Trading Cycles algorithm (Abdulkadiroğlu and Sönmez 2003) when all schools are unconstrained, and becomes equivalent to Gale and Shapley's celebrated student optimal stable matching algorithm (David E. Gale and Lloyd S. Shapley 1962) when all schools are stability constrained. STC operates like Top Trading Cycles, however it imposes the stability constraint on constrained schools.

STC is not strategy-proof for students. That is, a student may prefer to report a false preference list when the assignment is determined by STC. However, there is no strategy-proof mechanism in the general model, unless it is one of the extreme one-sided or two-sided models.

We discuss the school choice programs in Boston and New York City in more detail below. We give a brief literature review next. We develop the formal model in Section 4, present our main result in Section 5, introduce STC in Section 6, discuss incentives in Section 7, give an extension in Section 8, and finally conclude by discussing further implications of our findings and future research questions in Section 9. The reader can jump to the model section directly.

2 Case Studies

In this section, we provide more detail on the cases of Boston and New York City.[3] Both examples clearly show the need for a more general matching

[3] See Abdulkadiroglu and Sonmez (2003) and Abdulkadiroglu, Pathak, Roth and Sönmez (2006) for a broader discussion of the Boston case; see Abdulkadiroglu, Pathak, Roth (2009) for the New York City case.

model that encompasses one-sided and two-sided matching models and their hybrid. In particular, the school choice programs in both cities consist of some schools at which respecting priorities arise as a major policy goal as well as other schools at which priority violations might be considered in order to promote student welfare.

2.1 Boston Public Schools (BPS)

BPS is the oldest school district in the US. It serves about 60,000 students from grades K–12 in almost 140 schools in three zones. For most regular schools, priorities are given to students in the following order:

1. Students who are guaranteed a space at the school by virtue of already attending that school or a feeder school (guaranteed priority),
2. students who have a sibling at the school and live in the walk zone of the school (sibling-walk priority),
3. students who have a sibling at the school (but who do not live in the walk zone of the school) (sibling priority),
4. students who live in the walk zone of the school (but who do not have a sibling at the school) (walk zone priority), and
5. other students in the zone.

A random lottery number for each student breaks ties in each category (random tie-breaker). Walk zone priorities do not apply for half of the seats at each school, and students are priority-ordered based on guaranteed and sibling priority and the random tie-breaker for those seats.

The choice between Pareto efficiency and stability has not been a trivial decision in Boston. In December 2003, the Boston School Committee initiated an evaluation of all aspects of student assignment. The final task force report recommended that BPS adopt the Pareto efficient[4] Top Trading Cycles mechanism. In July 2005, the Boston School Committee voted to adopt Gale and Shapley's student optimal stable mechanism.

[4]A matching is Pareto efficient if there is no other matching that makes every student weakly better off and some strictly better off.

There are also several exam high schools that process applicants separately. Students who apply to the exam schools are ranked according to their exam scores and grade point averages in Math and English in the previous year. The history of admissions to Boston exam schools shows political and legal difficulties with violating exam score rankings in Boston. The exam schools in Boston have been at the center of debates on school desegregation. In the 1970s, Federal Court Judge Arthur Garrity ordered that "at least 35% of each of the entering classes at the three exam schools in September 1975 shall be composed of black and Hispanic students." However, racial preferences in exam school admissions, a cause for violating exam score rankings, were disallowed following litigation beginning with the decision in McLaughlin v. Boston Sch. Committee (1996) court case.

From a theoretical point of view, processing admissions to regular schools and exam schools together improves student welfare.[5] However, the legal and political implications of student rankings at regular schools and rankings at exam schools are substantially different. While the BPS considered the option of violating priorities at regular schools for the sake of student welfare, violating score ranking at exam schools proved to be a legal challenge even when the majority of the public might have approved the underlying cause.

2.2 New York City High School match

New York City is the biggest school district in the nation, serving about one million students annually. Admissions to high schools are centralized. Each fall, about 90,000 eligible students fill out a high school application form for the following school year.

In late October or early November, students who are interested in attending one of the exam schools – also known as Specialized High Schools – in New York City take the Specialized High School Admissions Test (SHSAT). Between 25,000 and 30,000 students take the exam every year. The Specialized High Schools account for between 4,000 and 5,000 school seats a year.

[5] To illustrate, consider a simple example with one exam school s_1 and one regular school s_2, each with one seat. i_1 prefers s_2, i_2 prefers s_1; i_1 has a higher exam score and i_2's priority at s_2 is higher. If admissions to one type of the schools is administered first and priorities are respected, then both students would be assigned their second choice. If the admissions were administered simultaneously, even when priorities are respected, both students would get their first choice.

In addition to the exam schools in New York City, there are three types of schools, known as mainstream schools:

1. Schools that actively evaluate applicants and submit a ranking. "Screened" and "audition" schools are examples of this type of school, at which the staff review applicants based on criteria ranging from seventh grade academic performance, attendance, and disciplinary actions to auditions, portfolio submissions, and interviews.

2. Schools that do not evaluate applicants, and instead employ priorities, which are determined not at the school, but by the Department of Education, to rank students. "Unscreened" schools are examples for this category. Priorities include geographic location, current middle school, or other criteria.

3. Schools, at which a fraction of seats are reserved for students who are explicitly ranked by the school, while the rest are automatically categorized into priority groupings set by the DOE. These schools are known as "educational option" or EdOpt schools.

In late November or early December, students submit their High School Application form for mainstream high school admissions. Students who have taken the SHSAT may also submit a separate rank order list expressing their preferences for the exam schools.

As in Boston, admissions to mainstream schools and admissions to exam schools are processed separately. And there is a clear theoretical case to combine both admissions processes to promote student welfare. Furthermore, in contrast to Boston, priorities at different types of mainstream schools have different policy implications.

More than half of the total district capacity in New York City is at schools where priorities are determined by geographic location of students and other exogenous criteria. On the other hand, the EdOpt schools may use students' academic records and rank students in a preference order. For instance, some EdOpt schools prefer students with low reading scores while others prefer higher scores or who had good attendance. If schools have different comparative advantages, allowing scope for their preferences by respecting priorities at those schools seem sensible.

Furthermore, prior to 2003, a major concern for the city was that some of the screened and EdOpt school principals strategically withheld capacity to match with students they preferred. Respecting priorities of such schools in assignment would limit the extent of such strategic behavior. In fact the New York City High School Match does not allow priority violations at *any* school. The choice of a stable assignment mechanism was mainly due to the presence of those latter schools despite the fact that they provide less than half of the district capacity.

While strategic behavior by principals and staff at screened and EdOpt schools is a concern, it is not relevant for the remaining schools. Likewise, when school priorities reflect comparative advantage of a school in educating certain populations, incorporating priorities of those schools in admissions would be a desirable policy goal; however allowing priority violations at other schools would have positive welfare consequences.

3 Literature Review

In this section, we only review the part of the matching literature that is relevant for our current work. See Abdulkadiroğlu and Pathak (in preparation) for a more extensive survey on school choice; see Roth and Sotomayor (1990) for an extensive survey on matching; and see Sönmez and Ünver (2009) and Abdulkadiroğlu and Sönmez (2010) for more recent developments in other market design applications of matching including school choice.

The literature on school choice is initiated by Abdulkadiroğlu and Sönmez (2003). Three policy goals shape the literature and the market design practice: Student welfare, stability and incentives.

When schools priorities do not reflect school preferences or when promoting student welfare to the extent possible is the guiding desideratum, a Pareto efficient matching can be found by the Top Trading Cycles (TTC) mechanism (Abdulkadiroğlu and Sönmez 2003): Initially every student and every school are available. In each round, every student points to the school she prefers most among the remaining schools, and every remaining school points to the student that has the highest priority at that school among all remaining students. A cycle is an ordered list of schools and students $\{s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1\}$ such that school s_k points to student i_k and student i_k points to school s_{k+1}, $k = 1, ..., K$. When such a cycle exists, student i_k is assigned to school s_{k+1}, the capacity of s_{k+1} is decreased by one, and the students in the cycle and schools with no more capacity are removed; the process is repeated with the remaining students and schools.

TTC is strategy-proof for the students, that is, submitting true preferences is a dominant strategy for every student (Abdulkadiroğlu and Sönmez 2003). TTC is a subclass of Pápai's (2000) hierarchical exchange rules, which constitute a subclass of Pycia and Ünver's (2010) trading cycles mechanisms. Abdulkadiroğlu and Che (2010) provide an axiomatic characterization for the specific use of priorities in TTC.

TTC is not stable. Given a school choice problem, multiple stable assignments may exist. When student preferences are strict and there are no ties in school priorities, there exists a unique student optimal stable assignment that every student weakly prefers to any other stable assignment (David E. Gale and Lloyd S. Shapley 1962). Gale-Shapley's celebrated student optimal stable mechanism (GS) finds that unique matching as follows: Every student applies to her most preferred school. Each school tentatively admits in the priority order up to its capacity from its applicants and rejects the remaining applicants. Each rejected student applies to her next most preferred school. Each school tentatively admits in the priority order up to its capacity from its new applicants and the ones it has tentatively admitted and rejects the remaining applicants. The tentative admissions are finalized when no more students are rejected by any school.

Under the assumption of strict preferences and priorities, GS is strategy-proof for the students (Lester E. Dubins and David Freedman 1981; Roth 1982). The specific use of priorities in GS is due to stability, which is also equivalent to eliminating justified envy (Balinski and Sönmez 1999).

However, school priorities usually involve ties among students in real life school choice programs. Both TTC and GS can be adopted directly after breaking ties via some tie-breaking rule, for example by assigning random numbers to students and breaking ties among them in favor of those with the smaller numbers. Both mechanisms continue to be strategy-proof and TTC continues to be Pareto efficient. However GS is no longer student optimal stable. Erdil and Ergin (2008) introduce a Stable Improvement Cycles algorithm, which removes inefficiencies from the GS assignment. Erdil and Ergin (2008) also show that no strategy-proof mechanism is student optimal stable. Abdulkadiroğlu, Pathak and Roth (2009) show that GS with a tie breaker is at the Pareto frontier of the strategy-proof mechanisms. That is, if matching mechanism commits to removing some of the inefficiency associated with GS with some tie breaker, then the mechanism opens room for preference manipulation.

The notion of stable transfer cycle benefits from the idea of stable improvement cycles (SIC) of Erdil and Ergin (2008). In particular, it imposes the stability constraint on particular schools like in stable improvement cycles. In contrast to Erdil and Ergin (2008), our algorithm incorporates both unconstrained and stability-constrained schools and operates on any pseudo-stable matching.

Ehlers and Westcamp (2010) address the question of exam schools and study a model with two types of schools: Exam schools rank students in strict preference order; regular schools are indifferent among all schools. They study stability, efficiency and incentives in their model. We extend their impossibility result for strategy-proof mechanisms directly to our setup.[6]

4 Model

A school choice problem consists of a finite set of students, I, and a finite set of schools S. As opposed to the standard matching model, S is partitioned into a set of **(stability-)constrained** schools S_c and a set of **unconstrained** schools S_u. Formally, $S = S_c \cup S_u$ and $S_c \cap S_u = \varnothing$.

Each school $s \in S$ has a capacity of q_s, which is the number of available seats at school s. Each student $i \in I$ has a strict preference relation $\succ_i$ over $S \cup \{\varnothing\}$, where $\varnothing$ represents the outside option of a student.[7] Let $s \succsim_i s'$ if and only if $s \succ_i s'$ or $s = s'$. Each school $s \in S$ has a weak priority relation $\succsim_s$ over $I \cup \{\varnothing\}$, where $\varnothing$ represents leaving a seat empty.[8] A student i is **eligible** for school s if $i \succ_s \varnothing$. We assume that either $i \succ_s \varnothing$ or $\varnothing \succ_s i$ for all i, s.

To simplify the notation, we will adopt the convention in the literature and replace $\{i\} \succ_s \{j\}$ with $i \succ_s j$. For any $A \subset I \cup S$, let $\succsim_A = (\succsim_a)_{a \in A}$. Also let $q = (q_s)_{s \in S}$. We fix I, S, q and $\succsim_S$ and determine a problem by $\succsim_I$.

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[6] Ehlers (2006) also studies the model with ties in schools priorities; Erdil and Ergin (2006) study a model with indifferences in student preferences and ties in school priorities. Also Abdulkadiroglu, Che and Yasuda (2008, 2009) study the incentives and efficiency from an ex ante point of view when school priorities involve ties. In that case, Kesten and Unver (2008) study stability from an ex ante point of view and Pathak and Sethuraman (2010) study tie breaking with TTC. Kesten (2010) also addresses the efficiency issue with GS within the standard framework with strict preferences and priorities.

[7] Formally, $\succ_i$ is a complete, irreflexive and transitive binary relation over $S \cup \{\varnothing\}$.

[8] We extend $\succsim_s$ over subsets of I as follows: Each $\succsim_s$ is **responsive** (to its restriction on $I \cup \{\varnothing\}$). That is, for every $I' \subset I$ and $i, j \in I \backslash I'$, (i) $I' \cup \{i\} \succsim_s I'$ if and only if $\{i\} \succsim_s \varnothing$, and (ii) $I' \cup \{i\} \succsim_s I' \cup \{j\}$ if and only if $\{i\} \succsim_s \{j\}$ (Roth 1985).

A **matching** of students to schools is a function $\mu : I \cup S \longrightarrow 2^{I\cup S}$ such that

- $\mu(i) \subset S$, $|\mu(i)| \leq 1$ for all $i \in I$,
- $\mu(s) \subset I$, $|\mu(s)| \leq q_s$ for all $s \in S$, and
- $s \in \mu(i)$ if and only if $i \in \mu(s)$ for all $i \in I$ and $s \in S$.

We will equivalently use $\mu(i) = s$ for $s \in \mu(i)$.

A matching μ is **feasible** if every student that is matched with a school is eligible for that school, i.e. $i \succsim_{\mu(i)} \varnothing$ for all $i \in I$. A matching μ is **individually rational** if $\mu(i) \succsim_i \varnothing$ for all $i \in I$ and $i \succsim_s \varnothing$ for all $i \in \mu(s)$, $s \in S$. To simplify the exposition, we will assume that a student can rank a school only if she is eligible for that school, that is, if $s \succ_i \varnothing$ then $i \succ_s \varnothing$. Then individual rationality implies feasibility.

Given $(\succsim_S, \succsim_I)$, μ **violates** i**'s priority at** s if there is a student $j \in \mu(s)$, $s \succ_i \mu(i)$ and $i \succ_s j$, that is, if μ assigns i's more preferred school s to a student with lower priority.[9]

In our model such violations cause a problem only if they occur at constrained schools, so we avoid such violations only at constrained school. Formally, a matching μ is **pseudo-stable** if it is individually rational and there is no $i \in I$, $s \in S_c$ and $j \in \mu(s)$ such that $s \succ_i \mu(i)$ and $i \succ_s j$. The null matching that does not match any student to a school is trivially pseudo-stable.

A matching μ **wastes a seat at** s if $|\mu(s)| < q_s$ and there is $i \in I$ such that $s \succ_i \mu(i)$ and $i \succ_s \varnothing$.[10] A matching μ is **stable** if it is pseudo-stable and it does not waste any seat. Although the null matching is trivially pseudo-stable, it wastes all the seats so it is not stable.

A matching μ **Pareto dominates** another matching ν if $\mu(i) \succsim_i \nu(i)$ for all $i \in I$ and $\mu(i) \succ_i \nu(i)$ for some $i \in I$.

A matching is **Pareto efficient** if it is not Pareto dominated by another matching; it is **student optimal stable** if it is pseudo-stable and not Pareto dominated by another pseudo-stable matching.

[9] In the standard two-sided matching literature, such (i, s) pair is said to block μ and it is referred as a blocking pair. The naming of violating priorities is due to Ergin (2002).

[10] Such a pair is also refered as a blocking pair in the two-sided matching literature. The naming of wastefulness is due to Balinski and Sönmez (1999), which is more appropriate in the context of one-sided matching. However, a matching that assigns two students each other's first choices could also be wastefull from the efficiency point of view.

Note that if a matching is pseudo-stable but not stable, then it is Pareto dominated by another pseudo-stable matching. So a student optimal stable matching does not waste any seat. That is, not wasting a seat and so stability are implied by optimality.[11] We note this as a lemma.

Lemma 2 *A student optimal stable matching is stable.*

Note that the model reduces to the standard two-sided matching model if $S = S_c$, that is $S_u = \varnothing$. In that case, our stability notion coincides with that in the two-sided matching. The model reduces to the standard one-sided matching model if $S_c = \varnothing$.

A (deterministic) **mechanism** selects a matching for every problem. The definitions for matching trivially extend for mechanism. For example, a mechanism is stable if it selects a stable matching for every problem. If φ is a mechanism, let $\varphi(\succsim_I)$ denote the matching selected by φ. A mechanism φ is **strategy-proof for students** if reporting true preferences is a dominant strategy for every student in the preference revelation game induced by φ, that is

$$\varphi(\succsim_I)(i) \succsim_i \varphi(\succsim_{I\backslash i}, \succ'_i)(i)$$

for all $\succsim_I$, $i \in I$ and $\succ'_i$, where $I\backslash i$ stands for $I\backslash\{i\}$. Strategy-proofness for schools is defined similarly.

Since the null matching is pseudo-stable, a pseudo stable matching exists for every problem. We are interested in finding a student optimal stable matching. But first the following follows easily from an earlier result:

Proposition 3 *No stable mechanism is strategy proof for schools.*

The proof of this result is a simple adoption of Roth (1982b). Given any problem, let $I' \subset I$ be such that $|I'| = \sum_{s \in S_c} q_s$, $I'' = I\backslash I$, $s \succ_i s' \succ_i \varnothing$ for all $s \in S_c$, $s \in S_u$ and $i \in I'$, and $i' \succ_s i''$ for all $s \in S_c$, $i' \in I'$ and $i'' \in I''$. Then stability implies that I' is matched with S_c and I'' is matched with I'', so the problem reduces to two separate subproblems. The one in which I' is matched with S_c is the standard many-to-one matching model for which Roth proves that no stable mechanism is strategy proof for schools. Hence the result follows.

[11] In other words, our definition of student optimal stable matching is equivalent to the following standard version, which involves some redundency in our case: A matching is student optimal stable if it is stable and not Pareto dominated by another stable matching.

5 Student Optimal Stable Matchings

Given $(\succsim_S, \succsim_I)$, let μ be a pseudo-stable matching. For each school $s \in S$, the set of students who prefer s to their assignment and who are eligible for enrolment at s is given by

$$P(s;\mu) = \{i \in I : s \succ_i \mu(i) \text{ and } i \succ_s \varnothing\}$$

Define the **waitlist** $W(s;\mu)$ for s at μ as follows: For an unconstrained school $s \in S_u$, $W(s;\mu) = P(s;\mu)$; and for a constrained school $s \in S_c$, $W(s;\mu)$ is the set of top $\succsim_s$-ranked students in $P(s;\mu)$:

$$W(s;\mu) = \{i \in P(s;\mu) : i \succsim_s j \text{ for all } j \in P(s;\mu)\}$$

We say that $i \in I$ is **waitlisted at** $s \in S$ **under** μ if $i \in W(s;\mu)$. Note that the only difference between a constrained school and an unconstrained one is in the definition of the waitlist $W(.;\mu)$. A student can be waitlisted for a constrained school $s \in S_c$ only if she is a top $\succsim_s$-ranked among all students who prefer s to their assignment; transferring her to s would not violate stability at s. In contrast, any student who prefers an unconstrained school can be tranferred to it without harming stability as long as she is eligible for it.

A **stable transfer cycle** is an ordered list of schools and students $c = (s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$ such that for all $k = 1, ..., K$,

(i) either $i_k \in \mu(s_k)$, or if $i_k \notin \mu(s_k)$ then $|\mu(s_k)| < q_{s_k}$ and $i_k \succ_{s_k} \varnothing$; and

(ii) $i_k \in W(s_{k+1};\mu)$

Given a stable transfer cycle $c = (s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$, define a new matching υ by

$$\upsilon(i) = \begin{cases} s_{k+1} \text{ if } i = i_k \text{ for some } k \\ \mu(i) \text{ otherwise} \end{cases}$$

υ clearly Pareto dominates μ. More importantly, if s_{k+1} is a constrained school, then i_k is the highest $\succsim_s$-ranked students in the waitlist, therefore υ continues to be stable. Our main result states that a stable transfer cycle exists if μ is not student optimal stable:

Theorem 4 *Given $(\succsim_S, \succsim_I)$, let μ be a pseudo-stable matching. If μ is not student optimal stable, then a stable transfer cycle exists.*

Proof. Consider any μ that is pseudo-stable but not student optimal stable. Let υ be a pseudo-stable matching which Pareto dominates μ.

If there is some $s \in S$ such that $|\mu(s)| < |\upsilon(s)|$, then there exists $i \in W(s;\mu)$ so that (s, i, s) forms a stable transfer cycle.

Suppose that $|\mu(s)| = |\upsilon(s)|$ for all $s \in S$. Let $I' = \{i \in I : \upsilon(i) \succ_i \mu(i)\}$ be the set of students who prefer their assignment under υ to their assignment under μ. For each $i \in I'$, $W(\upsilon(i);\mu) \neq \varnothing$. Start with some $i_1 \in I'$. Given i_k, pick $i_{k+1} \in W(\upsilon(i_k);\mu) \neq \varnothing$. By finiteness, this yields a stable transfer cycle. ■

Since a pseudo-stable matching always exists (e.g. the null matching), and students are made weakly better off after implementing a stable transfer cycle, by finiteness, a student optimal stable matching exists.

Corollary 5 *A student optimal stable matching exists.*

These findings also generalize two earlier results in the standard two-sided matching, in which all schools are stability-constrained. First, existence of a stable matching in that set-up becomes a corollary. Second, Gale-Shapley's student proposing matching mechanism (GS) finds the unique student optimal stable matching when student preferences and school priorities are strict. Therefore GS can be equivalently formulated via stable transfer cycles.

Also, when school priorities involve ties, GS can still be applied after breaking the ties, for example by a lottery. However, GS is no longer student optimal stable in this two-sided matching framework. In that set up, Erdil and Ergin (2008) propose a Stable Improvement Cycles algorithm that removes inefficiencies from the GS matching after a tie breaking. More importantly, a stable transfer cycle processes assignments to both stability-constrained and unconstrained schools.

6 Stable Transfer Cycles

Our result in the previous section provides a general approach to finding a student optimal stable matching. Namely, start with any pseudo-stable matching. If it is not student optimal stable, find a stable transfer cycle and update the matching by transferring the students of the cycle to the schools they point to in the cycle. There may exist multiple cycles and there is no unique way of selecting a cycle.

In this section, we introduce a new algorithm that reduce to TTC when all schools are unconstrained and to GS when all schools are stability constrained. By over using the notion, we will refer it as Stable Transfer Cycles algorithm (STC). We define STC less formally and give the main result here by deferring the proof to the appendix. In the appendix, we give a more formal definition to guide the proof of the main result.

First we need to introduce the following: Initially all students and all schools are **available**. The null matching does not match any student with any school, so it is trivially pseudo-stable. A pseudo-stable matching represents **tentative** assignments of all **available students** and the **permanent** assignments of all **unavailable students**. Given a pseudo-stable matching, if a student is assigned her most preferred school among all available schools, her assignment is finalized; she and her seat becomes **unavailable**. A school becomes unavailable if all of its seats become unavailable.

Given a pseudo-stable matching, the waitlist at an available unconstrained school is composed of all *available* students who prefer that school to their current tentative assignment. The waitlist at every constrained school s is composed of *all* students – available or not – who prefer that school to their tentative or permanent assignment and who are top $\succsim_s$-ranked among such students.

Recursively, an available student is a **transfer student** if she is in the waitlist of a transfer school, and a school is a **transfer school** if it currently enrolls transfer student or it has an empty seat and there is a transfer student eligible for the school. The set of transfer student students and the set of transfer schools can be computed recursively as follows: Mark all available students as transfer students and all available schools as transfer schools initially, and recursively unmark those students who are not in the waitlist of any transfer school, and eliminate any school which is left with no assigned transfer student and with no unassigned but eligible transfer student in case it has an empty seat.

Initially all students and all schools are available. STC starts with the null matching, then proceeds via the following algorithm:

1. Given a tentative pseudo-stable matching μ and the set of available students and schools,

(a) Finalize the assignments of students who are assigned their first choice among all available schools. Repeat this step until no more students are permanently assigned.

(b) Construct the waitlist at available schools.

(c) Find the transfer students and transfer schools.

(d) Every transfer school s points to the transfers students that are tentatively assigned s and if s has an empty seat it points to a top $\succsim_s$-ranked transfer student that is not assigned s; if there are more than one such student, the ties among such students is broken by some tie-breaker, e.g. a lottery.

(e) Every student points to her most preferred school among all transfer schools at which she is in the waitlist.

(f) A "top" stable transfer cycle is an ordered list of transfer schools and transfer students $c = (s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$, such that s_k points to i_k and i_k points to s_{k+1}, $k = 1, ...K$. A stable transfer cycle is **implemented** by transferring every student in that cycle to the school she points to.

(g) Two cycles are disjoint if they do not share a student. All disjoint cycles are implemented.

2. Repeat Step 1 until no more cycle exists.

The resulting assignment in step 1 is a Pareto improvement, since every student in the cycle is assigned a better school. It is also pseudo-stable because any student that is transferred to a constrained school is the highest ranked student in the waitlist of that school. The algorithm is repeated with the new assignment and it terminates when no more cycles are found.

Let us demonstrate STC via the following examples.

Example 6 *Consider a school choice problem with three students $\{i_1, i_2, i_3\}$ and three schools $\{s_1, s_2, s_3\}$ each with one seat and all of which are stability constrained. Student preferences and school priorities are given by*

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$
s_2	s_1	s_1
s_1	s_2	s_2
s_3	s_3	s_3

and

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_1	i_2	i_2
i_3	i_1	i_1
i_2	i_3	i_3

The unique student optimal stable assignment $\mu = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$, which can be computed by Gale and Shapley's student optimal stable matching algorithm (GS) since all schools are constrained in this example.

Let I^k and S^k be the set of available students and schools, respectively, in the kth round of STC. Initialize $I^{-1} = \{i_1, i_2, i_3\}$ and $S^{-1} = \{s_1, s_2, s_3\}$, that is all students and schools are available initially.

STC starts with the null matching μ^0, where $\mu^0(s) = \varnothing$ for each s. Given the null matching, all students in I^{-1} and all schools in S^{-1} remain available, that is $I^0 = \{i_1, i_2, i_3\}$ and $S^0 = \{s_1, s_2, s_3\}$. The waitlists at every available school are given by $W(s_1; \mu^0) = \{i_1\}$ and $W(s_2; \mu^0) = W(s_3; \mu^0) = \{i_2\}$; all students are transfer students and all schools are transfer schools.

s_1 points to i_1, who is the top $\succ_{s_1}$-ranked student among I^0, and s_2 and s_3 point to i_2. i_1 is waitlisted at s_1 only, so he points to s_1; i_2 is waitlisted at both s_2 and s_3, she points to s_2 since she prefers s_2 to s_3. Two disjoint cycles exists, (s_1, i_1, s_1) and (s_2, i_2, s_2) so STC updates μ^0 to μ^1 by assigning i_1 to s_1 and i_2 to $s_2 : \mu^1 = (i_1 - s_1, i_2 - s_2, i_3 - \varnothing)$

Given μ^1, I^0 and S^0, no student is assigned her first choice among available schools so every student in I^0 and every school in S^0 remain available, i.e. $I^1 = I^0$ and $S^1 = S^0$. The waitlists are given by $W(s_1; \mu^1) = \{i_3\}$, $W(s_2; \mu^1) = \{i_1\}$ and $W(s_3; \mu^1) = \{i_3\}$. i_2 is not in the waitlist of a transfer school that she prefers, so she is not a transfer student, so her current assigned school s_2 is not a transfer school as it does not have any empty seat either. Given that s_2 is not a transfer school, i_1 is not in the waitlist of a transfer school that he prefers, so he is not a transfer student either, and his current assigned school s_1 is not a transfer school as it does not have any empty seat. So i_3 is the only transfer student and s_3 is the only transfer school.

s_3 points to i_3, who points back to s_3, forming a cycle. i_3 is assigned s_3 so that $\mu^2 = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$.

Given μ^2, I^1 and $S1$, no student is assigned his/her first choice so every student in I^1 and every school in S^1 remain available, i.e. $I^2 = I^1$ and $S^2 = S^1$. The waitlists are given by $W(s_1; \mu^1) = \{i_3\}$, $W(s_2; \mu^1) = \{i_1\}$ but there are no transfer students and no transfer schools, so the STC matching is given by $\mu^{STC} = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$, which is the student optimal stable matching.

In this example, the STC outcome coincides with GS outcome. This is not a coincidence. As we will state as a theorem at the end of this section, STC produces a student optimal stable matching. So when all schools are stability constrained and school priorities are strict, it produces the unique student optimal stable matching, therefore it is an equivalent formulation of GS via top stable transfer cycles.

Example 7 *Consider Example 6; assume that all the schools are unconstrained. As above, initialize $I^{-1} = \{i_1, i_2, i_3\}$ and $S^{-1} = \{s_1, s_2, s_3\}$, that is all students and schools are available initially.*

STC starts with the null matching μ^0, where $\mu^0(s) = \varnothing$ for each s. Given the null matching, all students in I^{-1} and all schools in S^{-1} remain available, that is $I^0 = \{i_1, i_2, i_3\}$ and $S^0 = \{s_1, s_2, s_3\}$. The waitlists are given by $W(s_1; \mu^0) = W(s_2; \mu^0) = W(s_3; \mu^0) = \{i_1, i_2, i_3\}$; all students are transfer students and all schools are transfer schools.

s_1 points to i_1, who is the top $\succ_{s_1}$-ranked student among I^0, and s_2 and s_3 point to i_2. All students are waitlisted at all schools, therefore every student points to his/her most preferred school: i_1 points to s_2, i_2 and i_3 point to s_1. $(s_1, i_1, s_2, i_2, s_1)$ is the only top stable transfer cycle so STC updates μ^0 to μ^1 by assigning i_1 to s_2 and i_2 to $s_1 : \mu^1 = (i_1 - s_2, i_2 - s_1, i_3 - \varnothing)$.

i_1 and i_2 are assigned their first choices among all available schools, so they become unavailable. Since their assigned schools do not have any more empty seats, those schools become unavailable as well. Therefore $I^1 = \{i_3\}$ and $S^1 = \{s_3\}$. Consequently, s_3 points to i_3, who points back to s_3, forming a cycle. i_3 is assigned s_3 so that $\mu^2 = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$. i_3 is assigned his most preferred school among all available schools, so he becomes unavailable and STC terminates with $\mu^{STC} = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$.

In this example, the STC assignment coincides with the Top Trading Cycles assignment (TTC). This is not by coincidence either. By design, when all schools are unconstrained, STC proceeds exactly like TTC. The following example involves both types of schools.

Example 8 *Consider Example 6; assume that s_1 is unconstrained and s_2 and s_3 are stability constrained. As above, initialize $I^{-1} = \{i_1, i_2, i_3\}$ and $S^{-1} = \{s_1, s_2, s_3\}$, that is all students and schools are available initially.*

STC starts with the null matching μ^0, where $\mu^0(s) = \varnothing$ for each s. Given the null matching, all students in I^{-1} and all schools in S^{-1} remain available, that is $I^0 = \{i_1, i_2, i_3\}$ and $S^0 = \{s_1, s_2, s_3\}$. The waitlists are given by $W(s_1; \mu^0) = \{i_1, i_2, i_3\}$ and $W(s_2; \mu^0) = W(s_3; \mu^0) = \{i_2\}$; all students are transfer students and all schools are transfer schools.

s_1 points to i_1, who is the top $\succ_{s_1}$-ranked student among I^0, and s_2 and s_3 point to i_2. All students are waitlisted at s_1, which is i_2 and i_3's most preferred school among all available. Therefore i_2 and i_3 point to s_1. Since i_1 is waitlisted only at s_1, i_1 points to s_1 as well. (s_1, i_1, s_1) is the unique cycle so STC updates μ^0 to μ^1 by assigning i_1 to $s_1 : \mu^1 = (i_1 - s_1, i_2 - \varnothing, i_3 - \varnothing)$.

Given μ^1, I^0 and S^0, no student is assigned her first choice among available schools so every student in I^0 and every school in S^0 remain available, i.e. $I^1 = I^0$ and $S^1 = S^0$. The waitlists are given by $W(s_1; \mu^1) = \{i_2, i_3\}$, $W(s_2; \mu^1) = \{i_1\}$ and $W(s_3; \mu^1) = \{i_3\}$. All students remain transfer students and all schools remain transfer schools.

s_1 points to i_1, who is currently assigned s_1. s_1 does not have an empty seat, so it does not point to any other student. s_2 and s_3 point to i_2. Since i_2 and i_3 are waitlisted at their first choice s_1, they point to s_1, and i_1 points to s_2 at which he is waitlisted. $(s_1, i_1, s_2, i_2, s_1)$ is the only top stable transfer cycle so STC updates μ^1 to μ^2 by assigning i_1 to s_2 and i_2 to $s_1 : \mu^2 = (i_1 - s_2, i_2 - s_1, i_3 - \varnothing)$.

i_1 and i_2 are assigned their first choices among all available schools, so they become unavailable. Since their assigned schools do not have any more empty seats, those schools become unavailable as well. Therefore $I^2 = \{i_3\}$ and $S^2 = \{s_3\}$. So s_3 points to i_3, who points back to s_3, forming a cycle. i_3 is assigned s_3 so that $\mu^3 = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$. i_3 is assigned his most preferred school among all available schools, so he becomes unavailable and STC terminates with $\mu^{STC} = (i_1 - s_1, i_2 - s_2, i_3 - s_3)$.

In this example, STC produces the TTC assignment by coincidence. However, it is not a coincidence that STC is student optimal stable:

Theorem 9 *STC is student optimal stable.*

The proof of this theorem, which is deferred to the Appendix, proceeds as follows: By finiteness, STC stops in finite time. The stability of the final matching follows easily. We prove in the appendix that, if a matching that is produced in an intermediate step of STC is not student optimal stable, then there exists a top stable transfer cycle.

This result implies the following:

Corollary 10 *When all schools are stability-constrained and school priorities are strict, the STC outcome coincides with the GS outcome.*

7 Incentives

STC is not strategy-proof, as demonstrated in the following example.

Example 11 *There are three students* $\{1,2,3\}$ *and three schools* $\{s_1, s_2, s_3\}$ *each with one seat.* s_1 *is an unconstrained school,* s_2 *and* s_3 *are constrained. Student preferences and school priorities are given as follows:*

$$\begin{array}{lccccc} 1: & s_1 & \succ_1 & s_3 & \succ_1 & s_2 \\ 2: & s_2 & \succ_2 & s_1 & \succ_2 & s_3 \\ 3: & s_1 & \succ_3 & s_2 & \succ_3 & s_3 \end{array} \quad \textit{and} \quad \begin{array}{lccccc} s_1: & 2 & \succ_{s_1} & 3 & \succ_{s_1} & 1 \\ s_2: & 1 & \succ_{s_2} & 2 & \succ_{s_2} & 3 \\ s_3: & 1 & \succ_{s_3} & 2 & \succ_{s_3} & 3 \end{array}$$

If the students report their preferences truthfully, STC produces

$$\mu = \begin{pmatrix} 1 & 2 & 3 \\ s_3 & s_2 & s_1 \end{pmatrix},$$

If 1 falsely reports $s_1 \succ'_1 s_2 \succ'_1 s_3$, *then STC produces*

$$\nu = \begin{pmatrix} 1 & 2 & 3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

which 1 prefers to μ *under her true preference relation. So STC is not strategy proof.*

However, unless all schools are stability-constrained or all schools are unconstrained, there is no strategy-proof mechanism.

Proposition 12 *When* $S_c \neq \varnothing$ *and* $S_u \neq \varnothing$, *no student optimal stable mechanism is strategy-proof for students.*

Proof. The proof is an adoptation of the counter example in the proof of Proposition 1 in Ehlers and Westcamp (2010) after setting an arbitrary preference relation for s_3.[12] ■

When school priorities involve ties, Erdil and Ergin (2008) shows that there is no strategy proof and student optimal stable mechanism. Although our result looks similar to Erdil and Ergin's and Ehlers and Westcamp's findings, it is substantively different. Our negative result holds even when school priorities do not involve ties and it is driven by the simultaneous presence of both stability-constrained and unconstrained schools.

[12] Ehlers and Westcamp (2010) studies a model in which there are exam schools with strict preferences and regular schools which are indifferent among all students.

8 Extension

We have stated reasons for different sources of priorities at different schools. In reality, even the sources and policy implications of different priority levels at a given single school might differ. For instance, where as violating sibling priority might be politically undesired, violating neighborhood priority for the sake of student welfare might be desirable. The model can be generalized further to capture such instances.

In the more general model, every school s has a list of priority levels, $p_1^s, ..., p_n^s$; priority level p_k^s is higher than p_{k+1}^s, that is, if i's priority is p_k^s and j's priority is p_{k+1}^s, then $i \succ_s j$. In addition, each priority level p_k^s also has a status $\sigma_k^s \in$ {unconstrained (uc), stability-constrained (c)}. Given a matching μ, student i's priority at s is violated if $s \succ_i \mu(i)$, $i \succ_s j$ for some $j \in \mu(s)$, and i's priority level at s is stability constrained. A matching is pseudo-stable if no student's priority is violated. The null matching is trivially pseudo-stable.

Let $\iota(s,i)$ denote the sub-index of the priority level of student i at s. Given $(\succsim_S, \succsim_I)$, let μ be a pseudo-stable matching. For each school $s \in S$, the set of students who prefer s to their assignment and who are eligible for enrolment at s is given by

$$P(s;\mu) = \{i \in I : s \succ_i \mu(i) \text{ and } i \succ_s \varnothing\}$$

For each school s, let $\pi(s,\mu)$ be the sub-index of the priority level of the lowest priority student assigned s at μ. If $\mu(s) = \varnothing$, then let $\pi(s,\mu) = 1$.

Define $k(s,\mu)$ as follows. If $p_{\pi(s,\mu)}^s$ is stability-constrained and there is $i \in P(s;\mu)$ with $\iota(s,i) = \pi(s,\mu)$, then $k(s,\mu) = \pi(s,\mu)$. Otherwise, $k(s,\mu) > \pi(s,\mu)$ is such that

- there is $i \in P(s;\mu)$ with $\iota(s,i) = k(s,\mu)$,
- $p_{k(s,\mu)}^s$ is stability constrained,
- and for any k such that $k(s,\mu) > k \geq \pi(s,\mu)$ either p_k^s is unconstrained or there is no $i \in P(s;\mu)$ with $\iota(s,i) = k$.

Then the **waitlist** $W(s;\mu)$ for s at μ is the set of students in $P(s;\mu)$ with priority level of $p_{k(s,\mu)}^s$ or higher. That is,

$$W(s;\mu) = \{i \in P(s;\mu) : \iota(s,i) \leq k(s,\mu)\ \}.$$

The stable transfer cycles and STC are defined as before with this new definition of the waitlists and all the relevant results follow. Note that this model reduces to our model if for every school, either every priority level is unconstrained – then the school is unconstrained in our earlier terminology–, or every priority level is stability constrained – then the school is stability constrained in our earlier terminology–.

9 Conclusion

We identify a problem that exhibits features of one-sided matching and two-sided matching simultaneously. We introduce a new model that encompasses both one-sided matching and two-sided matching models as well as their hybrid. We also introduce a natural stability notion for the new model, characterize student optimal stable matchings and offer a new student optimal stable matching algorithm, which reduces to Top Trading Cycles when the problem is a one-sided matching problem and becomes equivalent to the Gale and Shapley's student optimal stable matching (GS) if it is a two-sided matching problem.

In a two-sided matching framework, the student optimal stable matching may not be Pareto efficient (Roth 1982). Furthermore, when school priorities involve ties, GS can still be applied after breaking ties. However GS may fail to be student optimal stable (Erdil and Ergin 2008; Abdulkadiroğlu, Pathak and Roth 2009). In contrast, we identify a substantially different source of inefficiency: When the model consists of schools at which priority violations are allowed and schools where priorities cannot be violated, respecting priorities at all schools causes inefficiency due the presence of the former type of schools.

The extent of inefficiency that is due to ties in school priorities in a two-sided matching framework has been studied by Erdil and Ergin (2008) via simulations and by Abdulkadiroğlu, Pathak and Roth (2009) with field data from New York City High School Match. In particular, Erdil and Ergin (2008) show that there is no strategy-proof and student optimal stable matching mechanism. They also provide the Stable Improvement Cycles algorithm to find a student optimal stable matching. Abdulkadiroğlu, Pathak and Roth (2009) show that GS with any fixed tie breaking is on the Pareto frontier of the strategy-proof mechanisms. They utilize a data set that was generated by such a mechanism and use Erdil and Ergin's stable improvement cycles to measure the cost of strategy-proofness as the inefficiency associated with the mechanism.

In a similar vein, even though there is no strategy-proof and student optimal stable mechanism in our general matching framework, our main characterization result and the STC can be used to study the trade-off between strategy-proofness and stability in the general framework. This question and the extent of efficiency gains with our algorithm is left as an empirical question for future investigation.

Furthermore, STC may become strategy-proof under certain informational assumptions in a manner similar to Erdil and Ergin (2008) and Featherstone and Niederle (2008). In particular, Erdil and Ergin (2008) show that there is no strategy proof and student optimal stable mechanism when school priorities involve ties; however their Stable Improvement Cycles mechanism becomes strategy-proof when students hold symmetric beliefs. Featherstone and Niederle (2008) show that the Boston mechanism, which is known to fail strategy-proofness, becomes strategy-proof in symmetric environments. We leave the question of whether STC becomes strategy-proof under certain informational assumptions as future research.

A Formal Definition of STC

A tie breaker at s is a one-to-one function $l_s : I \to \mathbb{R}$. l_s is potentially obtained by a fair lottery, it may be the same at all schools (single tie breaking), or different (multiple tie breaking). Given the tie breaker l_s, $\succ_s$ is obtained from $\succsim_s$ as follows: $i \succ_s j$ if and only if $[i \succsim_s j$ and (not $j \succsim_s i$ or $l_s(i) < l_s(j))]$. When we say i is top $\succ_s$-ranked among I', we mean that $i \succ_s j$ for all $j \in I' \backslash \{i\}$.

μ^0 is the null matching, i.e. $\mu^0(s) = \varnothing$ for all s, initialize the set of available students $I^{-1} = I$ and the set of available schools $S^{-1} = S$.

Given $(\mu^n, I^{n-1}, S^{n-1})$, compute the new set of available students I^n and schools S^n as follows: Set $\tilde{I}_0 = I^{n-1}$ and $\tilde{S}_0 = S^{n-1}$. Given $\tilde{I}_k$, $\tilde{S}_k$, let

$$\tilde{I}_{k+1} = \left\{ i \in \tilde{I}_k : \exists s \in \tilde{S}_k \text{ such that } s \succ_i \mu^n(i) \right\},$$
$$\tilde{S}_{k+1} = \left\{ s \in \tilde{S}_k : |\mu^n(s) \cap (I \backslash \tilde{I}_{k+1})| < q_s \right\}$$

and $I^n = \lim_k \tilde{I}_k$, $S^n = \lim_k \tilde{S}_k$. Note that $\tilde{I}_0$ and $\tilde{S}_0$ can be initialized as $\tilde{I}_0 = I$ and $\tilde{S}_0 = S$, but that would slow down the algorithm.

Note that $I^n \subset I^{n-1}$ and $S^n \subset S^{n-1}$. Furthermore, $I \backslash I^n$ is the set of students who are permanently assigned at μ^n. Also a school becomes unavailable when all of its seats are permanently assigned.[13]

For an available unconstrained school $s \in S_u \cap S^n$, the waitlist is given by the set of *available* students who prefer it to their assignment:

$$W(s; \mu^n) = \{i \in I^n : s \succ_i \mu^n(i)\}$$

For each available constrained school $s \in S_c \cap S^n$, the waitlist is given by the set of students who are top $\succsim_s$-ranked among *all* students – available or not – who prefer s to their assignment:

$$W(s; \mu^n) = \{i \in I : s \succ_i \mu^n(i) \text{ and } i \succsim_s j \text{ for all } j \in I \text{ such that } s \succ_j \mu^n(j)\}$$

The transfer students $TrStu(\mu^n)$ and transfer schools $TrSch(\mu^n)$ are determined recursively as follows: Set $\hat{I}_0 = I^n$ and $\hat{S}_0 = S^n$. Given $\hat{I}_k$, $\hat{S}_k$,

[13] Also note for the sake of speed that once a student is permanently assigned, the algorithm does not need to revisit that student at later steps; likewise when all the seats at a school are permanently assigned, the algorithm does not need to revisit that school.

let

$$\hat{I}_{k+1} = \left\{ i \in \hat{I}_k : \exists s \in \hat{S}_k \text{ such that } i \in W(s;\mu^n) \right\},$$

$$\hat{S}_{k+1} = \left\{ \begin{array}{l} s \in \hat{S}_k : \mu^n(s) \cap \hat{I}_{k+1} \neq \varnothing \text{ or} \\ \qquad\qquad \text{if } |\mu^n(s)| < q_s \text{ then } \exists i \in \hat{I}_{k+1} \backslash \mu^n(s) \text{ such that } i \succ_s \varnothing \end{array} \right\}$$

and $TrStu(\mu^n) = \lim_k \hat{I}_k$, $TrSch(\mu^n) = \lim_k \hat{S}_k$.

STC starts with the null matching μ^0, the set of available students $I^{-1} = I$ and the set of available schools $S^{-1} = S$. Given $(\mu^n, I^{n-1}, S^{n-1})$, STC computes (μ^{n+1}, I^n, S^n) as follows:

1. Find the set of available students I^n and the available schools S^n.
2. Find the waitlist $W(s;\mu^n)$ at every available school $s \in S^n$.
3. Find the transfer students $TrStu(\mu^n)$ and transfer schools $TrSch(\mu^n)$.
4. *Construct the following directional graph:*
   (a) Each transfer school $s \in TrSch(\mu^n)$ points to every transfer student i assigned s, i.e. $i \in \mu^n(s) \cap TrStu(\mu^n)$; if s has an empty seat, s points to the top $\succ_s$-ranked transfer student among $TrStu(\mu^n)$ if one exists (note that we apply tie breaking here).
   (b) Each student $i \in TrStu(\mu^n)$ points to her most preferred transfer school at which she is waitlisted, i.e. her most preferred school among $\{s \in TrSch(\mu^n) : i \in W(s;\mu^n)\}$.
   (c) A **top stable transfer cycle** is an ordered list of transfer schools and transfer students, $(s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$, such that s_k points to i_k and i_k points to s_{k+1}, $k = 1, ...K$
   (d) A cycle $(s_1, i_1, ..., s_K, i_K, s_{K+1} \equiv s_1)$ is **implemented** by assigning i_k to s_k, $k = 1, ..., K$.
   (e) Multiple cycles may exist. Two cycles are disjoint if they do not share a student. Implement all disjoint cycles.
5. Repeat from Step 1 until no more stable transfer cycles exist.

Proposition 13 *Let μ^n be the matching at the beginning of n-th round of STC. If μ^n is not student optimal stable, then there exists a top stable transfer cycle.*

Proof. We will prove that via the following claims:

Claim 1: $TrStu(\mu^n) \neq \varnothing$ if and only if $TrSch(\mu^n) \neq \varnothing$.

Proof of the claim: This follows from the definitions of $TrStu(\mu^n)$ and $TrSch(\mu^n)$. By finiteness, there exist k such that $TrStu(\mu^n) = \hat{I}_k$, $TrSch(\mu^n) = \hat{S}_k$. If either $\hat{I}_k = \varnothing$ or $\hat{S}_k = \varnothing$, then by definition, $\hat{I}_{k+1} = \varnothing$ and $\hat{S}_{k+1} = \varnothing$, which proves the claim.

Consider the set of available students I^n and the available schools S^n in step n of STC.

Claim 2: If $TrStu(\mu^n) \neq \varnothing$, or equivalently $TrSch(\mu^n) \neq \varnothing$, then there exists a top stable transfer cycle.

Proof of the claim: Any $i \in TrStu(\mu^n)$ points to a school in $TrSch(\mu^n)$ and any $s \in TrSch(\mu^n)$ points to a student in $TrStu(\mu^n)$. Therefore, by finiteness, a top stable transfer cycle exists if $TrStu(\mu^n) \neq \varnothing$.

Claim 3: If μ^n is not student optimal stable, then $TrStu(\mu^n) \neq \varnothing$.

Proof of the claim: Suppose that μ^n is not student optimal stable. Let υ be a pseudo-stable matching which Pareto dominates μ^n. Let $I' = \{i \in I : \upsilon(i) \succ_i \mu^n(i)\}$ and $S' = \{\mu^n(i) : i \in I'\}$.

Suppose that $I' \cap TrStu(\mu^n) = \varnothing$ and $S' \cap TrSch(\mu^n) = \varnothing$.

For each $x \in I' \cup S'$, let σ_x is the step of STC at which x becomes unavailable, i.e. $x \in I^{\sigma_x - 1} \cup S^{\sigma_x - 1}$ but $x \notin I^{\sigma_x} \cup S^{\sigma_x}$; set $\sigma_x = \infty$ if x never becomes unavailable.

Let $x \in I' \cup S'$ be such that $\sigma_x \leq \sigma_y$ for all $y \in I' \cup S'$.

Suppose that $\sigma_x \leq n$.

If x is a student, then by definition $\upsilon(x)$ must become unavailable at step σ_x or earlier, i.e. $\sigma_{\upsilon(x)} \leq \sigma_x$. By choice of x, $\sigma_{\upsilon(x)} = \sigma_x$. But then $\upsilon(x)$ must be fully assigned at step σ_x of STC, i.e. $|\mu^{\sigma_x}(\upsilon(x))| = q_{\upsilon(x)}$ and every $i \in \mu^{\sigma_x}(\upsilon(x))$ must be permanently assigned $\upsilon(x)$ and become unavailable at that step. So $\mu^n(i) = \mu^{\sigma_x}(i) = \upsilon(x)$ for all $i \in \mu^{\sigma_x}(\upsilon(x))$. Also $|\mu^{\sigma_x}(\upsilon(x))| = q_{\upsilon(x)}$, so that $|\mu^n(\upsilon(x))| = q_{\upsilon(x)}$. Since $\upsilon(x) \succ_x \mu^n(x)$ and $\upsilon(x)$ is fully assigned at μ^n, there exists $y \in \mu^{\sigma_x}(\upsilon(x))$ such that $\upsilon(y) \neq \mu^n(y)$, which implies $\upsilon(y) \succ_y \mu^n(y)$ so that $y \in I'$. But since $y \in \mu^{\sigma_x}(\upsilon(x))$ and $\upsilon(x)$ becomes unavailable in step σ_x, y must become unavailable in step $\sigma_y \leq \sigma_x$ as well. By choice of x, we have $\sigma_y = \sigma_x$.

Now consider all students $I'' \subset I'$ that become unavailable in step σ_x. Let $\{\tilde{I}_k, \tilde{S}_k\}$ be the sequence that produces $\{I^{\sigma_x}, S^{\sigma_x}\}$, the sets of available students and schools in step σ_x. For each $i \in I''$, let π_i be such that $i \in \tilde{I}_{\pi_i - 1}$ and $i \notin \tilde{I}_{\pi_i}$, $\pi_i < \infty$ since i becomes unavailable. Let $I''' \subset I''$ be such that for all $i \in I'''$, $\pi_i \leq \pi_j$ for all $j \in I''$ and $\pi_i < \pi_j$ for all $j \in I'' \backslash I'''$. Then $I''' \neq \varnothing$.

For each $i_t \in I''' \subset I'' \subset I'$, $\upsilon(i_t) \succ_{i_t} \mu^n(i_t)$ so that there exists $i_{t+1} \in \mu^n(\upsilon(i_t))$ such that $\upsilon(i_{t+1}) \neq \mu^n(i_{t+1})$, which implies $\upsilon(i_{t+1}) \succ_{i_{t+1}} \mu^n(i_{t+1})$ so that $i_{t+1} \in I'$. Since i_t cannot become unavailable before $\upsilon(i_t)$ becomes unavailable, i.e. $\pi_{\upsilon(i_t)} \leq \pi_{i_t}$, and $\upsilon(i_t)$ cannot become unavailable before i_{t+1} becomes unavailable, i.e. $\pi_{i_{t+1}} \leq \pi_{\upsilon(i_t)} \leq \pi_{i_t}$, $i_{t+1} \in I'''$ must hold as well. By finiteness, there exists a set of students $\{i_1, ..., i_T, i_{T+1} \equiv i_1\} \subset I'''$ such that i_t does not become unavailable if i_{t+1} does not become unavailable, $t = 1, ..., T$. But then by construction, $\{i_1, ..., i_T\}$ never become unavailable, which contradicts $\{i_1, ..., i_T\} \subset I'''$.

So $\sigma_x > n$. In other words, student in I' and schools in S' are available in step n of STC, i.e. $I' \subset I^n$ and $S' \subset S^n$.

Now let $\{\hat{I}_k, \hat{S}_k\}$ be the sequence of sets that produce $TrStu(\mu^n)$ and $TrSch(\mu^n)$. Since $I' \cap TrStu(\mu^n) = \varnothing$ and $S' \cap TrSch(\mu^n) = \varnothing$, for each $x \in I' \cup S'$, there exists $\tau_x < \infty$ such that $x \in \hat{I}_{\tau_x - 1} \cup \hat{S}_{\tau_x - 1}$ and $x \notin \hat{I}_{\tau_x} \cup \hat{S}_{\tau_x}$. Let $J \subset I'$ be such that for all $i \in J$, $\tau_i \leq \tau_j$ for all $j \in I'$ and $\tau_i < \tau_j$ for all $j \in I' \backslash J$. Then $I'' \neq \varnothing$.

For each $i_t \in J \subset I'$, $\upsilon(i_t) \succ_{i_t} \mu^n(i_t)$ so that there exists $i_{t+1} \in \mu^n(\upsilon(i_t))$ such that $\upsilon(i_{t+1}) \neq \mu^n(i_{t+1})$, which implies $\upsilon(i_{t+1}) \succ_{i_{t+1}} \mu^n(i_{t+1})$ so that $i_{t+1} \in I'$. Since i_t cannot dropped from $\hat{I}_{\tau_{i_t} - 1}$ before $\upsilon(i_t)$ is dropped from $\hat{S}_{\tau_{\upsilon(i_t)} - 1}$, i.e. $\tau_{\upsilon(i_t)} \leq \tau_{i_t}$, and $\upsilon(i_t)$ cannot be dropped from $\hat{S}_{\tau_{\upsilon(i_t)} - 1}$ before i_{t+1} dropped from $\hat{I}_{\tau_{i_{t+1}} - 1}$, i.e. $\tau_{i_{t+1}} \leq \tau_{\upsilon(i_t)} \leq \tau_{i_t}$, $i_{t+1} \in J$ must hold as well. By finiteness, there exists a set of students $\{i_1, ..., i_T, i_{T+1} \equiv i_1\} \subset J$ such that i_t is not dropped if i_{t+1} is not dropped, $t = 1, ..., T$. But then by construction, $\{i_1, ..., i_T\}$ is never dropped from $\hat{I}_{\tau_{i_1} - 1}$, which contradicts $\{i_1, ..., i_T\} \subset J$.

Therefore $I' \cap TrStu(\mu^n) \neq \varnothing$ which implies $TrStu(\mu^n) \neq \varnothing$. This completes the proof of Claim 3.

Then by proof of the proposition follows by Claim 3 and Claim 2. ■

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